Some 2-coloured 4-cycle decompositions

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Abstract

There are six distinct ways in which the vertices of a 4-cycle may be coloured with two colours, called *colouring types*. Let C be the set of these colouring types and let S be a non-empty subset of C. Suppose we colour the vertices of K_v with two colours. If D is a 4-cycle decomposition of K_v such that the colouring type of each 4-cycle is in S, then D is said to have a *colouring of type* S. Furthermore, the colouring is said to be *proper* if every colouring type in S is represented in D. For all possible S of size one, two or three, exluding three cases already settled, we completely settle the existence question for 4-cycle decompositions of K_v with a colouring of type S.

1 Introduction

Let G and H be graphs. A G-decomposition of H is a set $\mathcal{G} = \{G_1, G_2, \ldots, G_p\}$ such that G_i is isomorphic to G for $1 \leq i \leq p$ and $\mathcal{G}$ partitions the edge set of H. Most commonly, $H = K_v$, the complete graph on v vertices. Another popular choice for H is $K_v - F$, the complete graph with the edges of a 1-factor removed. The problem of determining all values of v for which there exists a G-decomposition of K_v is called the *spectrum problem* for G.

An m-cycle, denoted by $(x_1, x_2, \ldots, x_m)$, is the graph with vertex set $\{x_1, x_2, \ldots, x_m\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_m, x_1\}\}$. The spectrum problem for m-cycles has recently been solved; see [3] and [7].

A variant of the spectrum problem for m-cycles arises when the vertices of K_v have been coloured and there are demands on how each m-cycle in

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the decomposition must be coloured. An m-cycle is said to be *monochromatic* if all m vertices are the same colour. Conversely, an m-cycle with m distinctly coloured vertices is said to be *polychromatic.* A *weak colouring* results in no m-cycle of the decomposition being monochromatic. A *strong colouring* requires that each m-cycle of the decomposition be polychromatic. Many existing papers have considered coloured 3-cycle decompositions of K_v (Steiner triple systems) and a fine survey can be found in [5].

While most work has considered weak colourings, new colouring systems are emerging. If the vertices of H have been coloured with k colours and each m-cycle in a decomposition of H has n_i vertices of colour c_i, for $1 \leq i \leq k$, and $|n_i - n_j| \leq 1$, for all $i, j \in \{1, 2, \ldots, k\}$, then the decomposition is said to be *equitably k-coloured.* A decomposition that can be equitably k-coloured is said to be *equitably k-colourable.* Thus far, only equitably 2 and 3-colourable m-cycle decompositions of K_v, $K_v - F$ and certain multipartite graphs have been considered for small values of m; see [1], [2], [6] and [10].

In this paper we consider coloured 4-cycle decompostions of K_v. As such, we often use the following well-known theorem.

Theorem 1.1 *A 4-cycle decomposition of K_v exists if and only if* $v \equiv 1 \pmod 8$.

There are clearly six distinct ways in which a 4-cycle may be coloured with two colours, say black and white (denoted by B and W respectively). Let $C_1C_2C_3C_4$ denote the colouring of the 4-cycle (x_1, x_2, x_3, x_4) which assigns the colour C_i to the vertex x_i, where $C_i \in \{B, W\}$ for $i = 1, \ldots, 4$,

Definition 1.2 *Let the colouring* BBBB *be denoted* Type A1, WWWW *be denoted* Type A2, BBBW *be denoted* Type B1, WWWB *be denoted* Type B2, BBWW *be denoted* Type C *and* BWBW *be denoted* Type D.

Let S be a subset of $C = \{$A1, A2, B1, B2, C, D$\}$. (For the sake of brevity we omit the word *Type*). Supposing that the vertices of K_v have been coloured with two colours, then a 4-cycle decomposition of this graph is said to be of colouring Type S if the colouring type of every 4-cycle in the decomposition is in S. Furthermore, the colouring is said to be *proper* if every colouring type in S is represented in the decomposition. Within this paper, every colouring is proper.

The cases $S = \{B1, B2\}$ and $S = \{B1, B2, D\}$ were solved by Quattrocchi in [8]. The case $S = \{C, D\}$ involves finding an equitably 2-coloured 4-cycle decomposition of K_v and this was solved in [2]. Here, we consider the remaining cases where $|S| \in \{1, 2, 3\}$. Note that some colouring types can be trivially obtained from other types by simply interchanging the colours of all vertices. If S_1 and S_2 are sets of such colouring types then in Table 1 and the constructions we write $S_1 \equiv S_2$.

Main Theorem There exist 2-coloured 4-cycle decompositions of K_v with colouring Type S with $|S| \in \{1,2,3\}$ if and only if the conditions in Table 1 are satisfied. Note that in every case, no decomposition with a proper colouring of Type S exists for $v = 1$.

S.	permissible v	reference
{A1} ≡ {A2}	$v \equiv 1 \pmod 8$, $v \neq 1$	Thm 2.1
{B1} ≡ {B2}	none	Thm 2.2
{C}, {D}	none	Thm 2.3
{A1, A2}	none	Thm 3.1
{A1, B1} ≡ {A2, B2}	$v \equiv 1 \pmod 8$, $v \neq 1$	Thm 3.5
{A1, B2} ≡ {A2, B1}	none	Thm 3.6
{A1, C} ≡ {A2, C}	none	Thm 3.7
{A1, D} ≡ {A2, D}	none	Thm 3.8
{B1, C} ≡ {B2, C}	$v \equiv 1 \pmod 8$, $v \neq 1$, $\sqrt{v} \in \mathbb{Z}$	Thm 3.15
{B1, D} ≡ {B2, D}	none	Thm 3.16
{B1, B2}	$v = 49 + 112\mu + 64\mu^2$, $\mu \geq 0$; $v = 1 + 16\mu + 64\mu^2$, $\mu \geq 1$	[8]
{C, D}	none	[2]
{A1, A2, B1} ≡ {A1, A2, B2}	$v \equiv 1 \pmod 8$, $v \neq 1, 9, 17, 25$	Thm 4.6
{A1, A2, C}	none	Thm 4.7
{A1, A2, D}	none	Thm 4.8
{A1, B1, B2} ≡ {A2, B1, B2}	$v \equiv 1 \pmod 8$, $v \neq 1, 9$	Thm 4.12
{A1, B1, C} ≡ {A2, B2, C}	$v \equiv 1 \pmod 8$, $v \neq 1$	Thm 4.15
{A1, B1, D} ≡ {A2, B2, D}	none	Thm 4.16
{A1, B2, C} ≡ {A2, B1, C}	$v \equiv 1 \pmod 8$, $v \neq 1, 9$	Thm 4.21
{A1, B2, D} ≡ {A2, B1, D}	$v \equiv 1 \pmod 8$, $v \neq 1, 9$	Thm 4.25
{A1, C, D} ≡ {A2, C, D}	$v \equiv 1 \pmod 8$, $v \neq 1$	Thm 4.28
{B1, B2, C}	$v \equiv 1 \pmod 8$, $v \neq 1$, $\sqrt{v} \in \mathbb{Z}$	Thm 4.30
{B1, C, D} ≡ {B2, C, D}	$v \equiv 1 \pmod 8$, $v \neq 1$	Thm 4.32
{B1, B2, D}	$v \equiv 1 \pmod 8$, $v \neq 1$	[8]

Table 1. The spectrum problem for 4-cycles with proper colouring type S, $|S| \in \{1,2,3\}$.

