Aspects of the linear groups $\mathrm{GL}(n,2), n < 7$

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Abstract

We provide tables which summarize various aspects of the finite linear groups $\mathrm{GL}(n,2), n < 7$, in their action upon the vector space $V_n = V(n,2)$ and upon the associated projective space $\mathrm{PG}(n-1,2)$. It is intended that the tabulated results should be immediately accessible to finite geometers, and to all others (design theorists, coding theorists, ...) who have occasional need of these groups. In the case $n = 4$ attention is also paid to the maximal subgroup $\Gamma\mathrm{L}(2,4)$. In the case $n = 6$ the maximal subgroups $\Gamma\mathrm{L}(2,8)$ and $\Gamma\mathrm{L}(3,4)$ are treated, as are class aspects of the tensor product structure $V_6 = V_2 \otimes V_3$, and of the exterior product structure $V_6 = \wedge^2 V_4$.

1 Introduction

We use $V_n = V(n,\mathbb{F})$ to denote an n-dimensional vector space over a field $\mathbb{F}$. If $\mathbb{F}$ is a finite field $\mathrm{GF}(q)$, and so $V_n = V(n,q)$, the general linear group $\mathrm{GL}(V_n)$ is a finite group, denoted $\mathrm{GL}(n,q)$. In fact, from section 3 onwards, we specialize to the case $q = 2$. If one is investigating an area of finite geometry, or of design or coding theory, where the base field is $\mathrm{GF}(2)$ then one is quite likely to require particular facts concerning the elements of one of the finite groups $\mathrm{GL}(n,2)$. *The main aim of the present paper is to make readily accessible (even to the non-expert) such facts for the finite group* $\mathrm{GL}(n,2)$, *acting upon* $V(n,2)$, or upon the associated projective space $\mathrm{PG}(n-1,2)$, *in the cases* $2 \le n \le 6$. See especially the material displayed in tables 1 - 5 in section 3. In these tables the rows refer to the different conjugacy classes of the group, the number of distinct classes of $\mathrm{GL}(n,2)$ being 3, 6, 14, 27, 60 according as $n = 2,3,4,5,6$. The columns in these tables convey information concerning such things as power maps, characteristic and minimal polynomials, fixed points, cycle type and centralizers. For a description of this information see section 3.

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In appendix A.2 we also provide information, see tables 6, 7, concerning certain maximal subgroups of $\mathrm{GL}(n,2)$, namely the maximal subgroups $\Gamma\mathrm{L}(2,4)$ of $\mathrm{GL}(4,2)$, and $\Gamma\mathrm{L}(3,4)$, $\Gamma\mathrm{L}(2,8)$ of $\mathrm{GL}(6,2)$. In the cases $n=4$ and $n=6$ we further provide, see theorems 2, 3 and 4 in appendix A.1, class information surrounding the tensor product and wedge product structures $V_4 = V_2 \otimes V_2$, $V_6 = V_2 \otimes V_3$ and $V_6 = V_4 \wedge V_4$.

Concerning the tables in section 3, it seems to us that information about fixed points and cycle types, for elements of $\mathrm{GL}(n,2)$ acting upon $\mathrm{PG}(n-1,2)$, is especially useful. Such information was certainly needed at several stages in the course of the classification, see [6], of all the partial spreads in $\mathrm{PG}(4,2)$. For example, if a partial spread $\mathcal{S}_r$ of r lines of $\mathrm{PG}(4,2)$ is *cyclic* — that is if there exists $A \in \mathrm{GL}(5,2)$ of order r such that $\mathcal{S}_r$ is of the form $\{\lambda, A(\lambda), \ldots A^{r-1}(\lambda)\}$ for some line λ of $\mathrm{PG}(4,2)$ — then it is clearly necessary that A, in its action upon $\mathrm{PG}(4,2)$, should have at least three cycles of length r. Upon glancing at table 4 in section 3 we immediately deduce that no cyclic $\mathcal{S}_8$ exists, and that in attempting to construct a cyclic $\mathcal{S}_6$ one must use an element $A \in \mathrm{GL}(5,2)$ of class 6B and not of class 6A. Using such an element A one quickly checks that a cyclic $\mathcal{S}_6$ in fact exists, see [13, equation (3.2)]; it is allocated to the $\mathrm{GL}(5,2)$-orbit VIa.1 in [6, Table B.2].

As a second example, consider the problem of classifying the r- *dimensional normalized linear sections*, denoted $\mathrm{NLS}_r(n,q)$'s, of $\mathrm{GL}(n,q)$. (Such a section is, by definition, an r-dimensional subspace of the n^2-dimensional vector space $\mathrm{End}(n,q)$ which contains the identity element $I_n \in \mathrm{GL}(n,q)$ and is such that every non-zero element lies in $\mathrm{GL}(n,q)$.) This classification problem was posed in [5], but was solved, for $n > 2$, only in the cases of $\mathrm{GL}(3,2)$ and $\mathrm{GL}(4,2)$. It is easy to see, [5, Lemma 2.1], that each element A of an $\mathrm{NLS}_r(n,q)$, other than the scalar multiples of I_n, must be fixed-point-free upon the points of $\mathrm{PG}(n-1,q)$. Thus, see table 3, the elements of an $\mathrm{NLS}_r(4,2)$ must be drawn solely from the classes 1A, 3A, 5A, 6A and 15A,B of $\mathrm{GL}(4,2)$ and, see table 4, the elements of an $\mathrm{NLS}_r(5,2)$ must be drawn solely from the classes 1A, 21A,B and 31A-31F of $\mathrm{GL}(5,2)$. See section 4.2 below for a few more details.

We think of the present paper as one of the EBUM genre: papers which contain Elementary But Useful Mathematics. *It seems to us that, far from being despised, certain EBUM papers should be welcomed!— namely those which summarize material, albeit of an 'elementary' and 'well-known' nature, which is nevertheless difficult and time-consuming to extract from the research literature.* For we believe that progress in a promising area of advanced research is all too often delayed, because the authors have to spend precious time digging out, or developing *ab initio*, certain not-readily-accessible EBUM material.

2 General considerations

Just for the moment we work over a general base field $\mathbb{F}$, and for $V_n = V(n, \mathbb{F})$ we consider the n^2-dimensional vector space $\mathrm{End}(V_n) = \mathrm{L}(V_n, V_n)$ consisting of all linear mappings of V_n into V_n. Take note that $\mathrm{End}(V_n)$ is in fact an associative algebra over $\mathbb{F}$, of dimension n^2.

Given $A \in \mathrm{End}(V_n)$, $B \in \mathrm{End}(V_k)$ we use $[A, B]$ to denote the set of linear maps $T : V_n \to V_k$ which *intertwine* A and B (in that order!):

$$[A, B] = \{T \in \mathrm{L}(V_n, V_k) \mid TA = BT\}.$$

The *intertwining space* $[A, B]$ is a vector subspace of the nk-dimensional vector space $\mathrm{L}(V_n, V_k)$. Two elements $A, B \in \mathrm{End}(V_n)$ are said to be *similar* whenever $[A, B]$ contains a nonsingular mapping T, in which case we write $A \sim B$. The *commutant* $[A]$ of $A \in \mathrm{End}(V_n)$ is defined to be $[A] := [A, A]$, and we denote by $\mathfrak{A}_A$ the subset of $\mathrm{End}(V_n)$ consisting of all polynomials in A over $\mathbb{F}$. Observe that we have the *subalgebra* inclusions

$$\mathfrak{A}_A \subseteq [A] \subseteq \mathrm{End}(V_n). \tag{1}$$

If the (monic) minimal polynomial $\mu_A \in \mathbb{F}[t]$ of $A \in \mathrm{End}(V_n)$ has degree m then the subalgebra $\mathfrak{A}_A$ has dimension m, with $\{I, A, \ldots, A^{m-1}\}$ a basis. Of course we have $m \leq n = \dim V_n = \deg \chi_A$, where $\chi_A \in \mathbb{F}[t]$ is the characteristic polynomial of A. (Recall that μ_A divides χ_A; moreover μ_A and χ_A share the same irreducible factors, although in general with different multiplicities.)

