Weighted edge-decompositions of graphs *

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Abstract

We prove tight estimates on the minimum weight of an edge decomposition of the complete graph into subgraphs of 3 or 4 edges, where the weight of a subgraph is the number of its vertices. We conjecture that the weighted edge decomposition problem on general graphs is NP-complete for every $k > 2$. This conjecture is shown to be true for every $k \leq 11$ except $k = 8$. The problem is motivated by the traffic grooming problem for optical networks.

1 Introduction and results

Let $k \geq 3$ be a fixed integer. A k-decomposition of a graph $G = (V, E)$ is a collection of subgraphs $G_i \subset G$ $(i = 1, 2, \ldots, m)$ such that

- $\bigcup_{i=1}^{m} E(G_i) = E(G)$
- $E(G_i) \cap E(G_j) = \emptyset$ for all $1 \leq i \neq j \leq m$
- $|E(G_i)| = k$ for all but possibly one of the G_i.

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Hence, by definition, the number m of subgraphs in a k-decomposition of G is precisely $\left\lceil \frac{|E(G)|}{k} \right\rceil$. For convenience, we shall assume that it is the last subgraph, G_m, that has fewer than k edges in case if $|E(G)|$ is not a multiple of k. The weight $w(G_i)$ of subgraph G_i in the decomposition is defined as the number of its vertices, $|V(G_i)|$.

Notation. We denote by $s_k(G)$ the smallest possible sum of the weights of the subgraphs G_i, taken over all k-decompositions $\{G_1, G_2, \ldots, G_m\}$ of G. The function $s_k(K_n)$ will be abbreviated as $s_k(n)$.

Decomposition into prescribed graphs. Given a family $\mathcal{H}$ of graphs, we say that G admits an $\mathcal{H}$-decomposition if G is the edge-disjoint union of some graphs each being isomorphic to some member of $\mathcal{H}$. If $\mathcal{H} = \{H\}$ is a single graph, we write H-decomposition instead of $\{H\}$-decomposition. In the above context, we shall be mostly interested in the cases where $\mathcal{H} = \mathcal{H}_k$ consists of all graphs with k edges and a minimum number of vertices.

1.1 Motivation

Weighted edge-decomposition of graphs is related to *traffic grooming*, an important problem in optical networks research that has received significant attention recently.

Much of today's network infrastructure is based on Synchronous Optical Network (SONET) rings, in which each fiber is able to carry multiple wavelengths simultaneously. Due to the huge bandwidth of a wavelength, SONET allows each wavelength to carry multiple unit circuits in a time division multiplexing fashion. A SONET add/drop multiplexer (SADM) is an electronic device used to multiplex and demultiplex the unit circuits of the wavelength. An exclusive SADM is required for each wavelength node pair such that a connection carried by the wavelength starts or ends at this node. Since SADMs are expensive and dominate cost of the system, minimizing the number of SADMs can greatly decrease the overall network design cost. An illustrative example is shown in Figure 1. Consider a ring network with 6 nodes and assume for the moment that the traffic pattern is unidirectional (traffic flows only in one direction), that each node pair has a connection between them and that the traffic of each connection is of unit size. Then the traffic forms a total of $\binom{6}{2} = 15$ circuit rings (see the traffic matrix of Figure 1). Suppose each wavelength can support 4 circuit rings. In Assignment 1, because each node is used as a start or end node for some connection under each wavelength, four SADMs are required at each node, yielding a total of 24 SADMs required for this network (note that the number of SADMs required for each wavelength cuqals the number of distinct nodes to which the wavelength is assigned). However, in Assignment 2, only 15 SADMs are required.