We now introduce some terminology and notation to be used throughout this paper. We say that an edge is *pure-coloured* if it connects two vertices of the same colour. Alternately, an edge is said to be *mixed-coloured* if it connects two vertices of different colours. We let the lower case letters b and w denote the number of black and white vertices in K_v respectively. We let $G - H$ denote the graph G with the edges of the graph H removed.

In Section 2 we consider the case $|S| = 1$, in Section 3 we have $|S| = 2$ and in Section 4 we deal with the case $|S| = 3$.

2 $|S| = 1$

2.1 $S = \{\mathrm{A1}\} \equiv S = \{\mathrm{A2}\}$

Theorem 2.1 *There exists a 4-cycle decomposition of K_v with colouring type $\{\mathrm{A1}\}$ if and only if $v \equiv 1 \pmod 8$, $v \neq 1$.*

Proof. The result follows immediately from Theorem 1.1 (colour every vertex black). □

2.2 $S = \{\mathrm{B1}\} \equiv S = \{\mathrm{B2}\}$

Theorem 2.2 *There exist no 4-cycle decompositions of K_v with colouring type $\{\mathrm{B1}\}$.*

Proof. Let $v > 1$ and suppose that the decomposition exists. A 4-cycle of Type B1 has two pure-coloured edges between black vertices, two mixed-coloured edges and no pure-coloured edges between white vertices. Hence $w = 1$ and, since $bw = b(b-1)/2$, $v = 4$. However, by Theorem 1.1, K_4 cannot be decomposed into 4-cycles. □

2.3 $S = \{\mathrm{C}\}$ and $S = \{\mathrm{D}\}$

Theorem 2.3 *There exist no 4-cycle decompositions of K_v with colouring type $\{\mathrm{C}\}$ or $\{\mathrm{D}\}$.*

Proof. Suppose a decomposition exists. Each 4-cycle has two black vertices and two white vertices. Hence, $b = w$ and so v is even. However, a cycle decomposition of K_v is possible only if v is odd. □

3 $|S| = 2$

3.1 $S = \{\mathrm{A1, A2}\}$

Theorem 3.1 *There exist no 4-cycle decompositions of K_v with colouring type $\{\mathrm{A1, A2}\}$.*

Proof. This follows immediately as neither a 4-cycle of either Type A1 nor Type A2 have any mixed-coloured edges. □

3.2 $S = \{\text{A1, B1}\} \equiv S = \{\text{A2, B2}\}$

We begin with some useful results.

Theorem 3.2 [9] *There exists an uncoloured 4-cycle decomposition of $K_{m,n}$ if and only if m and n are both even, $m \geq 2$, $n \geq 2$ and $4|mn$.*

Corollary 3.3 *There exists a 4-cycle decomposition of $K_{m,n}$ with colouring type* $\{\text{A1}\}$ *if and only if m and n are both even, $m \geq 2$, $n \geq 2$ and $4|mn$.*

Lemma 3.4 *There exists a 4-cycle decomposition of K_9 with colouring type* $\{\text{A1, B1}\}$.

Proof. Let the vertex set of K_9 be $\mathbb{Z}_9$. Colour the vertices $1, 2, \ldots, 8$ black and colour the vertex 0 white. A suitable decomposition is given by: $(3,4,1,5)$, $(3,7,2,8)$, $(1,2,5,6)$, $(2,4,8,6)$, $(4,5,8,7)$, $(1,3,2,0)$, $(3,6,4,0)$, $(5,7,6,0)$ and $(7,1,8,0)$. □

Theorem 3.5 *There exists a 4-cycle decomposition of K_v with colouring type* $\{\text{A1, B1}\}$ *if and only if $v \equiv 1 \pmod 8$, $v \neq 1$.*

Proof. The necessary conditions follow from Theorem 1.1 and the fact that the colouring is proper.

Let $v = 8x + 1$, for $x \geq 1$. Let the vertex set of K_v be $(\cup_{i=1}^{x} X_i) \cup \{\infty\}$, where $|X_i| = 8$ for $i = 1, 2, \ldots, x$. Colour all vertices in X_i black, for $i = 1, 2, \ldots, x$, and colour the vertex ∞ white.

By Lemma 3.4, we can place a copy of the decomposition of K_9 with colouring type $\{\text{A1, B1}\}$ on $X_i \cup \{\infty\}$. By Corollary 3.3, we can place a copy of the decomposition of $K_{8,8}$ with colouring type $\{\text{A1}\}$ on $X_i \cup X_j$, for $1 \leq i < j \leq x$. The result is a 4-cycle decomposition of K_v with colouring type $\{\text{A1, B1}\}$. □

3.3 $S = \{\text{A1, B2}\} \equiv S = \{\text{A2, B1}\}$

We begin with some observations. A 4-cycle of Type A1 has four pure-coloured edges, each of which connects two black vertices. A 4-cycle of Type B2 has two pure-coloured edges, each of which connects two white vertices, and two mixed-coloured edges.

Theorem 3.6 *There exist no 4-cycle decompositions of K_v with colouring type* $\{\text{A1, B2}\}$.

Proof. Let $v > 1$ and suppose that the decomposition exists. We now look for a contradiction. From the above observations, $w(w-1)/2 = bw$, so $w = 2b + 1$ is odd. Now consider a black vertex x in K_v. Edges in K_v connecting x and any white vertices can only be accounted for in the decomposition using 4-cycles of Type B2 and, within each such cycle, x is adjacent to two white vertices. Hence, w is even. Thus we have a contradiction. □

3.4 $S = \{$A1, C$\} \equiv S = \{$A2, C$\}$

Theorem 3.7 *There exist no 4-cycle decompositions of K_v with colouring type* {A1, C}.

Proof. Suppose $v > 1$ and that the decomposition exists. Pure-coloured edges between two white vertices and mixed-coloured edges occur only in 4-cycles of Type C, so $bw = w(w-1)$ and hence $b = w - 1$. Furthermore, as $v = b + w$, then $b = (v-1)/2$ and $w = (v+1)/2$. Also, as the colouring is proper, the number of mixed-coloured edges must be less than twice the number of 4-cycles in the decomposition. Thus $bw < v(v-1)/4$. However, substituting for b and w gives $bw = (v-1)(v+1)/4 > v(v-1)/4$, a contradiction. □

3.5 $S = \{$A1, D$\} \equiv S = \{$A2, D$\}$

Theorem 3.8 *There exist no 4-cycle decompositions of K_v with colouring type* {A1, D}.

Proof. Let $v > 1$ and suppose the decomposition exists. There can be no pure-coloured edges between white vertices, so $w \leq 1$. However, at least one 4-cycle in the decomposition must be of Type D, which contains two white vertices. □

3.6 $S = \{$B1, C$\} \equiv S = \{$B2, C$\}$

This case is the most involved thus far. We begin with a number of existence results.

Lemma 3.9 *There exists a 4-cycle decomposition of K_9 with colouring type* {B1, C}.

Proof. Let the vertex set of K_9 be $\mathbb{Z}_9$. Colour the vertices $0, 1, \ldots, 5$ black and the vertices 6, 7 and 8 white. A suitable decomposition is given by: $(2,3,1,6)$, $(3,0,4,6)$, $(2,4,3,7)$, $(5,1,4,7)$, $(0,2,1,8)$, $(3,5,2,8)$, $(1,0,6,7)$, $(5,0,7,8)$ and $(5,4,8,6)$. □

Lemma 3.10 *There exists a 4-cycle decomposition of $K_9 - K_{2,2}$ with colouring type* {B1, C}.