If $A, B \in \mathrm{GL}(V_n)$ then $A \sim B$ if and only if they belong to the same conjugacy class of $\mathrm{GL}(V_n)$. The *centralizer* $\{X \in \mathrm{GL}(V_n) \mid XA = AX\}$ in $\mathrm{GL}(V_n)$ of an element $A \in \mathrm{GL}(V_n)$ is denoted $C(A)$. Note that we have the *subgroup* inclusions

$$\langle A \rangle \subseteq C(A) \subseteq \mathrm{GL}(V_n), \tag{2}$$

where $\langle A \rangle \cong Z_r$, $r =$ order of A.

A subspace $V_r \subseteq V_n$ is *invariant* under the action of $A \in \mathrm{End}(V_n)$ if $Av \in V_r$ for all $v \in V_r$. Given $e \in V$ the subspace $W = \mathfrak{A}_A e$, which is spanned by the vectors $A^s e$, $s \geq 0$, is an example of an A-invariant subspace; such an invariant subspace is termed the *cyclic subspace* for A *generated* by the vector e. If $\mathfrak{A}_A e = V_n$ then the vector e is termed a *cyclic vector* for A.

If there exists a non-zero proper subspace $V_r \subset V_n$ which is invariant under the action of $A \in \mathrm{End}(V_n)$ then A is said to be *reducible*. If V_n admits a non-trivial direct sum decomposition $V_n = V_r \oplus V_s$ where both V_r and V_s are A-invariant, then $A \in \mathrm{End}(V_n)$ is said to be *decomposable*, and we write $A = A_r \oplus A_s$ where A_r and A_s are the restrictions of A to the subspaces

V_r and V_s. Here, and often below, we use lower indices attached to a linear mapping to indicate the dimension; thus $A_n, B_n, J_n, S_n, \ldots \in \mathrm{End}(V_n)$. The identity mapping is denoted I_n. (However Z_n and D_n will denote the cyclic and dihedral groups of the indicated order.)

Consider an element $N_n \in \mathrm{End}(V_n)$ which is nilpotent of index n, satisfying $(N_n)^n = 0$ and $(N_n)^{n-1} \neq 0$. The minimal and characteristic polynomials of $N = N_n$ are $\mu_N = \chi_N = t^n$. Choosing $e_1 \in V_n$ such that $N^{n-1}e_1 \neq 0$, then e_1 is a cyclic vector for N: if $e_i = N^{i-1}e_1$, $i = 2, 3, \ldots, n$, then $\{e_1, \ldots, e_n\}$ is a basis for V_n upon which the effect of N_n is $e_1 \mapsto e_2 \mapsto \ldots \mapsto e_n \mapsto 0$. Then $J_n := I_n + N_n$ is an element of $\mathrm{GL}(V_n)$ which is unipotent of index n. In matrix terms, relative to $\{e_1, \ldots, e_n\}$ as basis, J_n is a Jordan matrix of size n with 1's down the main diagonal:

$$J_n = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} = I_n + N_n, \quad \text{where} \quad N_n = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}. \tag{3}$$

Throughout we will reserve the notations N_n and $J_n = I_n + J_n$ for elements of $\mathrm{End}(V_n)$ of the preceding kind.

See [4, Section 2.2] for a collection of relevant results for linear mappings of V_n which hold for a general base field $\mathbb{F}$. In particular it is there demonstrated that one may adopt a somewhat unorthodox choice for the form of the indecomposable "blocks" B which occur in the classical canonical form for an element $A \in \mathrm{GL}(V_n)$, namely

$$B = J_u \otimes C_d, \qquad \text{satisfying } \mu_B = \chi_B = (\chi_{C_d})^u, \tag{4}$$

where $J_u = I_u + N_u \in \mathrm{GL}(V_u)$ is as above and C_d is an irreducible element of $\mathrm{GL}(V_d)$. Besides being economically expressed, the choice (4) has the virtue of being matrix-free; moreover it is also helpful, *cf.* [4, Lemma 3.9], in the calculation of centralizers.

We will have frequent recourse to [4] as the main fall-back reference for material omitted from the present paper. In particular, *suppose now that the base field is finite:* $\mathbb{F} = \mathrm{GF}(q)$ for some prime power q; then consult [4, Section 3] for a collection of results relevant to the determination of canonical forms for elements of $\mathrm{GL}(n, q)$. Of crucial importance is the fact that the finite group $\mathrm{GL}(n, q)$ always possesses *Singer elements,* that is elements $S \in \mathrm{GL}(n, q)$ whose order $\mathrm{o}(S)$ is $q^n - 1$. For any Singer element $S \in \mathrm{GL}(n, q)$ one can show that:

(i) $\mathfrak{A}_S$ is a field $\cong \mathrm{GF}(q^n)$, with S a primitive element;
(ii) S is irreducible and χ_S is irreducible;
(iii) S permutes the $q^n - 1$ nonzero vectors of $V(n, q)$ in a single cycle;
(iv) if $A = S^r \neq I$ then A is fixed-point-free on $V(n, q) \setminus \{0\}$.

In each dimension $k \leq n$, let us choose a Singer element $S_k \in \mathrm{GL}(k,q)$ and, for $2 \leq k \leq n$, let us also choose a unipotent element $J_k \in \mathrm{GL}(k,q)$ of index k. It is demonstrated in [4, Section 3] that:

> *representatives for each conjugacy class of* $\mathrm{GL}(n,q)$ *can be simply constructed solely from the* $2n-1$ *elements* $\{S_1, \ldots, S_n;\ J_2, \ldots J_n\}$.

The construction merely involves:
(i) forming *ordinary powers*, see [4, Section 3.1], of the S_k;
(ii) forming tensor products, *cf.* (4), of a J_u with ordinary powers of an S_k;
(iii) taking direct sums of elements of the kinds (i) and (ii).

For the number of conjugacy classes of $\mathrm{GL}(n,q)$, and for material of a more advanced nature, see [3], [7], [9], [14].