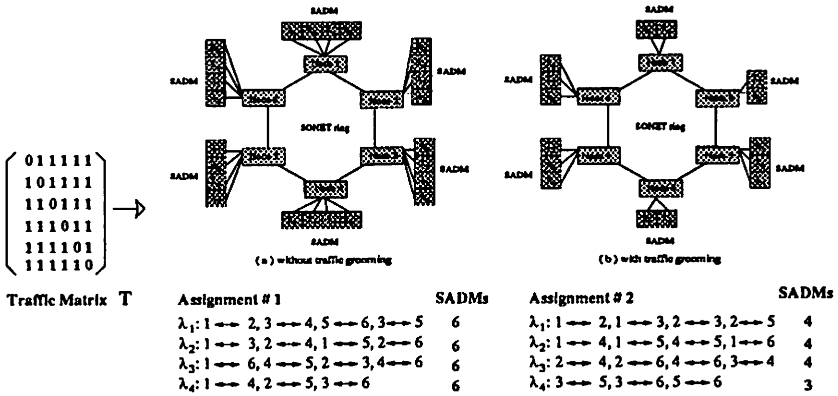

Figure 1: Traffic grooming in SONET rings. $i \rightarrow j$ represents a traffic connection from i to j; $i \leftrightarrow j$ represents two connections: $i \leftarrow j$ and $j \leftarrow i$.

In the above example, we assume uniform traffic model that means all node pairs have similar traffic loads and thus the traffic matrix is symmetric. We want to formulate the minimization of SADMs as a graph decomposition problem. Given a ring network with n nodes under the uniform traffic model, if we treat node i in K_n as node i in the ring network and edge (i, j) in K_n as the connection $i \leftrightarrow j$, then we can view a subgraph S_m of a decompositon of K_n as an assignment of connections to wavelength λ_m. Therefore, minimizing the number of SADMs is equivalent to finding a minimum weighted edge decomposition of K_n (see Figure 2). More generally, minimizing SADMs with traffic between each of the node

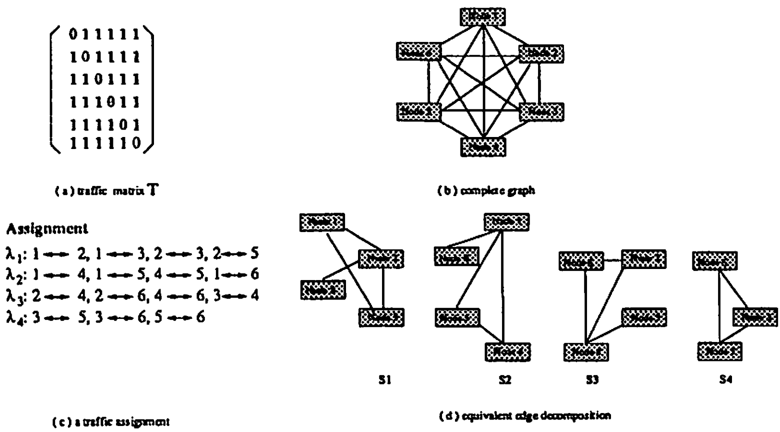

Figure 2: Reduction of a traffic assignment to an edge decomposition.

pairs equal to b circuit units is equivalent to decomposing a multi-graph K_n, in which each pair of nodes has an edge of multiplicity b, into subgraphs with exactly k edges.

1.2 Tight bounds

Regarding numerical estimates, the main results of our paper are tight bounds on the functions $s_k(n)$ for $k = 3$ and $k = 4$, as follows.

Theorem 1

(i) *For n odd,*

$$s_3(n) = \binom{n}{2} + c_n$$

where $c_n = 0$ if $n \equiv 1$ or $3 \pmod 6$ and $c_n = 2$ if $n \equiv 5 \pmod 6$.

(ii) *For n even,*

$$s_3(n) = \binom{n}{2} + \frac{n}{4} + c_n$$

where $c_n = 0$ if $n \equiv 4$ or $12 \pmod{24}$, and $c_n \leq 8$ for every n.

This result will be proved in Section 2, at the end of which a more precise formulation will also be given for n even.

Somewhat unexpectedly, the exact solution for $k = 4$ can be determined with much less effort; its proof is presented in Section 3.