Proof. Let the vertex set of K_9 be $\mathbb{Z}_9$. Colour the vertices $0, 1, 2, 3$ and 8 black and the vertices $4, 5, 6, 7$ white. Let the edges of the copy of $K_{2,2}$, be $\{0, 4\}$, $\{0, 5\}$, $\{1, 4\}$ and $\{1, 5\}$. A suitable decomposition is given by: $(0, 1, 7, 6)$, $(0, 8, 5, 7)$, $(1, 2, 4, 6)$, $(8, 2, 7, 4)$, $(3, 8, 6, 5)$, $(2, 3, 4, 5)$, $(2, 0, 3, 6)$ and $(8, 1, 3, 7)$. □

Lemma 3.11 *There exists a 4-cycle decomposition of $K_{8,8}$ with colouring type* {C}.

Proof. Let the vertex set of $K_{8,8}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 7_i\}$. Colour the vertices 0_i, 1_i, 2_i and 3_i black for $i \in \{1, 2\}$. Colour the remaining vertices white. A suitable decomposition is given by: $(0_1, i_2, 4_1, (i + 4)_2)$, $(1_1, i_2, 5_1, (i + 4)_2)$, $(2_1, i_2, 6_1, (i + 4)_2)$ and $(3_1, i_2, 7_1, (i + 4)_2)$, where $i = 0, 1, 2, 3$. □

Lemma 3.12 *There exists a 4-cycle decomposition of $K_{8,8}$ with colouring type* {B1, C}.

Proof. Let the vertex set of $K_{8,8}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 7_i\}$. Colour the vertices $0_1, 1_1, 2_1, 3_1$, $0_2, 1_2, \ldots, 4_2$ black and the remaining vertices white. A suitable decomposition is given by: $(0_1, i_2, 4_1, (i+3)_2)$, $(1_1, i_2, 5_1, (i+3)_2)$, $(2_1, i_2, 6_1, (i + 3)_2)$ and $(3_1, i_2, 7_1, (i + 3)_2)$ for $i \in \{2, 3, 4\}$, along with $(0_2, i_1, 1_2, (i + 4)_1)$ for $i \in \{0, 1, 2, 3\}$. □

Lemma 3.13 *Let G be the graph $K_{8,8,8}$ with vertex set $\cup_{i=1,2,3}\{0_i, 1_i, \ldots, 7_i\}$. Colour the vertices $0_i, 1_i, \ldots 4_i$ black for $i = 1, 2$. In the third part, colour the vertices 0_3, 1_3, 2_3 and 3_3 black. All remaining vertices are coloured white. Let H be the graph formed by adding four edges to G, so $H = G \cup \{\{0_3, 4_3\}, \{0_3, 5_3\}, \{1_3, 4_3\}, \{1_3, 5_3\}\}$. There exists a 4-cycle decomposition of H with colouring type* {B1, C}.

Proof. A suitable decomposition is given by:

$(2_1, 2_2, 4_3, 6_2)$, $(0_1, 3_2, 4_3, 5_2)$, $(3_1, 0_2, 5_3, 5_2)$, $(4_1, 1_2, 5_3, 6_2)$, $(2_1, 4_2, 6_3, 5_2)$,
$(3_1, 2_3, 6_2, 6_3)$, $(4_1, 2_3, 7_2, 6_3)$, $(0_1, 3_3, 5_2, 7_3)$, $(1_1, 3_3, 6_2, 7_3)$, $(2_1, 3_3, 7_2, 7_3)$,
$(2_2, 0_1, 4_3, 7_1)$, $(4_2, 1_1, 4_3, 5_1)$, $(3_2, 2_1, 5_3, 7_1)$, $(0_2, 4_1, 5_3, 5_1)$, $(1_2, 1_1, 6_3, 7_1)$,
$(0_2, 2_3, 5_1, 6_3)$, $(1_2, 2_3, 6_1, 6_3)$, $(0_2, 3_3, 7_1, 7_3)$, $(1_2, 3_3, 5_1, 7_3)$, $(4_2, 3_3, 6_1, 7_3)$,
$(0_3, 2_1, 7_2, 5_1)$, $(0_3, 3_1, 6_2, 6_1)$, $(1_3, 0_1, 6_2, 5_1)$, $(1_3, 1_1, 5_2, 6_1)$, $(2_3, 1_1, 7_2, 7_1)$,
$(0_3, 4_2, 6_1, 7_2)$, $(0_3, 0_2, 7_1, 6_2)$, $(0_3, 1_2, 5_1, 5_2)$, $(1_3, 4_2, 7_1, 5_2)$, $(0_1, 1_2, 6_1, 5_3)$,
$(2_1, 0_2, 6_1, 4_3)$, $(4_2, 0_1, 7_2, 4_3)$, $(2_2, 4_1, 7_2, 5_3)$, $(0_2, 0_1, 0_3, 4_3)$, $(3_2, 1_1, 0_3, 5_3)$,
$(1_2, 2_1, 1_3, 4_3)$, $(4_2, 3_1, 1_3, 5_3)$, $(1_3, 0_2, 1_1, 6_2)$, $(1_3, 1_2, 3_1, 7_2)$, $(2_3, 4_2, 4_1, 5_2)$,
$(3_1, 3_2, 4_1, 4_3)$, $(3_1, 2_2, 1_1, 5_3)$, $(0_1, 2_3, 2_1, 6_3)$, $(3_1, 3_3, 4_1, 7_3)$, $(2_2, 0_3, 3_2, 6_3)$,
$(2_2, 1_3, 3_2, 7_3)$, $(2_2, 2_3, 3_2, 5_1)$, $(2_2, 3_3, 3_2, 6_1)$, $(0_3, 4_1, 1_3, 7_1)$.

□

Lemma 3.14 *Let $v_k = (2k+1)^2$, $k \geq 2$. There exists a 4-cycle decomposition of $K_{v_k} - K_{v_{(k-1)}}$ with colouring type* {B1, C}.

Proof. Arrange the vertices into one group of $8k$ vertices, of which $4k+1$ are coloured black and $4k-1$ are coloured white, and one group of $(2k-1)^2$ vertices, of which $k(2k-1)$ are coloured black and $(k-1)(2k-1)$ are coloured white. The hole will be placed on the second group of vertices.

Consider the group of $8k$ vertices. Divide these vertices into k subgroups of eight vertices each: one group, labelled W, with five black and three white vertices, and $(k-1)$ groups, labelled $X_1, X_2, \ldots, X_{(k-1)}$, each containing four black and four white vertices.

Now consider the group of $(2k-1)^2$ vertices. Divide these vertices into a single vertex and $k(k-1)/2$ subgroups of eight vertices each. Let the single vertex be labelled ∞ and coloured black. Let $k-1$ subgroups, labelled $Y_1, Y_2, \ldots, Y_{(k-1)}$, each contain five black and three white vertices, and the remaining $(k-1)(k-2)/2$ subgroups of eight vertices, labelled $Z_1, Z_2, \ldots, Z_{(k-1)(k-2)/2}$, each contain four black and four white vertices; see Figure 1.