3 Tables for $\mathrm{GL}(n,2)$, $n = 2,3,4,5,6$

From now on we specialize to the case $q = 2$. In the following we summarize, see tables 1, 2, 3, 4 and 5a,b, information concerning the groups $\mathrm{GL}(n,2)$, for n taking the values $2,3,4,5$ and 6, respectively. The rows of the tables refer to the distinct conjugacy classes of the group, and for each class we display a representative element of the kind described at the end of the last section, and also the associated characteristic and minimal polynomials χ and μ. The latter are given in terms of the following irreducible, see [8, Ch.10], polynomials $f_d, g_d, \ldots$ in $\mathrm{GF}(2)[t]$ of degree $d \leq 6$:

$d \leq 2:\quad f_1 = t+1\ (=\hat{f}_1);\quad f_2 = t^2+t+1\ (=\hat{f}_2);$
$d = 3:\quad f_3 = t^3+t+1,\ \hat{f}_3 = t^3+t^2+1;$
$d = 4:\quad f_4 = t^4+t+1,\ \hat{f}_4 = t^4+t^3+1;\quad g_4 = t^4+t^3+t^2+t+1\ (=\hat{g}_4);$
$d = 5:\quad f_5 = t^5+t^2+1,\ \hat{f}_5 = t^5+t^3+1;\quad g_5 = t^5+t^4+t^3+t+1,$
$\hat{g}_5 = t^5+t^4+t^2+t+1;\quad h_5 = t^5+t^4+t^3+t^2+1,\ \hat{h}_5 = t^5+t^3+t^2+t+1;$
$d = 6:\quad f_6 = t^6+t^5+1,\ \hat{f}_6 = t^6+t+1;\quad g_6 = t^6+t^5+t^2+t+1,$
$\hat{g}_6 = t^6+t^5+t^4+t+1;\quad h_6 = t^6+t^4+t^3+t+1,\ \hat{h}_6 = t^6+t^5+t^3+t^2+1;$
$k_6 = t^6+t^4+t^2+t+1,\ \hat{k}_6 = t^6+t^5+t^4+t^2+1;\quad l_6 = t^6+t^3+1\ (=\hat{l}_6).$

Here we have set $\hat{f}_d(t) := t^d f_d(t^{-1})$. (If $A \in \mathrm{GL}(d,2)$ has characteristic polynomial f_d then A^{-1} has characteristic polynomial $\hat{f}_d$.) In the tables, in order to save space, we abbreviate a paired entry such as $(f_1)^2 f_3$, $(f_1)^2 \hat{f}_3$ by $(f_1)^2 f_3\,\&\,(\hat{\ })$.

For $2 < n < 6$ the labels given to the classes are as in [2]. For a class in $\mathrm{GL}(n,2)$ with representative A_n the length $|\mathrm{GL}(n,2)|/|C(A_n)|$ of the class is found once the centralizer order $|C(A_n)|$ has been determined. (However since the class 2A in each of our tables consists of transvections its length, namely $(2^n-1)(2^{n-1}-1)$, is more easily determined directly, as in [4, Lemma 4.2].

In the column headed "F.p.'s" we indicate the fixed points of a representative A_n of a class of GL$(n,2)$ *in projective language*, that is in terms of the natural action of GL$(n,2)$ upon the projective space PG$(n-1,2)$ (which last may be identified with $V_n \setminus \{0\}$). Thus if the vectors fixed by A_n form a V_2, this is reported in the table as the existence of a *line* of fixed points. If there are no fixed points, we indicate this by an entry f.p.f. (= fixed-point-free).

The column headed "Cycle type" refers to the permutational action of $A_n \in$ GL$(n,2)$ when acting upon PG$(n-1,2)$. (The cycle type is readily computed and is of considerable use, see for example section 4.1.) By way of illustration, consider the permutational action of the element $A_4 = S_2 \oplus J_2 \in$ GL$(4,2)$ of class 6B (see table 3) upon the 15 points of PG$(3,2) = V_4 \setminus \{0\}$. In projective terms a decomposition $V_4 = V_2 \oplus V_2$ determines two skew lines in PG$(3,2)$, and S_2 acts as a 3-cycle on one line, and J_2 as a 2-cycle on the other line. It follows that A_4 acts as the product of one 6-cycle, two 3-cycles, one 2-cycle and one 1-cycle (fixed point) on PG$(3,2)$. We will accordingly record the cycle type of A_4 on PG$(3,2)$ as $6^1\,3^2\,2^1\,1^1$, and use a similar notation for general elements of GL$(n,2)$.

In tables 3, 4 and 5a,b the column headed '(Class)^p' gives *power map* information, see [2, Section 7.3]. For an element $A \in$ GL$(n,2)$ of composite order r it is of interest to know the class of A^p for those primes p which divide r. In many cases the required information is readily obtained from the representative column of the tables. For example class 21B of GL$(6,2)$ has, see [4, Eq. (5.11)], representative $A = S_2 \otimes S_3^{-1}$, and so $A^3 = I_2 \otimes S_3^4 \sim I_2 \otimes S_3$ is, see table 5a, of class 7A; thus $(21\text{B})^3 = 7\text{A}$, and similarly $(21\text{A})^3 = 7\text{B}$. In some cases such *power maps* may be obtained by using the cycle type column. For example, class 8A of GL$(6,2)$ has, see table 5b, representative A with CT$(A) = 8^4 4^6 2^2 1^3$, whence CT$(A^2) = 4^8 2^{12} 1^7$, which cycle type is peculiar to class 4B; thus $(8\text{A})^2 = 4\text{B}$. However in the case of $A \in$ class 8B of GL$(6,2)$ we have CT$(A) = 8^6 4^3 2^1 1^1$, and so CT$(A^2) = 4^{12} 2^6 1^3$, but this last cycle type is shared by classes 4C and 4E. The ambiguity is easily resolved: for since $\mu_A = (t+1)^6 = (t^2+1)^3$, it follows that $\mu_{A^2} = (t+1)^3$, which is the minimal polynomial for class 4C, but not for 4D; so $(8\text{B})^2 = 4\text{C}$. In the second column of tables 3, 4 and 5a,b we provide the power map information in all cases where the order is composite, the prime divisors of the order being taken in the order $p < p' < p'' < \ldots$ Thus, in table 5b, the entry BAC for class 30B conveys the power map information $(30\text{B})^2 = 15\text{B}$, $(30\text{B})^3 = 10\text{A}$ and $(30\text{B})^5 = 6\text{C}$.

For information on the Singer elements of GL$(5,2)$ and GL$(6,2)$, and on their powers, see [4, end of Section 4].

The centralizer information in the tables was obtained as in [4]. In this connection it should be pointed out that *the tables 1 - 5 in [4] contain an extra 'Notes' column with specific references.*

Table 1. Aspects of the linear group GL(2, 2)								
Class	Reptve.	χ	μ	F.p.'s	Cycle type	$C(A)$	$\|C(A)\|$	Length
1A	I_2	$(f_1)^2$	f_1	line	1^3	GL(2, 2)	6	1
2A	J_2	$(f_1)^2$	$(f_1)^2$	point	$2^1\,1^1$	$\langle J_2\rangle \cong Z_2$	2	3
3A	S_2	f_2	f_2	f.p.f.	3^1	$\langle S_2\rangle \cong Z_3$	3	2

Table 2. Aspects of the linear group GL(3, 2)								
Class	Reptve.	χ	μ	F.p.'s	Cycle type	$C(A)$	$\|C(A)\|$	Length
1A	I_3	$(f_1)^3$	f_1	plane	1^7	GL(3, 2)	168	1
2A	$J_2 \oplus I_1$	$(f_1)^3$	$(f_1)^2$	line	$2^2\,1^3$	D_8	8	21
3A	$S_2 \oplus I_1$	$f_1 f_2$	$f_1 f_2$	point	$3^2\,1^1$	Z_3	3	56
4A	J_3	$(f_1)^3$	$(f_1)^3$	point	$4^1\,2^1\,1^1$	Z_4	4	42
7A	S_3	f_3	f_3	f.p.f.	7^1	Z_7	7	24
7B	S_3^{-1}	$\hat{f}_3$	$\hat{f}_3$	f.p.f.	7^1	Z_7	7	24

Table 3. Aspects of the linear group GL(4, 2)