Theorem 2 *If $n \equiv 0, 1, 3, 6 \pmod 8$, then $s_4(n) = \binom{n}{2}$; and $s_4(n) = \binom{n}{2} + 1$ if $n \equiv 2, 4, 5, 7 \pmod 8$.*

For larger values of k, we only have some asymptotic estimates. To formulate them, we need to introduce a notation. For any $k \geq 3$, let

$$\alpha_k = \frac{t_k}{k}$$

where t_k is the smallest integer such that

$$k \leq \binom{t_k}{2}.$$

The relevance of this parameter is shown by the following simple observation.

Proposition 1 *For every $k \geq 3$ and every graph $G = (V, E)$,*

$$s_k(G) \geq \alpha_k \, |E| \, .$$

Proof In any k-decomposition $G_1, \ldots, G_m$ of G, each G_i $(1 \leq i < m)$ has precisely k edges and thus at least t_k vertices. Therefore, the relative weight of an edge in a k-edge subgraph is at least α_k. Observing that α_k is a decreasing function of k, the same lower bound on the edge weights remains valid in G_m as well, independently of whether its size is k or smaller. ◇

Theorem 3 *For every $k \geq 3$,*

$$s_k(n) = \left(\tfrac{1}{2}\,\alpha_k + o(1)\right) n^2 .$$

Proof Consider the hypergraph whose vertex set is $E(K_n)$, and its edges correspond to the subgraphs of K_n with precisely k edges and t_k vertices. In this hypergraph, each vertex (representing an edge of K_n) is contained in $\Theta(n^{t_k-2})$ edges, while each pair of vertices belongs to $O(n^{t_k-3})$ edges only. On applying a theorem of Frankl and Rödl [4], we obtain that the hypergraph has an almost perfect packing of edges; thus, all but $o(n^2)$ edges of K_n can be partitioned into mutually edge-disjoint subgraphs of t_k vertices and k edges. In those subgraphs, the average weight of an edge is $t_k/k = \alpha_k$. Moreover, any $\lfloor t_k/2 \rfloor$ of the remaining edges can be covered with a k-edge subgraph on t_k vertices, hence the average weight of those $o(n^2)$ edges is constant (less than 3). ◇

1.3 Complexity results

We also study the time complexity of determining $s_k(G)$ for general input graphs G. Our results are summarized in the following assertion.

Theorem 4 *For an unrestricted input graph G, it is NP-complete to find the value of $s_k(G)$ for each $3 \leq k \leq 7$, and more generally also for every k of the form $\binom{t}{2} - 1$, $\binom{t}{2}$, and $\binom{t}{2} + 1$, where $t \geq 4$ is any integer. Moreover, for those values of k it is NP-complete to decide whether $s_k(G) = \alpha_k\,|E|$.*

In this way, some range of small values of k is covered. The smallest missing cases are $k = 8, 12, 13, 17, 18, 19$.

2 Tight bounds for $k = 3$

In this section we prove the upper and lower bounds for Theorem 1.

Proof for n odd A K_3-decomposition for $n \equiv 1, 3 \pmod 6$ is just a Steiner Triple System. For $n \equiv 5 \pmod 6$, K_n can be decomposed into triangles and one cycle of length four [9], and we can partition the latter into a path of length 3 and a K_2 with total weight 6. This proves the upper bounds.

The lower bounds follow by Proposition 1, because $t_3 = 3$, $\alpha_3 = 1$, and there exists no Steiner Triple System for $n \equiv 5 \pmod 6$.

Lower bound for n even Suppose that $G_1, \ldots, G_m$ is a 3-decomposition of K_n with total weight $s_3(n)$. The average weight of an edge in a G_i is 1 if and only if $G_i \cong K_3$. Let us re-number the G_i (if necessary) so that, for some ℓ, the subgraphs $G_i \cong K_3$ if and only if $\ell < i \leq m$. We denote by G the graph $G_1 \cup \cdots \cup G_\ell$.

