Radius-Essential Edges in a Graph

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Abstract

The graph resulting from contracting edge e is denoted G/e. An edge e is *radius-essential* if $rad(G/e) < rad(G)$. Let $c_r(G)$ denote the number of radius-essential edges in graph G. In this paper, we study realizability questions relating to the number of radius-essential edges, give bounds on $c_r(G)$ in terms of radius, and order, and we characterize various classes of graphs achieving extreme values of $c_r(G)$.

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1 Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$.The *distance* $d(u, v)$ between vertices u and v is the length of a shortest path (*geodesic*) joining u and v. The *eccentricity* $e(v)$ of v is the distance to a farthest vertex from v. Vertex u is an *eccentric vertex* of v if $d(u, v) = e(v)$. The *diameter* $diam(G)$ is the maximum eccentricity and the *radius* $rad(G)$ is the minimum eccentricity among the vertices of G. The *center* $C(G)$ and *periphery* $P(G)$ of graph G consist of the sets of vertices of minimum and maximum eccentricity, respectively. Vertices of $C(G)$ are called *central vertices*, and those of $P(G)$ are called are called *peripheral vertices.* A *radial path* is a geodesic from a central vertex to one of its eccentric vertices. An *elementary contraction* of edge $e = uv$ in G is obtained by removing u and v, inserting a new vertex w, and inserting an edge between w and any vertex to which either u or v (or both) were adjacent. The graph resulting from such contraction is denoted G/e.

Definition 1 An edge e is called *radius-essential (r-essential)* if $rad(G/e) < rad(G)$; otherwise edge e is *r-contractible.*

In previous works, researchers have studied radius alteration from the perspective of deleting edges (Gliviak [5,6], Dutton, Medidi, and Brigham [4], Walikar, Buckley, and Itagi [12]), adding edges (Dutton, Medidi, and Brigham [4] and Gliviak [7]) and deleting vertices (Gliviak [5], and Dutton, Medidi, and Brigham [4]; also see the excellent survey by Gliviak [8] for additional references on vertex deletion problems).

Let $c_r(G)$ denote the number of radius-essential edges in graph G, that is, $c_r(G) = \{e \in E(G) : rad(G/e) < rad(G)\}$. It is clear from the definition that $0 \leq c_r(G) \leq q$, where q is the number of edges in G. In this paper, we study realizability questions relating to the number of radius-essential edges, give bounds on $c_r(G)$ in terms of radius and order, and characterize various classes of graphs achieving extreme values of $c_r(G)$. An analogous study was done by Walikar, Buckley, and Itagi [11] for diameter-essential edges, that is, edges whose contraction decreases the diameter of a graph.

2 Existence and Bounds

In this section, we determine precisely when there exists a graph G with given number of edges and given value of $c_r(G)$ .We then obtain a tight upper bound for $c_r(G)$ in terms of order and radius.

First we need two definitions. Suppose that A and B are graphs with $u \in V(A)$ and $v \in V(B)$. Then the *coalition of u with v* is an identification of those two vertices so as to produce a new graph G consisting of $(A - u) \cup (B - v)$ together with a new vertex w that is adjacent to all of the former neighbors of u and of v. The most common instances of this operation are the attachment of a pendant edge or, more generally, a pendant path where one endvertex of the path is identified with a specific vertex in some graph. As described in Buckley and Lewinter [,p.75], the *sequential join* $G_1+G_2+\cdots+G_k$ of graphs $G_1, G_2, \ldots, G_k$ is the graph formed by taking one copy of each of the graphs $G_1, G_2, \ldots, G_k$ and adding additional edges from each vertex of G_i to each vertex of G_{i+1}, for $1 \le i \le k-1$. Often with sequential joins, one graph is repeated numerous times at the beginning of a sum, at the end of a sum, or at both the beginning and end of a sum. We need a shorthand notation for these. Thus let $G+^kH$ be the sequential join $G+G+\cdots+G+H$ where G appears k consecutive times; let $H+_kG$ be the sequential join $H+G+G+\cdots+G$ where G appears k consecutive times; and so $G+^kH+_kG$ is $G+G+\cdots+G+H+G+G+\cdots+G$ where G appears k times on each side of H. Let $G_{q,n}$ denote a graph with q edges, precisely n of which are r-essential.