Class	(Class)^p	Reptve.	χ	μ	F.p.'s	Cycle type	$C(A)$	$\lvert C(A)\rvert$	Length
1A		I_4	$(f_1)^4$	f_1	PG(3, 2)	1^{15}	GL(4, 2)	20,160	1
2A		$J_2 \oplus I_2$	$(f_1)^4$	$(f_1)^2$	plane	$2^4\,1^7$	[4, Lemma 3.8]	192	105
2B		$J_2 \oplus J_2$	$(f_1)^4$	$(f_1)^2$	line	$2^6\,1^3$	[4, Eq. (5.3)]	96	210
3A		$S_2 \oplus S_2$	$(f_2)^2$	f_2	f.p.f.	3^5	GL(2, 4)	180	112
3B		$S_2 \oplus I_2$	$(f_1)^2 f_2$	$f_1 f_2$	line	$3^4 1^3$	$Z_3 \times \mathrm{GL}(2,2)$	18	1120
4A	A	$J_3 \oplus I_1$	$(f_1)^4$	$(f_1)^3$	line	$4^2\,2^2\,1^3$	[4, Eq. (3.19)]	16	1260
4B	B	J_4	$(f_1)^4$	$(f_1)^4$	point	$4^3\,2^1\,1^1$	[4, Eq. (2.10)]	8	2520
5A		$(S_4)^3$	g_4	g_4	f.p.f.	5^3	$\langle S_4\rangle \cong Z_{15}$	15	1344
6A	AB	$J_2 \otimes S_2$	$(f_2)^2$	$(f_2)^2$	f.p.f.	$6^2\,3^1$	$Z_3 \times (Z_2)^2$	12	1680
6B	BA	$S_2 \oplus J_2$	$(f_1)^2 f_2$	$(f_1)^2 f_2$	point	$6^1\,3^2\,2^1\,1^1$	$Z_3 \times Z_2 = Z_6$	6	3360
7A		$S_3 \oplus I_1$	$f_1 f_3$	$f_1 f_3$	point	$7^2\,1^1$	Z_7	7	2880
7B		$S_3^{-1} \oplus I_1$	$f_1 \hat{f}_3$	$f_1 \hat{f}_3$	point	$7^2\,1^1$	Z_7	7	2880
15A	AA	S_4	f_4	f_4	f.p.f.	$(15)^1$	Z_{15}	15	1344
15B	AA	S_4^{-1}	$\hat{f}_4$	$\hat{f}_4$	f.p.f.	$(15)^1$	Z_{15}	15	1344

Table 4 Aspects of the linear group GL(5, 2)								
Class	(Class)^p	Representative	χ	μ	F.p.'s	Cycle type	$\vert C(A)\vert$	Length
1A		I_5	$(f_1)^5$	f_1	PG(4, 2)	1^{31}	9,999,360	1
2A		$J_2 \oplus I_3$	$(f_1)^5$	$(f_1)^2$	PG(3, 2)	$2^8\,1^{15}$	21,504	465
2B		$J_2 \oplus J_2 \oplus I_1$	$(f_1)^5$	$(f_1)^2$	plane	$2^{12}\,1^7$	1,536	6,510
3A		$S_2 \oplus I_3$	$(f_1)^3 f_2$	$f_1 f_2$	plane	$3^8\,1^7$	504	19,840
3B		$S_2 \oplus S_2 \oplus I_1$	$f_1(f_2)^2$	$f_1 f_2$	point	$3^{10}\,1^1$	180	55,552
4A	A	$J_3 \oplus I_2$	$(f_1)^5$	$(f_1)^3$	plane	$4^4\,2^4\,1^7$	384	26,040
4B	A	$J_3 \oplus J_2$	$(f_1)^5$	$(f_1)^3$	line	$4^4\,2^6\,1^3$	128	78,120
4C	B	$J_4 \oplus I_1$	$(f_1)^5$	$(f_1)^4$	line	$4^6\,2^2\,1^3$	32	312,480
5A		$(S_4)^3 \oplus I_1$	$f_1 g_4$	$f_1 g_4$	point	$5^6\,1^1$	15	666,624
6A	AA	$S_2 \oplus J_2 \oplus I_1$	$(f_1)^3 f_2$	$(f_1)^2 f_2$	line	$6^2\,3^4\,2^2\,1^3$	24	416,640
6B	BB	$(J_2 \otimes S_2) \oplus I_1$	$f_1(f_2)^2$	$f_1(f_2)^2$	point	$6^4\,3^2\,1^1$	12	833,280
7A,B		$S_3^{\pm 1} \oplus I_2$	$(f_1)^2 f_3\,\&\,(\hat{\ })$	$f_1 f_3\,\&\,(\hat{\ })$	line	$7^4\,1^3$	42	238,080
8A	B	J_5	$(f_1)^5$	$(f_1)^5$	point	$8^2\,4^3\,2^1\,1^1$	16	624,960
12A	AA	$S_2 \oplus J_3$	$(f_1)^3 f_2$	$(f_1)^3 f_2$	point	$12^1\,6^1\,4^1\,3^2\,2^1\,1^1$	12	833,280
14A,B	AA, AA	$S_3^{\pm 1} \oplus J_2$	$(f_1)^2 f_3\,\&\,(\hat{\ })$	$(f_1)^2 f_3\,\&\,(\hat{\ })$	point	$14^1\,7^2\,2^1\,1^1$	14	714,240
15A,B	AB, AB	$S_4^{\pm 1} \oplus I_1$	$f_1 f_4\,\&\,(\hat{\ })$	$f_1 f_4\,\&\,(\hat{\ })$	point	$15^2\,1^1$	15	666,624
21A,B	BA, AA	$S_3^{\pm 1} \oplus S_2$	$f_2 f_3\,\&\,(\hat{\ })$	$f_2 f_3\,\&\,(\hat{\ })$	f.p.f.	$21^1\,7^1\,3^1$	21	476,160
31A,B		$S_5,\ S_5^{-1}$	$f_5,\ \hat{f}_5$	$f_5,\ \hat{f}_5$	f.p.f.	31^1	31	322,560
31C,D		$(S_5)^5,\ (S_5)^{-5}$	$\hat{g}_5,\ g_5$	$\hat{g}_5,\ g_5$	f.p.f.	31^1	31	322,560
31E,F		$(S_5)^{-6},\ (S_5)^6$	$\hat{h}_5,\ h_5$	$\hat{h}_5,\ h_5$	f.p.f.	31^1	31	322,560

Remark. In some cases the structure of the centralizer $C(A)$ is immediately deduced from the Representative column. Thus for classes 7A,B we have $C(A) \cong Z_7 \times \mathrm{GL}(2,2)$, and for classes 21A,B we have $C(A) \cong Z_7 \times Z_3 = Z_{21}$. In other cases much more work has to be done. See [4, Table 4] for more information.