For n even, all vertex degrees of K_n are odd. Once a vertex v_i is involved in some K_3, two degrees will be reduced from $d(v_i)$. Since $d(v_i)$ is odd, at least one edge ended at v_i will be in G, which means each node occurs in G at least once. Moreover, if G_i has n_i vertices and m_i edges, then its weight is $m_i + \lceil n_i/4 \rceil$. (Here the possible values of n_i are 2, 3, and 4.) Thus,

$$s_3(n) = \binom{n}{2} + \sum_{i=1}^{\ell} \lceil \frac{n_i}{4} \rceil \geq \binom{n}{2} + \lceil \frac{\sum_{i=1}^{\ell} n_i}{4} \rceil \geq \binom{n}{2} + \lceil \frac{n}{4} \rceil .$$

Upper bound for n even If n is not too large, say $n \leq 70$, the estimate $\binom{n}{2} + \frac{n}{4} + 8$ follows from the asymptotically weaker inequality

$$s_3(n) \leq s_3(n-1) + \lceil 4(\frac{n-1}{3}) \rceil$$

$$= \binom{n}{2} + C_{n-1} + \lceil \frac{n-1}{3} \rceil \leq \binom{n}{2} + \frac{n}{4} + 8$$

on applying the previous upper bounds for odd $n-1$. This recursion is obtained by taking a 3-decomposition of K_{n-1} and decomposing the star centered at the nth vertex into stars of at most three edges. If n is a multiple of 6, then the decomposition of K_{n-1} contains an edge, that can be completed to a K_3 with the nth vertex, and the edges uncovered so far form a star of degree $n-3$; hence only $\frac{1}{3}n - 1$ stars will be needed to complete the 3-decomposition in this case.

For a construction with n large (say, $n > 70$), the basic ingredients are the following configurations.

- A proper edge coloring of the complete graph K_s with s colors, s odd.
- A Steiner Triple System STS(v) or a Partial Triple System PTS(v) covering all the pairs but the edges of a 4-cycle.
- A resolvable Group Divisible Design GDD(s, s, s) with three groups, each of size s. We fix a resolution with parallel classes $P_1, \ldots, P_s$.

Each of these is known to exist for all $s, v \geq 3$.

Given n, we choose s as the largest *odd* integer not exceeding $n/4$, and $v = n - 3s$. Moreover, let $t = \frac{1}{2}(v - s)$.[1] We partition the vertex set V of K_n as $V = A \cup B_1 \cup B_2 \cup B_3$, $|A| = v$ and $|B_1| = |B_2| = |B_3| = s$. A 3-decomposition of K_n is defined as follows.

[1] so $0 \leq t \leq 3$.

1. Inside A, we take the decomposition as described above, derived from the blocks of an STS(v) or PTS(v).
2. We specify t (possibly zero) mutually vertex-disjoint blocks $T_1, \dots, T_t$ in A. Denote by $v_{t+1}, \dots, v_s$ the vertices not occurring in the T_i.
3. If $t > 0$, then we also specify t blocks $S_1, \dots, S_t$ in GDD(s, s, s), one for each T_i, in such a way that each S_i belongs to a distinct parallel class (say, $S_i \in P_i$), and the S_i are mutually vertex-disjoint.
4. Each block in $P_{t+1} \cup \dots \cup P_s$ defines a K_3 in the 3-decomposition of K_n.
5. We partition the set $(B_1 \cup B_2 \cup B_3) \setminus (S_1 \cup \dots \cup S_t)$ into triples $T_{t+1} \dots, T_s$ so that $|T_i \cap B_j| = 1$ for all $t < i \le s$ and $1 \le j \le 3$, and then each 3-edge star with center v_i and leaf set T_i ($t < i \le s$) will be taken as a subgraph in the 3-decomposition. Moreover, if xy is an edge in the unique color class in the edge s-coloring of B_j that does not meet T_i, then the K_3 induced by $\{v_i, x, y\}$ is also chosen for the 3-decomposition.
6. Inside each $T_i \cup S_i$ we take a decomposition into three K_3's, one path of length 3, and one star with 3 edges.
7. For $1 \le i \le t$ and $1 \le j \le 3$, let $C(i,j)$ be the color class disjoint from S_i in B_j. We also denote the elements of S_i by $s(i,j)$ ($j = 1, 2, 3$). In the 3-decomposition of K_n, each $s(i,j)$ induces a K_3 with $T_i \setminus B_j$ and with each edge in $C(i,j)$.