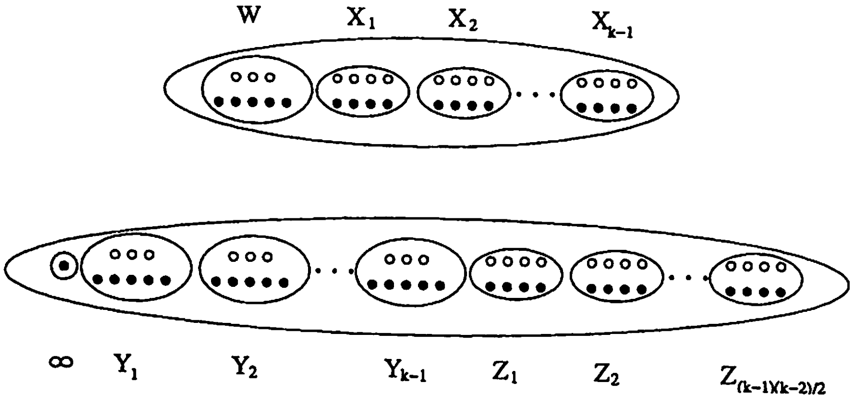

Figure 1: The vertex set of the graph $K_{v_k} - K_{v_{(k-1)}}$, where $v_k = (2k+1)^2$, $k \geq 2$.

By Lemma 3.9, we can place a 4-cycle decomposition of K_9 and colouring type {B1, C} on $W \cup \{\infty\}$. By Lemma 3.10, we can place a 4-cycle decomposition of $K_9 - K_{2,2}$ with colouring type {B1, C} on $X_i \cup \{\infty\}$, for $1 \leq i \leq k-1$. Here the edges of each copy of $K_{2,2}$ are entirely within X_i, for $1 \leq i \leq k-1$. By Lemma 3.13, we can also place a 4-cycle decomposition of H, as defined in Lemma 3.13, with colouring type {B1, C} on $W \cup Y_i \cup X_i$, for $1 \leq i \leq k-1$. Place a copy of the decomposition of $K_{8,8}$ given in Lemma 3.11 on $X_i \cup X_j$, for $1 \leq i < j \leq k-1$, and on $X_i \cup Z_j$, for $1 \leq i \leq k-1$ and

$1 \leq j \leq (k-1)(k-2)/2$. Place a copy of the decomposition of $K_{8,8}$ given in Lemma 3.12 on $W \cup Z_i$, for $1 \leq i \leq (k-1)(k-2)/2$, and on $X_i \cup Y_j$, for $1 \leq i, j \leq k-1$, excluding the case $i = j$. □

Theorem 3.15 *There exists a* 4*-cycle decomposition of* K_v *with colouring type* {B1, C} *if and only if* $v \equiv 1 \pmod 8$, $v \neq 1$ *and* $\sqrt{v} \in \mathbb{Z}$.

Proof. Let there be b black and w white vertices in K_v. As a 4-cycle of colouring Type B1 or C always contains two mixed-coloured edges, then a decomposition is possible only if the number of mixed-coloured edges in K_v is twice the number of 4-cycles in the decomposition. Consequently, $bw = b(v-b) = v(v-1)/4$. From this, we can obtain a quadratic expression in b, the solution of which is $b = (v \pm \sqrt{v})/2$. Without loss of generality let $b = (v + \sqrt{v})/2$, so $w = (v - \sqrt{v})/2$. For b and w to be integers, we require that $\sqrt{v} \in \mathbb{Z}$. This fact, combined with Theorem 1.1, provides the necessary conditions for a 4-cycle decomposition of K_v with colouring type {*B1*, *C*}.

To prove sufficiency, we use an inductive method. Let $v_k = (2k+1)^2$, for $k \geq 2$. The appropriate 4-cycle decomposition exists for K_9, by Lemma 3.9. By Lemma 3.14, there exists a 4-cycle decomposition of $K_{v_k} - K_{v_{(k-1)}}$, with colouring type {B1, C}. Thus, the decomposition exists for all $v = v_k$, $k \geq 1$. □

3.7 $S = \{\text{B1, D}\} \equiv S = \{\text{B2, D}\}$

Theorem 3.16 *There exist no* 4*-cycle decompositions of* K_v *with colouring type* {B1, D}.

Proof. The proof mirrors that for Theorem 3.8. □

4 $|S| = 3$

4.1 $S = \{\text{A1, A2, B1}\} \equiv S = \{\text{A1, A2, B2}\}$

Lemma 4.1 *There exists a* 4*-cycle decomposition of* $K_{33} - K_9$ *with colouring type* {A1, A2, B1}.

Proof. Let the vertex set of $K_{33} - K_9$ be $\{\infty_1, 0_1, 1_1, \ldots, 7_1\} \cup (\cup_{i=1,2,3} \{0_{2i}, 1_{2i}, \ldots, 7_{2i}\})$. The hole is on the vertices with subscript 1. Colour the vertices in the hole white and all other vertices black. Let the vertices $0_{2i}, 1_{2i}, \ldots, 7_{2i}$ be contained in the set Y_i, $i = 1, 2, 3$.

By Lemma 3.4 we can place a 4-cycle decomposition of K_9 with colouring type {A1, B1} on $\{\infty_1\} \cup Y_i$, for $i = 1, 2, 3$. The remaining 4-cycles in the decomposition are: $(0_{2i}, j_{2(i+1)}, 1_{2i}, j_1)$, $(2_{2i}, j_{2(i+1)}, 3_{2i}, j_1)$, $(4_{2i}, j_{2(i+1)}, 5_{2i}, j_1)$ and $(6_{2i}, j_{2(i+1)}, 7_{2i}, j_1)$, where $i = 1, 2, 3$ and $j = 0, 1, \ldots, 7$. □

Lemma 4.2 *There exists a 4-cycle decomposition of K_{33} with colouring type* {A1, A2, B1}.

Proof. Let the vertex set of K_{33} be $X \cup Y$, where $|X| = 24$ and $|Y| = 9$. Colour all the vertices in X black and all the vertices in Y white. By Theorem 2.1, we can place a 4-cycle decomposition of K_9 with colouring type {A2} on Y. By Lemma 4.1, we can place a 4-cycle decomposition of $K_{33} - K_9$ with colouring type {A1, A2, B1} on $X \cup Y$, where the hole is on the vertices in Y. □

Lemma 4.3 *There exists a 4-cycle decomposition of $K_{8,8}$ with colouring type* {B1}.

Proof. Let the vertex set of $K_{8,8}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 7_i\}$. Colour the vertices 4_1, 5_1, 6_1 and 7_1 white and the remaining vertices black. A suitable decomposition is given by: $(i_2, 0_1, (i+1)_2, 4_1)$, $(i_2, 1_1, (i+1)_2, 5_1)$, $(i_2, 2_1, (i+1)_2, 6_1)$ and $(i_2, 3_1, (i+1)_2, 7_1)$, where $i \in \{0, 2, 4, 6\}$. □

Lemma 4.4 *There exists a 4-cycle decomposition of $K_{41} - K_{33}$ with colouring type* {A1, B1}.

Proof. Let the vertex set of $K_{41} - K_{33}$ be $(\{\infty\} \cup X_1 \cup X_2 \cup Y_1 \cup Y_2) \cup Z$, where $|X_i| = |Y_i| = 8$ for $i = 1, 2$ and $|Z| = 8$. Let the hole be on the vertices $\{\infty\} \cup X_1 \cup X_2 \cup Y_1 \cup Y_2$. Colour the vertex ∞ white. Colour all vertices in X_1, X_2 and Z black. Colour four vertices of Y_i black and four white, for $i = 1, 2$.