Table 5a. Aspects of the linear group GL(6, 2): elements of order < 8

Class	(Class)^p	Representative	χ	μ	F.p.'s	Cycle type	$\lvert C(A)\rvert$	Length
1A		I_6	$(f_1)^6$	f_1	PG(5, 2)	1^{63}	$\lvert$GL(6, 2)$\rvert$	1
2A		$I_4 \oplus J_2$	$(f_1)^6$	$(f_1)^2$	PG(4, 2)	$2^{16}\,1^{31}$	10,321,920	1,953
2B		$I_2 \oplus J_2 \oplus J_2$	$(f_1)^6$	$(f_1)^2$	PG(3, 2)	$2^{24}\,1^{15}$	147,456	136,710
2C		$J_2 \oplus J_2 \oplus J_2$	$(f_1)^6$	$(f_1)^2$	plane	$2^{28}\,1^{7}$	86,016	234,360
3A		$S_2 \oplus S_2 \oplus S_2$	$(f_2)^3$	f_2	f.p.f.	3^{21}	181,440	111,104
3B		$I_4 \oplus S_2$	$(f_1)^4 f_2$	$f_1 f_2$	PG(3, 2)	$3^{16}\,1^{15}$	60,480	333,312
3C		$I_2 \oplus S_2 \oplus S_2$	$(f_1)^2 (f_2)^2$	$f_1 f_2$	line	$3^{20}\,1^{3}$	1,080	18,665,472
4A	A	$I_3 \oplus J_3$	$(f_1)^6$	$(f_1)^3$	PG(3, 2)	$4^{8}\,2^{8}\,1^{15}$	43,008	468,720
4B	A	$I_1 \oplus J_2 \oplus J_3$	$(f_1)^6$	$(f_1)^3$	plane	$4^{8}\,2^{12}\,1^{7}$	2,048	9,843,120
4C	B	$J_3 \oplus J_3$	$(f_1)^6$	$(f_1)^3$	line	$4^{12}\,2^{6}\,1^{3}$	1,536	13,124,160
4D	B	$I_2 \oplus J_4$	$(f_1)^6$	$(f_1)^4$	plane	$4^{12}\,2^{4}\,1^{7}$	768	26,248,320
4E	B	$J_2 \oplus J_4$	$(f_1)^6$	$(f_1)^4$	line	$4^{12}\,2^{6}\,1^{3}$	256	78,744,960
5A		$I_2 \oplus (S_4)^3$	$(f_1)^2 g_4$	$f_1 g_4$	line	$5^{12}\,1^{3}$	90	223,985 ,664
6A	AB	$S_2 \oplus (J_2 \otimes S_2)$	$(f_2)^3$	$(f_2)^2$	f.p.f.	$6^{8}\,3^{5}$	576	34,997,760
6B	BA	$I_2 \oplus J_2 \oplus S_2$	$(f_1)^4 f_2$	$(f_1)^2 f_2$	plane	$6^{4}\,3^{8}\,2^{4}\,1^{7}$	576	34,997,760
6C	CA	$J_2 \oplus S_2 \oplus S_2$	$(f_1)^2 (f_2)^2$	$(f_1)^2 f_2$	point	$6^{5}\,3^{10}\,2^{1}\,1^{1}$	360	55,996,416
6D	BB	$J_2 \oplus J_2 \oplus S_2$	$(f_1)^4 f_2$	$(f_1)^2 f_2$	line	$6^{6}\,3^{4}\,2^{6}\,1^{3}$	288	69,995,520
6E	CB	$I_2 \oplus (J_2 \otimes S_2)$	$(f_1)^2 (f_2)^2$	$f_1 (f_2)^2$	line	$6^{8}\,3^{4}\,1^{3}$	72	279,982,080
6F	CC	$J_2 \oplus (J_2 \otimes S_2)$	$(f_1)^2 (f_2)^2$	$(f_1)^2 (f_2)^2$	point	$6^{9}\,3^{2}\,2^{1}\,1^{1}$	24	839,946,240
7A,B		$S_3 \oplus S_3$, $S_3^{-1} \oplus S_3^{-1}$	$(f_3)^2$, $(\hat{f}_3)^2$	f_3, $\hat{f}_3$	f.p.f.	7^{9}	3,528	5,713,920
7C,D		$I_3 \oplus S_3$, $I_3 \oplus S_3^{-1}$	$(f_1)^3 f_3$, $(f_1)^3 \hat{f}_3$	$f_1 f_3$, $f_1 \hat{f}_3$	plane	$7^{8}\,1^{7}$	1,176	17,141,760
7E		$S_3 \oplus S_3^{-1}$	$f_3 \hat{f}_3$	$f_3 \hat{f}_3$	f.p.f.	7^{9}	49	411,402,240

Table 5b. Aspects of the linear group GL(6, 2): elements of order ≥ 8

Class	(Class)^p	Representative	χ	μ	F.p.'s	Cycle type	$\lvert C\rvert$	Length
8A	B	$I_1 \ominus J_5$	$(f_1)^6$	$(f_1)^5$	line	$8^4\, 4^6\, 2^2\, 1^3$	64	314,979,840
8B	C	J_6	$(f_1)^6$	$(f_1)^6$	point	$8^6\, 4^3\, 2^1\, 1^1$	32	629,959,680
9A	A	$(S_6)^7$	l_6	l_6	f.p.f.	9^7	63	319,979,520
10A	AA	$J_2 \oplus (S_4)^3$	$(f_1)^2 g_4$	$(f_1)^2 g_4$	point	$10^3\, 5^6\, 2^1\, 1^1$	30	671,956,992
12A	AC	$J_3 \otimes S_2$	$(f_2)^3$	$(f_2)^3$	f.p.f.	$12^4\, 6^2\, 3^1$	48	419,976,960
12B	BA	$I_1 \oplus S_2 \oplus J_3$	$(f_1)^4 f_2$	$(f_1)^3 f_2$	line	$12^2\, 6^2\, 4^2\, 3^4\, 2^2\, 1^3$	48	419,976,960
12C	DD	$S_2 \oplus J_4$	$(f_1)^4 f_2$	$(f_1)^4 f_2$	point	$12^3\, 6^1\, 4^3\, 3^2\, 2^1\, 1^1$	24	839,946,240
14A,B	AC, BC	$J_2 \otimes S_3,\ J_2 \otimes S_3^{-1}$	$(f_3)^2,\ (\hat{f}_3)^2$	$(f_3)^2,\ (\hat{f}_3)^2$	f.p.f.	$14^4\, 7^1$	56	359,976,960
14C,D	CA, DA	$I_1 \oplus J_2 \oplus S_3^{\pm 1}$	$(f_1)^3 f_3$ & $(\hat{\ })$	$(f_1)^2 f_3$ & $(\hat{\ })$	line	$14^2\, 7^4\, 2^2\, 1^3$	56	359,976,960
15A,B	AC, AC	$I_2 \oplus S_4,\ I_2 \oplus S_4^{-1}$	$(f_1)^2 f_4$ & $(\hat{\ })$	$f_1 f_4$ & $(\hat{\ })$	line	$15^4\, 1^3$	90	223,985,664
15C	AB	$S_2 \oplus (S_4)^3$	$f_2 g_4$	$f_2 g_4$	f.p.f.	$15^3\, 5^3\, 3^1$	45	447,971,328
15D,E	AA, AA	$S_2 \oplus S_4,\ S_2 \oplus S_4^{-1}$	$f_2 f_4$ & $(\hat{\ })$	$f_2 f_4$ & $(\hat{\ })$	f.p.f.	$15^4\, 3^1$	45	447,971,328
21A,B	BA, AA	$(S_6)^{-3}, (S_6)^3$	$k_6,\ \hat{k}_6$	$k_6,\ \hat{k}_6$	f.p.f.	21^3	63	319,979,520
21C,D	DB, CB	$I_1 \oplus S_2 \oplus S_3^{\pm 1}$	$f_1 f_2 f_3$ & $(\hat{\ })$	$f_1 f_2 f_3$ & $(\hat{\ })$	point	$21^2\, 7^2\, 3^2\, 1^1$	21	959,938,560
28A,B	CA, DA	$J_3 \oplus S_3,\ J_3 \oplus S_3^{-1}$	$(f_1)^3 f_3$ & $(\hat{\ })$	$(f_1)^3 f_3$ & $(\hat{\ })$	point	$28^1\, 14^1\, 7^2\, 4^1\, 2^1\, 1^1$	28	719,953,920
30A,B	AAC, BAC	$J_2 \oplus S_4,\ J_2 \oplus S_4^{-1}$	$(f_1)^2 f_4$ & $(\hat{\ })$	$(f_1)^2 f_4$ & $(\hat{\ })$	point	$30^1\, 15^2\, 2^1\, 1^1$	30	671,956,992
31A...F		$I_1 \oplus S_5, \ldots, I_1 \oplus (S_5)^6$ (see table 4)	$f_1 x_5,\ f_1 \hat{x}_5$ $(x_5 = f_5, \hat{g}_5, \hat{h}_5)$	$f_1 x_5,\ f_1 \hat{x}_5$	point	$31^2\, 1^1$	31	650,280,960
63A,B	BA, AA	$S_6,\ S_6^{-1}$	$f_6,\ \hat{f}_6$	$f_6,\ \hat{f}_6$	f.p.f.	63^1	63	319,979,520
63C,D	BA,AA	$(S_6)^{-5},\ (S_6)^5$	$g_6,\ \hat{g}_6$	$g_6,\ \hat{g}_6$	f.p.f.	63^1	63	319,979,520
63E,F	BA,AA	$(S_6)^{11},\ (S_6)^{-11}$	$h_6,\ \hat{h}_6$	$h_6,\ \hat{h}_6$	f.p.f.	63^1	63	319,979,520