To determine precisely when there exists a graph G with a given number of edges and given value of $c_r(G)$, we need to be able to quickly determine the value of $c_r(G)$.We also need to construct infinite classes of graphs with q edges and $c_r(G) = n$

where n is bounded by some function of q. For small values of q, graph tables as well as the following lemma will aid us.

Lemma 1 *For any nontrivial connected graph G, if $diam(G) = 2\ rad(G)$ then $c_r(G) = 0$.*

Proof. Let G be a nontrivial connected graph with $diam(G) = 2\ rad(G)$. Then each central vertex v in G is on a diametral path composed of two edge-disjoint paths of length $rad(G)$. In order to decrease the radius by edge contraction, an edge e must be on a radial path. For any central vertex v let e be on a radial path P and let P' be an edge disjoint path such that P and P' together comprise a diametral path whose endvertices are u and w. By contracting e, the distance from v to one of u or w will decrease by one, but its distance to the other vertex will remain the same. Thus the eccentricity of any central vertex of G will remain unchanged. Hence no edge of G is r-essential, that is, $c_r(G) = 0$. ∎

Theorem 1 *For any pair of integers n and q where $0 \leq n \leq q$, there exists a graph G having q edges such that $c_r(G) = n$, except for $q = 1$, $n = 0$; $q = 2$, $n = 1$ or 2; $q = 3$, $n = 1$ or 2; $q = 4$, $n = 1$ or 3; and $q = 6$, $n = 3$.*

Proof. First, for each $q \leq 2$, the connected graph is unique and a path: $c_r(P_1) = c_r(P_3) = 0$, and $c_r(P_2) = 1$. So $q = 1$, $n = 0$ and $q = 2$, $n = 1$ or 2 are impossible. For the three connected graphs when $q = 3$: $c_r(K_3) = c_r(K_{1,3}) = 0$ and $c_r(P_4) = 3$. Thus $q = 3$, $n = 1$ or 2 are impossible. For the five connected graphs when $q = 4$: $c_r(P_5) = c_r(K_{1,4}) = c_r(K_1 + K_1 + K_2) = 0$, $c_r(K_1 +^3 \bar{K}_2) = 2$, and $c_r(C_4) = 4$. Thus $q = 4$, $n = 1$ or 3 are impossible. When $q \geq 5$, we shall focus on the parity of n.
<u>Case 1</u> (n is even, $q \geq 5$). For $n = 0$, use $G_{q,0} = K_{1,q}$; for $n = q$, use $G_{q,q} = C_q$; for $n = q - 1$, let $G_{q,q-1}$ be C_{q-1} with a pendant edge; and for all other pairs q, n where n is even, $q \geq 5$, let $G_{q,n} = K_1 +^{n+1} \bar{K}_{q-n}$.

Case 2 (n is odd, $q \geq 5$). For $n = 1$, use $G_{q,1} = \bar{K}_2 + K_1 + K_1 + \bar{K}_{q-3}$; for $n = 3$, let $G_{5,3}$ be the graph having a pendent edge at two distinct vertices of K_3. Using a table of graphs, such as in Buckley and Harary [1] or Read and Wilson [9], one can verify that none of the twenty connected graphs with six edges has precisely three r-essential edges. Thus $G_{6,3}$ does not exist, that is, $q = 6$, $n = 3$ is impossible. For $n = 3$ and $q \geq 7$, let $G_{q,3}$ be the graph in Figure 1(a) if q is odd and Figure 1(b) if q is even.

For n odd, $q \geq 5$, and $n \geq 5$, use $G_{q,q} = P_{q+1}$; and for $n < q$, let $G_{q,n}$ consist of the graph formed from the coalition of the ("one of the", when $n = q - 1$) center vertex of $K_{1,q-n}$ with a vertex at distance two from an endvertex of P_{n+1}. (See Figure 1(c).) ∎