By Lemma 3.4, we can place a 4-cycle decomposition of K_9 with colouring type {A1, B1} on $\{\infty\} \cup Z$. By Corollary 3.3, we can place a 4-cycle decomposition of $K_{8,8}$ with colouring type {A1} on $X_i \cup Z$, for $i = 1, 2$. Finally, by Lemma 4.3, we can place a 4-cycle decomposition of $K_{8,8}$ with colouring type {B1} on $Y_i \cup Z$, for $i = 1, 2$. □

Lemma 4.5 *There do not exist 4-cycle decompositions of K_9, K_{17} and K_{25} with colouring type* {A1, A2, B1}.

Proof. Each 4-cycle of Type A1 (Type A2) contains four pure-coloured edges between two black (white) vertices. Each 4-cycle of Type B1 contains two mixed-coloured edges and two edges connecting two black vertices.

Consequently, the number of edges connecting two black vertices must exceed the number of mixed-coloured edges, so $b > 2w + 1$. Furthermore, the number of pure-coloured edges between white vertices in K_v must be a multiple of four. Hence, $w \equiv 0, 1 \pmod 8$. However, as every white vertex in a 4-cycle of Type A1, A2 or B1, is adjacent to either two black vertices or two white vertices, w is odd. Hence, $w \equiv 1 \pmod 8$, $w \geq 9$. Clearly, for each of K_9, K_{17} and K_{25} it is not possible to have both $w \equiv 1 \pmod 8$, $w \geq 9$ and $b > 2w + 1$. □

Theorem 4.6 *There exists a 4-cycle decomposition of K_v with colouring type* {A1, A2, B1} *if and only if* $v \equiv 1 \pmod 8$, $v \neq 1, 9, 17, 25$.

Proof. The necessary conditions arise from Theorem 1.1 and Lemma 4.5.

Now, suppose $v \geq 33$. Let $v = 8x + 33$, where $x \geq 0$. The case $x = 0$ is covered in Lemma 4.2. Let the vertex set of K_v be $(\cup_{i=1}^{x} X_i) \cup Y$, where $|X_i| = 8$ for $i = 1, 2, \ldots, x$ and $|Y| = 33$. Colour all the vertices in X_i black, for $i = 1, 2, \ldots, x$ and colour twenty-four vertices in Y black and nine white.

By Corollary 3.3, we can place a 4-cycle decomposition of $K_{8,8}$ with colouring type {A1} on $X_i \cup X_j$, for $1 \leq i < j \leq x$. By Lemma 4.2, we can place a 4-cycle decomposition of K_{33} with colouring type {A1, A2, B1} on Y. Finally, by Lemma 4.4, we can place a 4-cycle decomposition of $K_{41} - K_{33}$ with colouring type {A1, B1} on $Y \cup X_i$, for $i = 1, 2, \ldots, x$, where the hole is on the vertices in Y. □

4.2 $S = \{$A1, A2, C$\}$

Theorem 4.7 *There exist no 4-cycle decompositions of K_v with colouring type* {A1, A2, C}.

Proof. Let $v > 1$ and suppose the decomposition exists. Without loss of generality let $b > w$. Mixed-coloured edges only occur in 4-cycles of Type C, each of which contains two such edges. Suppose that within the decomposition there are n 4-cycles of Type C, then $n = bw/2$. Furthermore, there must be more than n pure-coloured edges between two white vertices and so $w(w-1) > 2n = bw$. Simplifying this we find that $b < w - 1$, which is a contradiction. □

4.3 $S = \{$A1, A2, D$\}$

Theorem 4.8 *There exist no 4-cycle decompositions of K_v with colouring type* {A1, A2, D}.

Proof. Suppose that $v > 1$ and that the decomposition exists. Without loss of generality, let b be odd and w be even, as v is odd. Consider a white vertex x, say. Then x is adjacent to an odd number of white vertices in K_v. However, x can only be adjacent to an even number of white vertices in the decomposition. □

4.4 $S = \{$A1, B1, B2$\} \equiv S = \{$A2, B1, B2$\}$

Lemma 4.9 *There does not exist a 4-cycle decomposition of K_9 with colouring type* {A1, B1, B2}.

Proof. Suppose the decomposition exists. There are nine 4-cycles in a 4-cycle decomposition of K_9, at least one of which is of Type A1. Hence, $b \geq 4$. By similar reasoning $w \geq 3$. Combining these two inequalities, $w \in \{3, 4, 5\}$. Only 4-cycles of Types B1 and B2 contain mixed-coloured edges. Indeed, each of these 4-cycles contain two such edges and so the number of mixed-coloured edges in K_v is less than twice the number of 4-cycles in the decomposition. Hence, $bw < 18$. However, this condition is not satisfied when $w \in \{3, 4, 5\}$. □

Lemma 4.10 *There exists a* 4*-cycle decomposition of* K_{17} *with colouring type* {A1, B1, B2}.

Proof. Let the vertex set of K_{17} be $\mathbb{Z}_{17}$. Colour the vertices $4, 5, \ldots, 16$ black and the vertices 0, 1, 2 and 3 white. A suitable decomposition is given by:

$(0,1,2,4)$,	$(0,2,3,5)$,	$(0,3,1,6)$,	$(15,6,13,9)$,	$(6,14,13,12)$,
$(14,5,8,12)$,	$(16,4,10,5)$,	$(12,4,5,7)$,	$(8,7,10,16)$,	$(14,15,13,8)$,
$(12,16,7,9)$,	$(0,7,4,8)$,	$(0,9,4,11)$,	$(0,10,6,16)$,	$(0,12,5,13)$,
$(0,14,4,15)$,	$(1,4,6,5)$,	$(1,7,6,8)$,	$(1,9,5,11)$,	$(1,10,8,15)$,
$(1,12,10,13)$,	$(1,14,9,16)$,	$(2,5,15,7)$,	$(2,6,9,8)$,	$(2,9,10,11)$,
$(2,10,14,16)$,	$(2,12,11,13)$,	$(2,14,11,15)$,	$(3,4,13,7)$,	$(3,6,11,8)$,
$(3,9,11,16)$,	$(3,10,15,12)$,	$(3,11,7,14)$,	$(3,13,16,15)$.	

□

Lemma 4.11 *There exists a* 4*-cycle decomposition of* $K_{25} - K_{17}$ *with colouring type* {A1, B1}.

Proof. Let the vertex set of $K_{25} - K_{17}$ be $\{0_1, 1_1, \ldots, 16_1\} \cup \{0_2, 1_2, \ldots, 7_2\}$, where the hole is on the vertices with subscript 1. Colour the vertices 13_1, 14_1, 15_1 and 16_1 white and the remaining vertices black. A suitable decomposition is given by:

$(13_1,0_2,0_1,1_2)$,	$(13_1,2_2,0_1,3_2)$,	$(14_1,0_2,1_1,1_2)$,	$(14_1,2_2,1_1,3_2)$,
$(15_1,0_2,2_1,1_2)$,	$(15_1,2_2,2_1,3_2)$,	$(16_1,0_2,9_1,4_2)$,	$(16_1,1_2,9_1,5_2)$,
$(13_1,4_2,0_2,5_2)$,	$(13_1,6_2,4_2,7_2)$,	$(14_1,4_2,5_2,6_2)$,	$(14_1,5_2,3_2,7_2)$,
$(15_1,4_2,2_2,7_2)$,	$(15_1,5_2,7_2,6_2)$,	$(16_1,6_2,1_2,2_2)$,	$(16_1,7_2,1_2,3_2)$,
$(10_1,0_2,6_2,2_2)$,	$(10_1,1_2,4_2,3_2)$,	$(11_1,6_2,3_2,0_2)$,	$(11_1,7_2,0_2,1_2)$,
$(12_1,0_2,2_2,3_2)$,	$(12_1,1_2,5_2,2_2)$,	$(3_1,0_2,6_1,1_2)$,	$(4_1,0_2,7_1,1_2)$,
$(5_1,0_2,8_1,1_2)$,	$(3_1,2_2,4_1,3_2)$,	$(5_1,2_2,6_1,3_2)$,	$(7_1,2_2,8_1,3_2)$,
$(9_1,2_2,11_1,3_2)$,	$(0_1,4_2,1_1,5_2)$,	$(2_1,4_2,3_1,5_2)$,	$(4_1,4_2,5_1,5_2)$,
$(6_1,4_2,7_1,5_2)$,	$(8_1,4_2,10_1,5_2)$,	$(12_1,4_2,11_1,5_2)$,	$(0_1,6_2,1_1,7_2)$,
$(2_1,6_2,3_1,7_2)$,	$(4_1,6_2,5_1,7_2)$,	$(6_1,6_2,7_1,7_2)$,	$(8_1,6_2,10_1,7_2)$,
$(12_1,6_2,9_1,7_2)$.			

□

Theorem 4.12 *There exists a* 4*-cycle decomposition of* K_v *with colouring type* {A1, B1, B2} *if and only if* $v \equiv 1 \pmod 8$, $v \neq 1, 9$.

Proof. The necessary conditions follow from Theorem 1.1, Lemma 4.9 and the fact that the colouring is proper.

Let $v = 8x + 17$, for $x \geq 0$. By Lemma 4.10 we know that the decomposition exists when $x = 0$, so suppose $x \geq 1$. Let the vertex set of K_v be $(\cup_{i=1}^{x} X_i) \cup Y$, where $|X_i| = 8$ for $i = 1, 2, \ldots, x$, and $|Y| = 17$. Colour all vertices in X_i black, for $i = 1, 2, \ldots, x$, and colour thirteen vertices of Y black and four white.

By Lemma 4.10, we can place a copy of the 4-cycle decomposition of K_{17} with colouring type {A1, B1, B2} on Y. By Lemma 4.11, we can place a copy of the 4-cycle decomposition of $K_{25} - K_{17}$ with colouring type {A1, B1} on $Y \cup X_i$, for $i = 1, 2, \ldots, x$, where the hole is on the vertices in Y. Finally, by Corollary 3.3, we can place a copy of the 4-cycle decomposition of $K_{8,8}$ with colouring type {A1} on $X_i \cup X_j$, for $1 \leq i < j \leq x$. The result is a 4-cycle decomposition of K_v with colouring type {A1, B1, B2}. □

4.5 $S = \{\text{A1, B1, C}\} \equiv S = \{\text{A2, B2, C}\}$

Lemma 4.13 *There exists a 4-cycle decomposition of K_9 with colouring type* {A1, B1, C}.

Proof. Let the vertex set of K_9 be $\mathbb{Z}_9$. Colour the vertices $0, 1, \ldots, 6$ black and the vertices 7 and 8 white. A suitable decomposition is given by: $(0,1,7,8)$, $(2,3,0,7)$, $(3,5,4,7)$, $(5,2,6,7)$, $(4,2,1,8)$, $(5,0,2,8)$, $(6,1,3,8)$, $(0,4,3,6)$ and $(1,4,6,5)$. □

Lemma 4.14 *There exists a 4-cycle decomposition of $K_{8,8}$ with colouring type* {A1, B1, C}.

Proof. Let the vertex set of $K_{8,8}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 7_i\}$. Colour the vertices $0_i, 1_i, \ldots, 5_i$ black and the vertices 6_i and 7_i white, for $i = 1, 2$. A suitable decomposition is given by:

$(0_1,0_2,6_1,6_2)$, $(0_1,1_2,6_1,7_2)$, $(1_1,0_2,7_1,6_2)$, $(1_1,1_2,7_1,7_2)$, $(2_1,2_2,3_1,3_2)$, $(2_1,4_2,3_1,5_2)$, $(4_1,2_2,5_1,3_2)$, $(4_1,4_2,5_1,5_2)$, $(2_1,0_2,3_1,6_2)$, $(4_1,0_2,5_1,6_2)$, $(2_1,1_2,3_1,7_2)$, $(4_1,1_2,5_1,7_2)$, $(2_2,0_1,3_2,6_1)$, $(4_2,0_1,5_2,6_1)$, $(2_2,1_1,3_2,7_1)$, $(4_2,1_1,5_2,7_1)$.

□

Theorem 4.15 *There exists a 4-cycle decomposition of K_v with colouring type* {A1, B1, C} *if and only if $v \equiv 1 \pmod 8$, $v \neq 1$.*

Proof. We use the construction from Theorem 3.5, with the vertex ∞ coloured black, six vertices coloured black and two coloured white in each X_i, and the designs given in Lemmas 4.13 and 4.14. □

4.6 $S = \{\text{A1, B1, D}\} \equiv S = \{\text{A2, B2, D}\}$

Theorem 4.16 *There exist no 4-cycle decompositions of K_v with colouring type* {A1, B1, D}.

Proof. Let $v > 1$ and suppose that the decomposition exists. No 4-cycle contains a pure-coloured edge between two white vertices, so $w < 2$. However, there is at least one 4-cycle of Type D in the decomposition, and so $w \geq 2$. □

4.7 $S = \{\text{A1, B2, C}\} \equiv S = \{\text{A2, B1, C}\}$

Lemma 4.17 *There does not exist a 4-cycle decomposition of K_9 with colouring type* {A1, B2, C}.

Proof. The proof mirrors that provided for Lemma 4.9. □

Lemma 4.18 *There exists a 4-cycle decomposition of K_{17} with colouring type* {A1, B2, C}.

Proof. Let the vertex set of K_{17} be $\mathbb{Z}_{17}$. Colour the vertices $0, 1, \ldots, 5$ black and the vertices $6, 7, \ldots, 16$ white. A suitable decomposition is given by:

$(0,1,2,3)$, $(0,2,6,7)$, $(0,4,6,8)$, $(0,5,6,9)$, $(1,3,6,10)$,
$(1,4,7,8)$, $(1,5,7,9)$, $(2,4,8,9)$, $(2,5,8,10)$, $(3,4,9,10)$,
$(3,5,9,11)$, $(4,5,10,11)$, $(0,6,11,12)$, $(0,10,7,11)$, $(0,13,6,14)$,
$(0,15,6,16)$, $(1,6,12,7)$, $(1,11,8,12)$, $(1,13,7,14)$, $(1,15,8,16)$,
$(2,7,15,11)$, $(2,8,13,12)$, $(2,13,9,14)$, $(2,15,9,16)$, $(3,7,16,12)$,
$(3,8,14,13)$, $(3,9,12,14)$, $(3,15,10,16)$, $(4,10,13,15)$, $(4,12,10,14)$,
$(4,13,11,16)$, $(5,11,14,15)$, $(5,12,15,16)$, $(5,13,16,14)$.

□

Lemma 4.19 *There exists a 4-cycle decomposition of $K_{25} - K_{17}$ with colouring type* {A1, B2, C}.