4 Further remarks

4.1 Cycle type

The tables show that the following result holds: given $A, B \in \mathrm{GL}(n,2)$, $2 \leq n \leq 6$, then

$$A \sim B \quad \Longleftrightarrow \quad \mathrm{CT}(A) = \mathrm{CT}(B) \text{ and } \mu_A = \mu_B, \tag{5}$$

where $\mathrm{CT}(A)$ denotes the cycle type of $A \in \mathrm{GL}(n,2)$ in its natural action on $\mathrm{PG}(n-1,2)$. Indeed *in many cases the cycle type alone suffices to distinguish between the classes* — which is useful since the cycle type of an element is easily determined. For example, from table 5a, we see that the cycle type alone distinguishes between the six classes of elements in $\mathrm{GL}(6,2)$ which have order 6. In fact, with two exceptions, given only that $\mathrm{CT}(A) = \mathrm{CT}(B)$, where $A, B \in \mathrm{GL}(n,2)$, $2 \leq n \leq 6$, it follows that A is conjugate to B^r, for some r. The two exceptions arise for $n = 6$: (i) classes 4C and 4E of $\mathrm{GL}(6,2)$ share the same cycle type $4^{12}2^{6}1^{3}$; (ii) classes 7A,B and 7E of $\mathrm{GL}(6,2)$ share the same cycle type 7^9.

4.2 F.p.f. elements and linear sections of $\mathrm{GL}(n,2)$

For certain purposes — see below for an example — it is of importance to know the f.p.f. classes of $\mathrm{GL}(n,2)$. Now, see [4, Theorem 3.5(iv)], any power $S^r \neq I$ of a Singer element $S \in \mathrm{GL}(n,2)$ is fixed-point-free on $\mathrm{PG}(n-1,2)$. In the case of $\mathrm{GL}(4,2)$, the f.p.f. classes 3A, 5A and 15A,B arise from the Singer elements and their powers, and, see table 3, there is only one further f.p.f. class, namely class 6A. In the case of $\mathrm{GL}(5,2)$, the Singer elements give rise solely to the f.p.f. classes 31A-31F, and, see table 4, there are just two further f.p.f. classes, namely classes 21A and 21B.

A far richer supply of f.p.f. elements is available in the case of $\mathrm{GL}(6,2)$. First of all the Singer elements and their powers provide us with twelve f.p.f. classes, namely classes 3A, 7A,B, 9A, 21A,B and 63A-63F. Secondly, see tables 5a and 5b, there exist eight further classes of f.p.f. elements, namely classes 6A, 7E, 12A, 14A,B, 15C and 15D,E.

In [5] the problem was posed of classifying the *r-dimensional normalized linear sections*, denoted $\mathrm{NLS}_r(n,q)$'s, of $\mathrm{GL}(n,q)$. (Such a section is, by definition, an r-dimensional subspace of the n^2-dimensional vector space $\mathrm{End}(n,q)$ which contains I_n and is such that every non-zero element lies in $\mathrm{GL}(n,q)$.) It is easy to see that $r \leq n$, and that $r = n$ is achieved by use of a Singer cyclic subgroup $\cong Z_{q^n-1}$. Also, [5, Lemma 2.1], each non-scalar element A of a $\mathrm{NLS}_r(n,q)$ must be f.p.f. upon the points of $\mathrm{PG}(n-1,q)$. The classification problem was solved in the case of $\mathrm{GL}(4,2)$, and it was found that, up to an appropriate notion of equivalence, there are just two classes $\mathcal{M}_3, \mathcal{M}'_3$ of maximal $\mathrm{NLS}_3(4,2)$'s and three classes $\mathcal{M}_4, \mathcal{M}'_4, \mathcal{M}''_4$, of $\mathrm{NLS}_4(4,2)$'s. The existence of the class $\mathcal{M}_3$ is, see [12], related to the existence of "7-clusters" in Alt(7), and any linear section $\mathcal{S} \in \mathcal{M}_3$ has the property that all six elements of $\mathcal{S} \setminus \{0, I\}$ belong to the non-Singer f.p.f. class 6A of GL(4, 2).

Concerning GL(5, 2), observe that $A \in \mathrm{GL}(5,2)$ is of class 21A (class 21B) according as $I + A$ is of class 21B (class 21A). (This follows from $A = S_3 \oplus S_2$, since a corresponding property holds for elements S_3 and $I_3 + S_3$ of the classes 7A and 7B of GL(3, 2).) Consequently all $\mathrm{NLS}_2(5,2)$'s of the kind $\{0, I, A, B\}$ with $\mathrm{o}(A) = \mathrm{o}(B) = 21$ are conjugate in GL(5, 2). Such a $\mathrm{NLS}_2(5,2)$ is not maximal. Indeed we have found, using MAGMA [1], that for $r = 3, 4$ and 5 there exist maximal $\mathrm{NLS}_r(5,2)$'s whose elements $\neq 0, I$ all belong to the non-Singer f.p.f. classes 21A,B.

The classification problem for maximal $\mathrm{NLS}_r(n,2)$'s is still open for dimension $n \geq 5$. In the case $n = 6$, granted the afore-mentioned rich supply of f.p.f. elements present in GL(6, 2), the existence of very many inequivalent kinds of maximal $\mathrm{NLS}_r(6,2)$'s seems likely.

Remark 1 *In the case of the real field $\mathbb{F} = \mathbb{R}$, the maximal dimension r for a linear section $\mathrm{NLS}_r(n,\mathbb{R})$ of $GL(n,\mathbb{R})$ is known for all values of n. See e.g. [10, after Theorem 13.68]. Of course $r = 1$ when n is odd, since then $\det(A + \lambda B) = 0$ always has a real root. A few sample values are:*

$n =$	4	5	6	8	10	16	24	32	40	60	64	126	128	130	2048
$r =$	4	1	2	8	2	9	8	10	8	4	12	2	16	2	24

While the proof of maximality is very tough, the attainment of these maximal values of m is easily achieved, granted a knowledge of the table of real Clifford algebras, cf. [10, Table 13.26]. Incidentally finite geometers are not always aware of the fact that the table of real Clifford algebras can be given a finite geometry derivation! See [12, Section 2.1] and [11].

A Appendix: further aspects

A.1 Tensor product and exterior product aspects

We may view V_4 as a tensor product $V_2 \otimes V_2$, and we may view V_6 not only as a tensor product $V_2 \otimes V_3$ but also as an exterior product $\wedge^2 V_4$. Class aspects of these three product structures are described in the following three theorems. See [4] for their proofs.