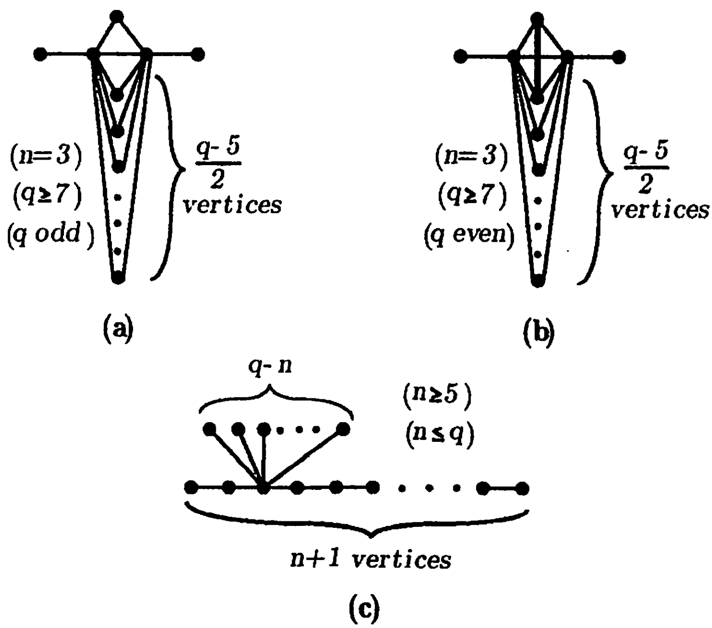


Figure 1

Theorem 2 *Every graph G can be embedded in a graph H of radius r such that $c_r(H) = 0$.*

Proof. If $G = K_1$, then already $c_r(H) = 0$ and there is nothing to prove. For $rad(G) = 1$, let $H = K_1 + G$. Then $rad(H) = 1$ and no edge of H is r-essential, so $c_r(H) = 0$. For $rad(G) > 1$, let $H = K_1 +^r H +_r K_1$, the sequential join with K_1 appearing r times on each side of G. Then $diam(H) = 2rad(H)$, so by Lemma 1, $c_r(H) = 0$. ∎

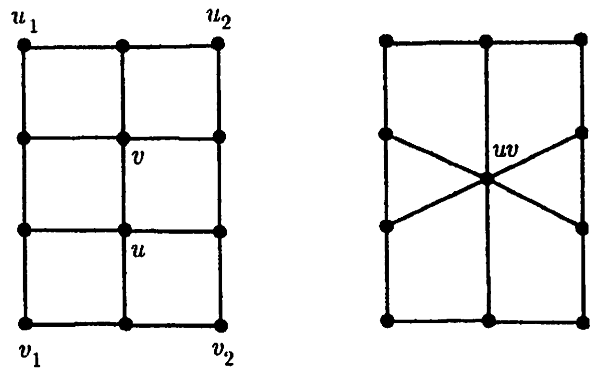
Theorem 3 *For any graph G, if $c_r(G) = 0$, then each central vertex v of G has at least two eccentric vertices that are joined to v by edge-disjoint radial paths..*

Proof. Suppose some central vertex v of a graph G with $c_r(G) = 0$ has only one eccentric vertex, say v'. Then contraction of any edge on a radial $v - v'$-path reduces the eccentricity of v by one and therefore $rad(G)$ by one, contradicting $c_r(G) = 0$. Thus suppose that each central vertex has at least two eccentric vertices. If some central vertex w has the property that no two of its eccentric vertices are joined to w by edge-disjoint radial paths, then for two of its eccentric vertices w' and w", all $w - w'$ and $w - w$" geodesics have an edge e in common. Then the eccentricity of w in G/e is one less than in G, so $rad(G/e) < rad(G)$, contradicting $c_r(G) = 0$. Thus each central vertex v has at least two eccentric vertices that are joined to v by edge-disjoint radial paths. ∎

Note that the converse to Theorem 3 does not hold. For example, the cartesian product $G = P_3 \times P_4$ depicted in Figure 2(a) has central vertices u and v with eccentric vertices u_1 and u_2 for u and v_1 and v_2 for v. There are edge- disjoint $u - u_1$ and $u - u_2$ radial paths and edge-disjoint $v - v_1$ and $v - v_2$ radial paths. Nevertheless, $c_r(G) > 0$. Indeed, $rad(G/uv) = 2 < rad(G) = 3$. So edge e is r-essential. See Figure 2(b).

Theorem 4 *For a tree T, $c_r(T) = 0$ if and only if T has just one central vertex.*

Proof. It is well known that a tree has either one or two central vertices and if two, they are adjacent. Suppose that T has two central vertices, say u and v. Then uv is on every radial path in T, $r(T/uv) < r(T)$, and $c_r(T) > 0$. On the other hand, if T has just one central vertex, then $diam(T) = 2rad(T)$, so by Lemma 1, $c_r(T) = 0$. ∎



$G = P_3 \times P_4$ $\qquad$ G/uv

Figure 2

What about the value of $c_r(T)$ for a bicentral tree? It is easy to show that for a bicentral tree T with q edges, $c_r(T) = q$ if and only if T is a path. Furthermore, $1 \leq c_r(T) \leq q$, with all integer values in the range achievable except that $c_r(T) \neq q - 1$ when $rad(T) = 3$. In fact, we have the following result.