Proof. Let the vertex set of $K_{25} - K_{17}$ be $\{0_1, 2_1, \ldots, 16_1\} \cup \{0_2, 1_2, \ldots, 7_2\}$, where the hole is on the vertices with subscript 1. Colour the vertices $0_1, 1_1, \ldots, 5_1$ and 0_2, 1_2 and 2_2 black and the remaining vertices white. A suitable decomposition is given by:

$(0_1,0_2,1_1,1_2)$, $(2_1,0_2,3_1,2_2)$, $(0_2,1_2,4_2,6_1)$, $(1_2,2_2,5_2,6_1)$,
$(2_2,0_2,3_2,6_1)$, $(4_1,0_2,4_2,5_2)$, $(5_1,0_2,5_2,6_2)$, $(2_1,1_2,3_2,5_2)$,
$(3_1,1_2,5_2,7_2)$, $(4_1,1_2,6_2,3_2)$, $(5_1,1_2,7_2,4_2)$, $(0_1,2_2,3_2,4_2)$,
$(1_1,2_2,4_2,6_2)$, $(4_1,2_2,6_2,7_2)$, $(5_1,2_2,7_2,3_2)$, $(6_2,6_1,7_2,0_2)$,
$(7_1,5_2,8_1,0_2)$, $(9_1,5_2,10_1,0_2)$, $(11_1,4_2,12_1,0_2)$, $(13_1,4_2,14_1,0_2)$,
$(15_1,4_2,16_1,0_2)$, $(7_1,6_2,8_1,1_2)$, $(9_1,6_2,14_1,1_2)$, $(10_1,3_2,13_1,1_2)$,
$(11_1,3_2,15_1,1_2)$, $(12_1,3_2,16_1,1_2)$, $(7_1,7_2,8_1,2_2)$, $(9_1,7_2,14_1,2_2)$,
$(10_1,7_2,13_1,2_2)$, $(11_1,5_2,15_1,2_2)$, $(12_1,6_2,16_1,2_2)$, $(3_2,14_1,5_2,0_1)$,
$(6_2,15_1,7_2,0_1)$, $(3_2,7_1,4_2,1_1)$, $(5_2,16_1,7_2,1_1)$, $(3_2,8_1,4_2,2_1)$,
$(6_2,11_1,7_2,2_1)$, $(3_2,9_1,4_2,3_1)$, $(5_2,13_1,6_2,3_1)$, $(4_2,10_1,6_2,4_1)$,
$(5_2,12_1,7_2,5_1)$.

□

Lemma 4.20 *There exists a 4-cycle decomposition of $K_{8,8}$ with colouring type* {A1, B2, C}.

Proof. Let the vertex set of $K_{8,8}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 7_i\}$. Colour the vertices 0_i, 1_i and 2_i black, for $i = 1,2$. Colour the remaining vertices white. A suitable decomposition is given by:

$(0_1,2_2,3_1,4_2)$, $(1_1,2_2,4_1,3_2)$, $(2_1,0_2,6_1,6_2)$, $(2_1,1_2,7_1,3_2)$, $(2_1,2_2,5_1,4_2)$,
$(3_2,5_1,6_2,0_1)$, $(5_2,3_1,7_2,0_1)$, $(6_2,7_1,7_2,1_1)$, $(4_2,4_1,5_2,1_1)$, $(5_2,6_1,7_2,2_1)$,
$(3_1,6_2,4_1,0_2)$, $(5_1,5_2,7_1,0_2)$, $(3_1,3_2,6_1,1_2)$, $(4_1,7_2,5_1,1_2)$, $(6_1,4_2,7_1,2_2)$,
$(0_1,0_2,1_1,1_2)$.

□

Theorem 4.21 *There exists a 4-cycle decomposition of K_v with colouring type* {A1, B2, C} *if and only if $v \equiv 1 \pmod 8$, $v \neq 1, 9$.*

Proof. We use the construction from Theorem 4.12, with three vertices coloured black and five coloured white in each X_i, six vertices coloured black and eleven coloured white in Y, and with the designs given in Lemmas 4.18, 4.19 and 4.20. □

4.8 $S = \{\text{A1, B2, D}\} \equiv S = \{\text{A2, B1, D}\}$

Lemma 4.22 *There does not exist a 4-cycle decomposition of K_9 with colouring type* {A1, B2, D}.

Proof. Only 4-cycles of Type A1 contain edges connecting two black vertices, so the number of pure-coloured edges between two black vertices in K_v must be a multiple of four. Hence $b \equiv 0, 1 \pmod 8$. However, as $b \geq 4$ and $w \geq 3$, then $b \in \{4, 5, 6\}$. Thus we have a contradiction. □

Lemma 4.23 *There exists a 4-cycle decomposition of K_{17} with colouring type* {A1, B2, D}.

Proof. Let the vertex set of K_{17} be $\mathbb{Z}_{17}$. Let the vertices $0, 1, \ldots, 8$ be coloured black and let the vertices $9, 10, \ldots, 16$ be coloured white. A suitable decomposition is given by developing the starter cycle $(0,1,8,5)$ modulo 9 and including the following cycles:

$(0,9,1,10)$, $(0,11,1,12)$, $(0,13,1,14)$, $(0,15,1,16)$, $(2,9,3,10)$,
$(2,11,3,12)$, $(2,13,3,14)$, $(2,15,3,16)$, $(4,9,5,10)$, $(4,11,5,12)$,
$(4,13,5,14)$, $(4,15,9,16)$, $(5,15,10,16)$, $(6,9,14,12)$, $(6,10,9,11)$,
$(6,13,16,15)$, $(6,14,11,16)$, $(7,9,13,10)$, $(7,11,10,14)$, $(7,12,13,15)$,
$(7,13,14,16)$, $(8,9,12,10)$, $(8,11,15,14)$, $(8,12,11,13)$, $(8,15,12,16)$.

□

Lemma 4.24 *There exists a 4-cycle decomposition of $K_{25} - K_{17}$ with colouring type* {A1, D}.

Proof. Let the vertex set of $K_{25} - K_{17}$ be $\{0_1, 2_1, \ldots, 16_1\} \cup \{0_2, 1_2, \ldots, 7_2\}$, where the hole is on the vertices with subscript 1. Colour the vertices $9_1, 10_1, \ldots, 16_1$ white and the remaining vertices black. A suitable decomposition is given by:

$(0_2,9_1,1_2,10_1)$, $(2_2,9_1,3_2,10_1)$, $(4_2,9_1,5_2,10_1)$, $(6_2,9_1,7_2,10_1)$,
$(0_2,11_1,1_2,12_1)$, $(2_2,11_1,3_2,12_1)$, $(4_2,11_1,5_2,12_1)$, $(6_2,11_1,7_2,12_1)$,
$(0_2,13_1,1_2,14_1)$, $(2_2,13_1,3_2,14_1)$, $(4_2,13_1,5_2,14_1)$, $(6_2,13_1,7_2,14_1)$,
$(0_2,15_1,1_2,16_1)$, $(2_2,15_1,3_2,16_1)$, $(4_2,15_1,5_2,16_1)$, $(6_2,15_1,7_2,16_1)$,
$(3_1,0_2,4_1,2_2)$, $(5_1,0_2,6_1,1_2)$, $(7_1,0_2,8_1,1_2)$, $(3_1,1_2,4_1,3_2)$,
$(0_1,2_2,1_1,3_2)$, $(0_1,4_2,1_1,5_2)$, $(3_1,4_2,4_1,5_2)$, $(3_1,6_2,4_1,7_2)$,
$(7_1,2_2,8_1,3_2)$, $(7_1,4_2,8_1,5_2)$, $(7_1,6_2,8_1,7_2)$, $(0_1,0_2,7_2,6_2)$,
$(0_1,1_2,2_2,7_2)$, $(2_1,2_2,4_2,0_2)$, $(2_1,3_2,0_2,1_2)$, $(6_1,4_2,6_2,2_2)$,
$(5_1,5_2,2_2,3_2)$, $(2_1,6_2,3_2,4_2)$, $(6_1,7_2,4_2,5_2)$, $(1_1,0_2,5_2,6_2)$,
$(1_1,1_2,5_2,7_2)$, $(5_1,2_2,0_2,6_2)$, $(6_1,3_2,1_2,6_2)$, $(5_1,4_2,1_2,7_2)$,
$(2_1,5_2,3_2,7_2)$.