It is a matter of routine to check that if the sets S_i, T_i can be properly selected for $1 \le i \le t$, then the subgraphs described above form a 3-decomposition of K_n. What remains to show is that the S_i and the T_i can be selected properly for all $n > 70$. This can be verified e.g. by the following rough estimates.

Having selected any two blocks from a GDD(s, s, s), their six points are contained in fewer than $6s$ blocks, and their parallel classes contain fewer than $2s$ further blocks. Thus, if $s \ge 8$ — that means $n \ge 9$ by parity reasons and hence $n \ge 36$ — some of the s^2 blocks are disjoint from those selected and also belong to a third parallel class.

On the other hand, in a (partial) triple system of order n, each point is contained in (at most) $\lfloor \frac{v-1}{6} \rfloor$ blocks, hence there are fewer than $3(v-1) - 2$ blocks sitting on them altogether. If $v \ge 18$, this upper bound is smaller than the number of blocks in the triple system used above, which is at least $\frac{1}{3}\left(\binom{v}{2} - 4\right)$. This requires, again due to parity, $v \ge 19$ and $s \ge 17$, hence $n = v + 3s \ge 70$ will suffice. ◇

The above construction yields that $s_3(n)$ does not exceed $\binom{n}{2} + \frac{n}{4}$ with more than $2t + 2 \le 8$, and also the term "$+2$" can be omitted unless $v \equiv 5 \pmod 6$. Moreover, if $v = s \not\equiv 5 \pmod 6$, $t = 0$. And we have $c_n = 0$ if $n \equiv 4$ *or* $12 \pmod{24}$.

3 Complete solution for $k = 4$

In this section we prove Theorem 2. Observe first that there are two graphs on 4 vertices where the average edge weight is precisely 1, namely the 4-cycle C_4 and the "paw" $PW = K_4 - P_3$. We shall prove that every K_n admits a 4-decomposition $G_1, \ldots, G_m$ where all the 4-vertex subgraphs (hence, by definition, all but possibly G_m) are 4-cycles or paws, and if $\binom{n}{2}$ is not a multiple of 4 then G_m is one of the graphs K_2, P_3, and K_3. Note that also the edges of K_3 have average weight 1.

Basic cases $n = 2, 3, 4, 6, 8$

Trivially, $m = 1$ and $G_1 = K_n$ for $n = 2, 3$. It is also immediate that K_4 is decomposable into PW and P_3. For $n = 6$, observe that $K_6 - K_3$ is decomposable into three paws. Indeed, we can distribute the three edges disjoint from the deleted K_3 among the three stars (of 3 edges each) incident to its vertices.

Finally, a PW-decomposition of K_8 can be obtained by labeling the vertices with $0, 1, 2, 3, 4, 5, 6, 7$ and selecting the subgraphs G_i with edge set $\{(7, i), (i, i+1), (i, i+3), (i+1, i+3)\}$, where addition is taken modulo 7.

A C_4-decomposition of $K_{2p,2q}$

This auxiliary decomposition of complete bipartite graphs will be needed for recursive constructions below. We partition the two vertex classes into disjoint pairs $g_1, \ldots, g_p$ and $h_1, \ldots, h_q$, respectively. Then the 4-cycles induced by $g_i \cup h_j$ $(1 \le i \le p, 1 \le j \le q)$ decompose $K_{2p,2q}$.

The $2k \to 2k + 8$ construction

The following recursion settles all cases of n even, by induction from $n - 8$ to n. We view K_{2k+8} as the union of K_{2k}, K_8, and $K_{2k,8}$. Take any optimal 4-decomposition of K_{2k} and of K_8, and decompose $K2k, 8$ into 4-cycles.