Theorem 5 *For a bicentral tree T, edge f is r-essential if and only if f is on every diametral path.*

Proof. Let T be a bicentral tree with central vertices u_1 and u_2. We know that $diam(T) = 2rad(T) - 1$.

($\Rightarrow$) Let edge f be r-essential in T. Then suppose that some diametral path P with end-vertices x and y in T does not contain

f. Since every diametral path contains the center, the distances from u_1 and u_2 to x and y are unchanged in T/f. Thus f is not r-essential, a contradiction. Hence every diametral path in T contains each r-essential edge.

($\Leftarrow$) Suppose that edge f is on every diametral path in bicentral tree T. Then $diam(T/f) = diam(T) - 1 = 2rad(T) - 2$. So T/f is unicentral, and in a unicentral tree the diameter is twice the radius. Thus $diam(T/f) = 2rad(T/f) = 2rad(T) - 2$, which implies that $rad(T/f) < rad(T)$. Hence edge f is r-essential. ∎

Note that Theorem 5 cannot be extended to graphs in general. For example, in the cycle C_6, every edge is r-essential, but no edge is on every diametral path. We now proceed toward obtaining a bound on $c_r(G)$ in terms of its order and radius. A *composition* of a positive integer n is an *ordered* partition $(a_1, a_2, \ldots, a_k)$ where $a_i \in N$ and $a_1 + a_2 + \cdots + a_k = n$. Note that $(2,5,6)$ and $(6,2,5)$ are distinct compositions of 13. The terms a_i are called *parts*. In [11], we proved the following.

Lemma 2 *Let n be a positive integer where $n \geq 2$. Then among all compositions of n into at least two parts, the one that maximizes $S = \sum_{i=1}^{k-1} a_i \cdot a_{i+1}$ is the composition $(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$.*

Theorem 6 *For any nontrivial graph G of order p and radius r, the number of r-essential edges is bounded as follows:*
(1) $c_r(G) = 1$ if $G = K_2$,
(2) $c_r(G) = 0$ if $r = 1$ and $G \neq K_2$,
(3) $c_r(G) = 2\lceil p/2 \rceil \lfloor (p-2)/2 \rfloor$ if $r = 2$, and
(4) $c_r(G) = \lceil (p-2r+2)/2 \rceil \lfloor (p-2r+2)/2 \rfloor + p - 1$ if $r \geq 3$.

Proof. When $r = 1$, an edge e is r-essential if its contraction makes $rad(G/e) = 0$. This only occurs when $G = K_2$. So for $r = 1$, $c_r(G) = 1$ if $G = K_2$, and $c_r(G) = 0$ otherwise.

Now suppose that $r = 2$. When p is even, we maximize $c_r(G)$ with $K_p - (p/2)K_2$, that is, K_p minus a pefect matching. All edges are r-essential. We can do no better since if any vertex had an additional adjacency, then $rad(G)$ would be 1 not 2. Graph $K_p - (p/2)K_2$ has $\binom{p}{2} - p/2 = p(p-2)/2$ edges, so $c_r(G) = p(p-2)/2$. When p is odd, we use $H = K_{p-1} - ((p-1)/2)K_2$ together with an additional vertex v joined to all but one of the vertices of H. Call the resulting graph G. Graph G has the maximum possible number of edges among graphs of radius 2 when p is odd. All edges except the edge joining the two vertices at distance two from the vertex of degree $p-3$ in G are r-essential. Graph G has $\binom{p}{2} - (p+1)/2 = (p^2 - 2p - 1)/2$ edges, so

$$c_r(G) = (p^2 - 2p - 1)/2 - 1 = (p+1)(p-3)/2.$$

Since both $p(p-2)/2$ (when p is even) and $(p+1)(p-3)/2$ (when p is odd) equal $2\lceil p/2 \rceil \lfloor (p-2)/2 \rfloor$, we can consolidate the two cases.