□

Theorem 4.25 *There exists a 4-cycle decomposition of K_v with colouring type* {A1, B2, D} *if and only if $v \equiv 1 \pmod 8$, $v \neq 1, 9$.*

Proof. We use the construction from Theorem 4.12, with all vertices coloured black in each X_i, nine vertices coloured black and eight coloured white in Y and with the designs given in Lemmas 4.23, 4.24 and Corollary 3.3. □

4.9 $S = \{\text{A1, C, D}\} \equiv S = \{\text{A2, C, D}\}$

Lemma 4.26 *There exists a 4-cycle decomposition of K_9 with colouring type* {A1, C, D}.

Proof. Let the vertex set of K_9 be $\mathbb{Z}_9$. Colour the vertices $0, 1, \ldots, 5$ black and the vertices 6, 7 and 8 white. A suitable decomposition is given by: $(0,1,6,7)$, $(2,5,6,8)$, $(3,4,7,8)$, $(0,2,1,3)$, $(0,4,1,5)$, $(2,3,5,4)$, $(2,6,3,7)$, $(0,6,4,8)$ and $(1,7,5,8)$. □

Lemma 4.27 *There exists a 4-cycle decomposition of $K_{8,8}$ with colouring type* {A1, C, D}.

Proof. Let the vertex set of $K_{8,8}$ be $\cup_{i=1,2}\{0_i, 1_i, \ldots, 7_i\}$. Colour the vertices $0_i, 1_i, \ldots, 5_i$ black and the vertices 6_i and 7_i white, for $i \in \{1, 2\}$. A suitable decomposition is given by:

$(0_1,0_2,6_1,6_2)$, $(0_1,1_2,6_1,7_2)$, $(1_1,0_2,7_1,6_2)$, $(1_1,1_2,7_1,7_2)$, $(2_1,6_2,3_1,7_2)$,
$(4_1,6_2,5_1,7_2)$, $(2_2,6_1,3_2,7_1)$, $(4_2,6_1,5_2,7_1)$, $(0_1,2_2,1_1,3_2)$, $(0_1,4_2,1_1,5_2)$,
$(2_1,0_2,3_1,1_2)$, $(2_1,2_2,3_1,3_2)$, $(2_1,4_2,3_1,5_2)$, $(4_1,0_2,5_1,1_2)$, $(4_1,2_2,5_1,3_2)$,
$(4_1,4_2,5_1,5_2)$.

□

Theorem 4.28 *There exists a 4-cycle decomposition of K_v with colouring type* {A1, C, D} *if and only if* $v \equiv 1 \pmod 8$, $v \neq 1$.

Proof. We use the construction from Theorem 3.5, with the vertex ∞ coloured white, six vertices coloured black and two coloured white in each X_i, and the designs given in Lemmas 4.26 and 4.27. □

4.10 $S = \{$B1, B2, C$\}$

Lemma 4.29 *There exists a 4-cycle decomposition of K_9 with colouring type* {B1, B2, C}.

Proof. Let the vertex set of K_9 be $\mathbb{Z}_9$. Colour the vertices $0, 1, \ldots, 5$ black and the vertices 6, 7 and 8 white. A suitable decomposition is given by: $(7,6,8,0)$, $(4,1,7,8)$, $(2,0,3,6)$, $(4,0,5,7)$, $(1,2,4,6)$, $(1,3,2,8)$, $(0,1,5,6)$, $(3,4,5,8)$ and $(2,5,3,7)$. □

Theorem 4.30 *There exists a 4-cycle decomposition of K_v with colouring type* {B1, B2, C} *if and only if* $v \equiv 1 \pmod 8$, $v \neq 1$ *and* $\sqrt{v} \in \mathbb{Z}$.

Proof. The proof mirrors that given for Theorem 3.15. In this case, however, we use the 4-cycle decomposition of K_9 with colouring type {B1, B2, C} given in Lemma 4.29, and the 4-cycle decomposition of $K_{v_k} - K_{v_{(k-1)}}$, for $k \geq 2$, with colouring type {B1, C}, given in Lemma 3.14. □

4.11 $S = \{$B1, C, D$\} \equiv S = \{$B2, C, D$\}$

Lemma 4.31 *There exists a 4-cycle decomposition of K_9 with colouring type* {B1, C, D}.

Proof. Let the vertex set of K_9 be $\mathbb{Z}_9$. Colour the vertices $0, 1, \ldots, 4$ black and the vertices 5, 6, 7 and 8 white. A suitable decomposition is given by: $(0,1,7,8)$, $(0,2,6,7)$, $(1,2,5,8)$, $(2,3,5,7)$, $(2,4,6,8)$, $(3,4,5,6)$, $(3,0,4,7)$, $(3,1,4,8)$ and $(0,5,1,6)$. □

Theorem 4.32 *There exists a 4-cycle decomposition of K_v with colouring type* {B1, C, D} *if and only if* $v \equiv 1 \pmod 8$, $v \neq 1$.

Proof. We use the construction from Theorem 3.5, with the vertex ∞ coloured black, four vertices coloured black and four coloured white in each X_i, and the designs given in Lemmas 4.31 and 3.11. □

References

[1] P. Adams, D. Bryant J. Lefevre and M. Waterhouse, Some Equitably 3-coloured Cycle Decompositions, *Discrete Math.*, (to appear).

[2] P. Adams, D. Bryant and M. Waterhouse, Some Equitably 2-coloured Cycle Decompositions, (submitted).

[3] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B*, **81** (2001) no. 1, 77-99.

[4] C. J. Colbourn, J. H. Dinitz and A. Rosa, Bicoloring Steiner triple systems, *Electron. J. Combin.*, **6** (1999) R25.

[5] C. J. Colbourn and A. Rosa, *Triple Systems*, Clarendon Press, Oxford (1999).

[6] J. Lefevre and M. Waterhouse, Some Equitably 3-coloured Cycle Decompositions of Multipartite Graphs, (submitted).

[7] M. Šajna, Cycle decompositions III: Complete graphs and fixed length cycles, *J. Combin Des.*, **10** (2002) no. 1, 27-78.

[8] G. Quattrocchi, Colouring 4-cycle systems with specified block colour patterns: the case of embedding P_3-designs, *Electron. J. Combin.*, **8** (2001) R24.

[9] D. Sotteau, Decomposition of $K_{m,n}$ ($K^*_{m,n}$) into Cycles (Circuits) of Length $2k$, *Combin. Theory Ser. B*, **30** (1981), 75-81.

[10] M. Waterhouse, Some Equitably 2-coloured Cycle Decompositions of Multipartite Graphs, *Utnil. Math.*, (to appear).