The $2k + 1 \to 4k + 1$ construction

Assume that an optimal 4-decomposition of K_{2k+1} is available, where G_m is one of C_4, PW, K_2, P_3, K_3. Let $V(K_{4k+1}) = X \cup Y \cup \{z\}$. In either case, we take a C_4-decomposition of the edge set joining X to Y in the way described above, with specified pairs g_1 and h_1. Moreover, inside $X \cup \{z\}$ and $Y \cup \{z\}$ we take optimal 4-decompositions. If G_m is one of C_4 and PW, then an optimal 4-decomposition of K_{4k+1} is already obtained. Otherwise some modifications in the last graphs $G_m(X)$ and $G_m(Y)$ (the copies of G_m) are needed. In this case we assume that $G_m(X)$ and $G_m(Y)$ contain z, and also that g_1, h_1 are edges in them if $G_m \cong P_3$.

If $G_m = K_2$, then $G_m(X) \cup G_m(Y) = P_3$ that we can take as the last subgraph in the 4-decomposition of K_{4k+1}.

If $G_m = P_3$, then $G_m(X) \cup G_m(Y)$ together with the 4-cycle induced by $g_1 \cup h_1$ is isomorphic to $K_5 - P_3$, decomposable into two paws.

If $G_m = K_3$, then $G_m(X) \cup G_m(Y)$ consists of two triangles incident to z, decomposable into PW and P_3.

The $2k+1 \rightarrow 4k+3$ construction

Also here, let X and Y be $2k$-element sets, now with three further vertices x, y, z. Optimal 4-decompositions are taken inside $X \cup \{z\}$ and $Y \cup \{z\}$, and C_4-decompositions for the edges joining X to Y and also $\{x, y\}$ to $X \cup Y$. We assume further that the copies $G_m(X), G_m(Y)$ of G_m contain z, and that g_1, h_h are edges in them unless $G_m = K_2$. The difference compared to the construction for $4k+1$ is that now a triangle xyz is also attached to z. Let F be the subgraph formed by $G_m(X) \cup G_m(Y) \cup xyz$ if $G_m \neq P_3$, and if $G_m = P_3$ then let F be the same subgraph together with the 4-cycle induced by $g_1 \cup h_1$.

If $G_m = K_2$, then F is a triangle with two pendant edges, decomposable into PW and P_2.

If $G_m = K_3$, then F consists of three triangles incident to z, decomposable into two paws and a K_2.

Finally, if $G_m = P_3$, then F is decomposable into two paws and the triangle xyz.

Starting from the basic cases, the theorem follows by induction on n, applying the recursive steps given above. ◇

4 NP-completeness

In this section we prove Theorem 4.

The common property of all values k listed in Theorem 4, except for $k = \binom{t}{2} + 1$, is that there is a *unique* k-edge graph H_k of minimum weight. In this situation, a subproblem of finding $s_k(G)$ is to decide whether a graph G with km edges admits an H_k-decomposition. This problem has been shown to be NP-complete for every complete graph, by Holyer [6], settling the cases of $k = \binom{t}{2}$; and for any graph containing a connected component with more than two edges, by Dor and Tarsi [3], hence giving the solution for $k = \binom{t}{2} - 1$.

What remains to prove is NP-completeness for $k = \binom{t}{2} + 1$. An important tool will be Holyer's theorem that we recall here:

Lemma 1 *It is NP-complete to decide whether a graph has a K_t-decomposition, for every $t \geq 3$.*

We shall need the following stronger variant of this result.

Lemma 2 *The K_t-decomposition problem remains NP-complete when restricted to regular graphs whose vertex degree is a multiple of $t(t-1)$.*

Proof Let $G = (V, E)$ be a general input graph, $V = \{v_1, \ldots, v_n\}$. We may assume without loss of generality that every vertex degree $d(v_i)$ is a multiple of $t-1$, because in graphs violating this divisibility condition the non-existence of a K_t-decomposition is decidable in linear time. Let us choose a multiple D of $t(t-1)$, such that $D \geq d(v_i)$ for all $1 \leq i \leq n$. We denote

$$D_i = D - d(v_i), \qquad k_i = \frac{D_i}{t-1}.$$

Let $k = \max k_i$.

We choose the smallest prime number $p \geq k$. Note that $p = k + o(k)$ as $k \to \infty$. ¿From the affine Galois plane $AG(p,2)$ we delete the p lines of a parallel class, and also all the points of $p - t$ of those lines. The configuration obtained is a resolvable Group Divisible Design with t groups of size p each. A resolution $\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_p$ is obtained from the parallel classes of $AG(p,2)$.