Finally suppose that $r \geq 3$. As in the even case of $r = 2$, we maximize $c_r(G)$ with each central vertex having a unique eccentric vertex. Then by contraction of an edge f on any radial path from a central vertex v, $e(v)$ will decrease, $rad(G/f) < rad(G)$ and f is r-essential. However, here the graph needed is more complex. Start with the sequential join

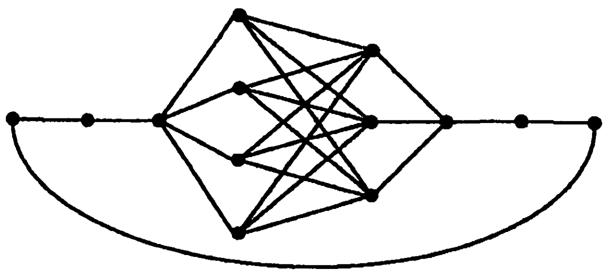
$$H = K_1 +^{r-1} \bar{K}_{\lceil (p-2r+2)/2 \rceil} + \bar{K}_{\lfloor (p-2r+2)/2 \rfloor} +_{k-1} K_1.$$

By Lemma 2, this construction will maximize the number of diametral paths (see [11]). To maximize the number of radial path, we add one additional edge, which does not change the radius. Let G be the graph formed from H by joining the two vertices of degree 1 by an edge. Then $rad(G) = r$, each edge of G is r-essential and for all $e \notin E(G)$ either e is not r-essential in $G + e$ or $rad(G+e) \neq r$.

Thus

$$\begin{aligned} c_r(G) &= \lceil (p-2r+2)/2 \rceil \lfloor (p-2r+2)/2 \rfloor \\ &\quad + \lceil (p-2r+2)/2 \rceil + \lfloor (p-2r+2)/2 \rfloor + 2r - 3 \\ &= \lceil (p-2r+2)/2 \rceil \lfloor (p-2r+2)/2 \rfloor \\ &\quad + p - 2r + 2 + 2r - 3 \\ &= \lceil (p-2r+2)/2 \rceil \lfloor (p-2r+2)/2 \rfloor + p - 1. \end{aligned}$$ ∎

Graph G is displayed in Figure 3 for $r = 4$ and $p = 13$.



$G = K_1 +^3 +\bar{K}_4 + \bar{K}_3 +_3 K_1$ with an extra edge.

Figure 3

Finally, we close with a result that shows that there are graphs for which $c_r(G) = 0$ whose periphery does not intersect the set of eccentric vertices of central vertices. Let $EC(G)$ denote the set of eccentric vertices of the central vertices. Thus $EC(G) = \{v \in V(G) : d(v,u) = r(G) \text{ for some } u \in C(G)\}$.

Theorem 7 *For $r \geq 4$, there exists a graph G of radius r where $c_r(G) = 0$ and $P(G) \cap EC(G) = \emptyset$.*

Proof. Let G be the graph displayed in Figure 4. Then $C(G) = x$, $P(G) = \{z_1, z_2\}$, and $EC(G) = \{y_1, y_2\}$. Contraction of an edge on a radial $x - y_1$ path does not alter $e(x)$, nor does the contraction of an edge on a radial $x - y_2$ path. Since those paths are the only ones that could decrease the radius, $c_r(G) = 0$. ∎

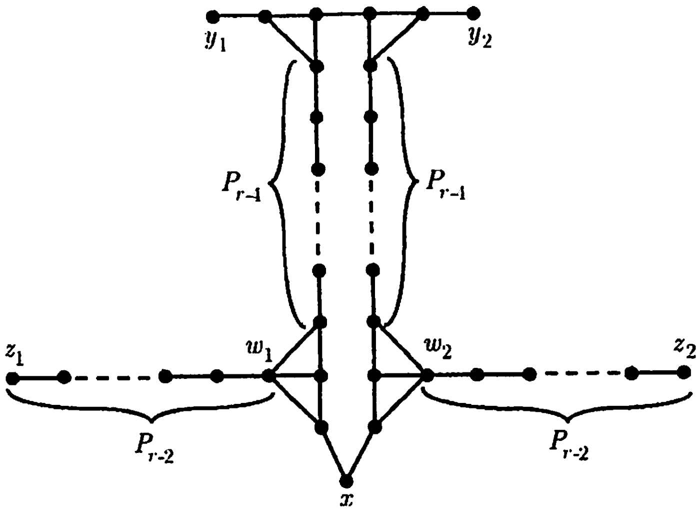


Figure 4

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