Theorem 2 *An element $A \in \mathrm{GL}(4,2)$ can be expressed in the form $A = B \otimes C$, for B, $C \in \mathrm{GL}(V_2)$, if and only if A belongs to one of the following classes in table 3: 1A, 2B, 3A, 3B, 6A.*

Theorem 3 *An element* $A \in \mathrm{GL}(6,2)$ *can be expressed in the form* $A = B \otimes C$, *for* $B \in \mathrm{GL}(V_2)$, $C \in \mathrm{GL}(V_3)$, *if and only if* A *belongs to one of the following classes in tables 5a, 5b:*

1A, 2B, 2C, 3A, 3C, 4C, 4E, 6A, 6F, 7A,B, 12A, 14A,B, 21A,B.

Theorem 4 *Under the injective mapping* $\mathrm{GL}(4,2) \to \mathrm{GL}(6,2) : A_4 \mapsto \wedge^2 A_4$ *the* 14 *classes of* GL(4,2) *in row 1 of the following table are mapped into* 11 *of the classes of* GL(6,2) *as indicated in row 2:*

1A	*2A*	*2B*	*3A*	*3B*	*4A*	*4B*	*5A*	*6A*	*6B*	*7A,B*	*15A,B*
1A	*2B*	*2B*	*3B*	*3C*	*4C*	*4E*	*5A*	*6D*	*6E*	*7E*	*15C*

A.2 Subgroups arising from field extensions

Choose a direct sum decomposition $V_4 = V_2 \oplus V_2$ and set $W = S_2 \oplus S_2$, $G_{2,4} = C(W)$ and $J = J_2 \oplus J_2$. Then, by [4, lemma 3.1], we may identify $\mathfrak{A}_W$ with the field GF(4), V_4 with a $V(2,4)$ and $G_{2,4}$ with GL(2,4). Since $JWJ^{-1} = W^2$ it follows that J is a σ-semilinear map of $V(2,4)$ with respect to the automorphism $\sigma : X \mapsto X^2$ of GF(4). Consequently the subgroup $\Gamma_{2,4} = G_{2,4} \cup JG_{2,4} \cong G_{2,4} \rtimes \langle J \rangle$ may be identified with $\Gamma\mathrm{L}(2,4)$, and we have a subgroup chain

$$\mathrm{SL}(2,4) < \mathrm{GL}(2,4) < \Gamma\mathrm{L}(2,4) < \mathrm{GL}(4,2) \tag{6}$$

where $\mathrm{SL}(2,4)(\cong \mathrm{Alt}(5))$ is the commutator subgroup $G'_{2,4}$ of $G_{2,4}$. In fact $\Gamma\mathrm{L}(2,4)$ is a maximal subgroup of GL(4,2). The groups SL(2,4), GL(2,4) and $\Gamma\mathrm{L}(2,4)$ have orders 60, 180 and 360, and possess 5, 15 and 12 conjugacy classes, respectively.

Information concerning the relation of these classes to those of GL(4,2) is provided in table 6. In particular the 4 classes 1A, 2B, 3A and 5A of GL(4,2) may be represented by elements of a SL(2,4) subgroup, the further 4 classes 3B, 6A and 15A,B may be represented if we use elements of $\mathrm{GL}(2,4) \setminus \mathrm{SL}(2,4)$, and class 4B may be represented by an element of $\Gamma\mathrm{L}(2,4) \setminus \mathrm{GL}(2,4)$. (The notation in the final two columns of the table is explained below, in the discussion of table 7.) The 5 classes 2A, 4A, 6B and 7A,B do not have representatives $\in \Gamma\mathrm{L}(2,4)$.

Table 6.	SL(2, 4), GL(2, 4) and ΓL(2, 4) classes inside GL(4, 2)				
GL(4, 2) class	Reptve.	Cycle type	SL(2, 4)	GL(2, 4)	ΓL(2, 4)
1A	I_4	1^{15}	1	1+0	1+0
2B	$J_2 \oplus J_2$	$2^6\,1^3$	1	1+0	1+1
3A	$S_2 \oplus S_2$	3^5	1	1+2	2+0
3B	$S_2 \oplus I_2$	$3^4 1^3$	—	0+2	1+0
4B	J_4	$4^3\,2^1\,1^1$	—	—	0+1
5A	$(S_4)^3$	5^3	2	2+0	1+0
6A	$J_2 \otimes S_2$	$6^2\,3^1$	—	0+2	1+1
15A,B	$S_4^{\pm 1}$	15^1	—	0+2, 0+2	1+0, 1+0
Total number of classes			5	15	12

Similarly, choosing a direct sum decomposition $V_6 = V_2 \oplus V_2 \oplus V_2$ and setting $W = S_2 \oplus S_2 \oplus S_2$, $G_{3,4} = C(W)$ and $J = J_2 \oplus J_2 \oplus J_2$, we may identify $\mathfrak{A}_W$ with the field GF(4), V_6 with a $V(3,4)$ and $G_{3,4}$ with GL(3, 4), and so arrive at a subgroup chain

$$\mathrm{SL}(3,4) < \mathrm{GL}(3,4) < \Gamma\mathrm{L}(3,4) < \mathrm{GL}(6,2). \qquad (7)$$

Here $\mathrm{SL}(3,4) = G'_{3,4}$ and $\Gamma\mathrm{L}(3,4) = G_{3,4} \cup JG_{3,4} \cong G_{3,4} \rtimes \langle J \rangle$. The groups SL(3, 4), GL(3, 4) and ΓL(3, 4) have orders 60, 480, 181, 440 and 362, 880, and possess 28, 60 and 39 classes, respectively.

Finally, choose instead a direct sum decomposition $V_6 = V_3 \oplus V_3$ and set $U = S_3 \oplus S_3$ and $G_{2,8} = C(U)$. Then, by [4, lemma 3.1], we may identify $\mathfrak{A}_U$ with the field GF(8), V_4 with a $V(2,8)$ and $G_{2,8}$ with GL(2, 8). Now the normalizer $N(\langle S_3 \rangle)$ of $\langle S_3 \rangle$ in GL(3, 2) has structure $\langle S_3 \rangle \rtimes Z_3$. Choosing K_3 in one of the Z_3 subgroups of $N(\langle S_3 \rangle)$ to satisfy $K_3 S_3 K_3^{-1} = (S_3)^2$, and setting $K = K_3 \oplus K_3$, then $KUK^{-1} = U^2$. It follows that K is a ψ-semilinear map of $V(2,8)$ with respect to the automorphism $\psi : X \mapsto X^2$ of GF(8). Consequently the subgroup $\Gamma_{2,8} = G_{2,8} \rtimes \langle K \rangle$ may be identified with ΓL(2, 8), and we have a subgroup chain

$$\mathrm{SL}(2,8) < \mathrm{GL}(2,8) < \Gamma\mathrm{L}(2,8) < \mathrm{GL}(6,2) \qquad (8)$$

where $\mathrm{SL}(2,8) = G'_{2,8}$. The groups SL(2, 8), GL(2, 8) and ΓL(2, 8) have orders 504, 3, 528 and 10, 584, and possess 9, 63 and 29 classes, respectively.

It is known that both ΓL(3, 4) and ΓL(2, 8) are maximal subgroups of GL(6, 2). (For a list of the maximal subgroups of GL(6, 2) consult [15].)