¿From the input graph G with n vertices, we are going to construct a larger graph G^+ with ptn vertices $v(i,j)$ $(1 \leq i \leq n,\ 1 \leq j \leq pt)$. For each j, let the vertex set $\{v(1,j), \ldots, v(n,j)\}$ induce a graph $G(j) \cong G$, where $v(i,j)$ corresponds to v_i under the isomorphism. On the other hand, to obtain the subgraphs $F(i)$ induced by $\{v(i,1), \ldots, v(i,pt)\}$ for each i, we consider $\mathcal{H}(i) = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_{k_i}$ and join two vertices $v(i,j)$ and $v(i,\ell)$ if and only if the jth and ℓth vertices of $\mathcal{H}(i)$ are contained in some block of $\mathcal{H}(i)$. Hence, $F(i)$ is D_i-regular, and thus G^+ is D-regular.

If G has a K_t-decomposition, then so does G^+, because the blocks of $\mathcal{H}(i)$ decompose $F(i)$ while the $G(j)$ inherit their decompositions from G. Also conversely, every K_t-decomposition of G^+ decomposes G, since any complete subgraph of G^+ is entirely contained in some $F(i)$ or some $G(j)$. Finally, the number of vertices in G^+ is $ptn \leq (1 + o(1))ktn \leq n^3 + o(n^3)$, and also $AG(p,2)$ has an explicit polynomial-time construction, therefore G^+ can be obtained from G in polynomial time. (Instead of searching for a prime, one may as well choose p to be the smallest power of 2 exceeding k.) ◇

Proof of Theorem 4 for $k = \binom{t}{2} + 1$. Let $G = (V, E)$ be any input graph for Lemma 2, with n vertices, regular of degree d, where d is a multiple of $t(t-1)$. We construct a graph $G^+ = (V^+, E^+)$ by joining $\frac{d}{t(t-1)}$ pendant vertices of degree 1 to each $v \in V$. This G^+ has $\frac{nd}{t(t-1)} + n < n^2$ vertices, i.e. its size is polynomial in the size of G. The proof will be done by showing that G admits a K_t-decomposition if and only if $s_k(G^+) = \alpha_k\,|E^+|$.

Suppose first that $s_k(G^+) = \alpha_k\,|E^+|$ holds. Then, in any k-decomposition $G_1, \ldots, G_m$ of G^+, each pendant edge belongs to a distinct subgraph G_i because $t_k = t + 1$, and $\binom{t-1}{2} + 2 < k$ for $t \geq 3$. Consequently, each G_i containing a pendant edge is a K_t plus one edge. In those G_i, the number of edges of the initial graph G is precisely

$$\frac{nd}{t(t-1)}\binom{t}{2} = \frac{nd}{2} = |E|,$$

i.e. the entire edge set of G is partitioned into complete subgraphs of order t.

Suppose next that G has a K_t-decomposition. Consider any number h of the K_t in such a decomposition. Assuming that they span n' vertices, the d-regularity of G implies

$$h\binom{t}{2} \leq \frac{dn'}{2},$$

$$n' \geq h\frac{t(t-1)}{d},$$

therefore the total number of pendant edges attached to them is not smaller than h. On applying Hall's theorem we obtain that G^+ admits a H-decomposition where H is obtained from K_t by attaching a pendant edge. Consequently, $s_k(G^+) = \alpha_k |E^+|$. ◇

5 Open problems

Below we mention some problems that remain open.

1. Determine the time complexity of finding $s_k(n)$ where both n and k are part of the input.
2. Prove that it is NP-complete to determine $s_k(G)$ for general input graphs G, for every fixed integer $k \geq 3$.
3. Find an asymptotically tight upper bound on the function

$$R_k(n) = \max_{\substack{G \\ |V(G)|=n}} \left(s_k(G) - \frac{1}{2}\alpha_k n^2 \right).$$

4. Describe a k-decomposition of K_n constructively, such that the total weight is asymptotically $s_k(K_n)$.

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