In table 7 we provide information concerning the relation of the classes of the foregoing subgroups of GL(6, 2) to those of GL(6, 2) itself. In the column headed GL(2, 8) an entry $r + s$ against class nX of GL(6, 2) means that class nX contains $r + s$ classes of GL(2, 8) of which r lie in SL(2, 8) and s lie in $\mathrm{GL}(2,8) \setminus \mathrm{SL}(2,8)$. Similarly in the column headed $\Gamma\mathrm{L}(2,8)$ an entry $r+s$ against class nX of $\mathrm{GL}(6,2)$ means that class nX contains $r+s$ classes of $\Gamma\mathrm{L}(2,8)$ of which r lie in $\mathrm{GL}(2,8)$ and s lie in $\Gamma\mathrm{L}(2,8) \setminus \mathrm{GL}(2,8)$. Similarly for the $\mathrm{GL}(3,4)$ and $\Gamma\mathrm{L}(3,4)$ columns. To save space, an entry $r+s$, $r+s$ against paired classes nX, Y is abbreviated to $(r+s)^2$.

In arriving at an understanding of table 7 it should be borne in mind that the above automorphisms σ, ψ of $\mathrm{GF}(4)$, $\mathrm{GF}(8)$ have periods 2, 3, respectively. Hence *the order of any element* $A \in \Gamma\mathrm{L}(3,4) \setminus \mathrm{GL}(3,4)$ *is necessarily even, and that of* $A \in \Gamma\mathrm{L}(2,8) \setminus \mathrm{GL}(2,8)$ *is a multiple of* 3. In fact, from the table, we have

$$\mathrm{o}(A) \in \begin{cases} \{2,4,6,8,14\} & \text{if } A \in \Gamma\mathrm{L}(3,4) \setminus \mathrm{GL}(3,4) \\ \{3,6,9\} & \text{if } A \in \Gamma\mathrm{L}(2,8) \setminus \mathrm{GL}(2,8)\,. \end{cases}$$

Also, *two distinct classes of* $\mathrm{GL}(3,4)$ *may well fuse to form a single class of* $\Gamma\mathrm{L}(3,4)$, *and three distinct classes of* $\mathrm{GL}(2,8)$ *may fuse to form a single class of* $\Gamma\mathrm{L}(2,8)$. (Three classes of $\mathrm{SL}(3,4)$ may also coalesce to form a single class of $\mathrm{GL}(3,4)$ — see the 4C and 12A entries of table 7.)

As an example, consider the elements of $\Gamma\mathrm{L}(2,8)$ of order 7. Since $3 \nmid 7$, such elements lie in $\mathrm{GL}(2,8)$. In fact $\mathrm{GL}(2,8)$ has 27 classes of elements of order 7, and these fuse in threes to form nine classes of $\Gamma\mathrm{L}(2,8)$. First of all there are the six singleton classes $\{U^i\}$, $i = 1, \ldots, 6$, of $\mathrm{GL}(2,8)$ which fuse to yield two classes of $\Gamma\mathrm{L}(2,8)$, namely $\{U, U^2, U^4\}$ and $\{U^{-1}, U^{-2}, U^{-4}\}$. Since $U = S_3 \oplus S_3$, the elements U^i, $i = 1,2,4$, belong to class 7A of $\mathrm{GL}(6,2)$ and the elements U^i, $i = 6,5,3$, belong to class 7B of $\mathrm{GL}(6,2)$. The remaining 21 classes have representatives of the form $S_3^i \oplus S_3^j (\sim_{\mathrm{GL}(2,8)} S_3^j \oplus S_3^i)$ with $0 \le i < j \le 6$. Since in each case the centralizer is $\cong Z_7 \times Z_7$, each of these 21 classes of $\mathrm{GL}(2,8)$ has length 72. Belonging to class 7A of $\mathrm{GL}(6,2)$ are the three classes with representatives $S_3 \oplus S_3^2$, $S_3^2 \oplus S_3^4$ and $S_3^4 \oplus S_3$. (Using $I_3 \oplus K_3^{-1}$, note that $S_3 \oplus S_3^2$ is similar to $S_3 \oplus S_3$; however $I_3 \oplus K_3^{-1} \notin \Gamma\mathrm{L}(2,8)$.) The inverses of these last three $\mathrm{GL}(2,8)$ classes accordingly belong to class 7B of $\mathrm{GL}(6,2)$. The six elements $I_3 \oplus S_3^i$, $i = 1, \ldots, 6$, represent six further classes of $\mathrm{GL}(2,8)$ which fuse in threes, $i = 1,2,4$ and $i = 6,5,3$, to form two classes of $\Gamma\mathrm{L}(2,8)$, with the first three belonging to class 7C, and the second three belonging to class 7D, of $\mathrm{GL}(6,2)$. Next there are the three classes of $\mathrm{GL}(2,8)$ with representatives $S_3^i \oplus S_3^{-i}$, $i = 1,2,3$; these lie in the subgroup $\mathrm{SL}(2,8)$, they belong to class 7E of $\mathrm{GL}(6,2)$ and they fuse to form a single class of $\Gamma\mathrm{L}(2,8)$. Finally there are six further classes of $\mathrm{GL}(2,8)$ which also belong to class 7E of $\mathrm{GL}(6,2)$; three of these have representatives $S_3 \oplus S_3^3$, $S_3^2 \oplus S_3^6$, $S_3^4 \oplus S_3^5$, which fuse to form a single class of $\Gamma\mathrm{L}(2,8)$, and the other three are obtained upon taking inverses.

Observe from the table that the following 24 classes of $\mathrm{GL}(6,2)$ cannot be represented by an element of $\Gamma\mathrm{L}(3,4)$ nor by an element of $\Gamma\mathrm{L}(2,8)$:

$$2A, 4A, 4B, 4D, 6B, 6C, 8B, 10A, 12B, 12C,$$
$$14C, 14D, 21C,D, 28A,B, 30A,B, 31A...F$$

If A belongs to one of these classes and $\mathcal{S}_{3,4}$ is a set of generators for the subgroup SL(3,4) of GL(6,2) it follows, since the normalizer $\Gamma\mathrm{L}(3,4)$ of SL(3,4) is maximal in GL(6,2), that $\langle \mathcal{S}_{3,4}, A\rangle = \mathrm{GL}(6,2)$. Similarly, if $\mathcal{S}_{2,8}$ is a set of generators for $\mathrm{SL}(2,8) < \mathrm{GL}(6,2)$, then $\langle \mathcal{S}_{2,8}, A\rangle = \mathrm{GL}(6,2)$. In particular these last statements hold for $A \in$ class 2A, that is if A is any transvection in GL(6,2). (Incidentally if $A \in \mathrm{GL}(3,4) < \mathrm{GL}(6,2)$ is a transvection *qua its action upon* $V(3,4)$ then A belongs to class 2B of GL(6,2), and if $A \in \mathrm{GL}(2,8) < \mathrm{GL}(6,2)$ is a transvection *qua its action upon* $V(2,8)$ then A belongs to class 2C of GL(6,2).)

Incidentally it is worth noting that the subgroup $G_\otimes = \{A_2 \otimes A_3 \mid A_i \in \mathrm{GL}(V_i)\} \cong \mathrm{GL}(V_2) \times \mathrm{GL}(V_3)$ of $\mathrm{GL}(V_6)$, associated with a tensor product structure $V_6 = V_2 \otimes V_3$, is (unlike the situation for $q > 2$) *not* a maximal subgroup of $\mathrm{GL}(V_6)$; indeed, see [4, Remark 7.1], $G_\otimes$ lies inside a $\Gamma\mathrm{L}(3,4)$ subgroup. Recall from theorem 3 that the following 16 classes of GL(6,2) may be represented by elements belonging to a $G_\otimes$ subgroup of GL(6,2):

$$1A,\ 2B,\ 2C,\ 3A,\ 3C,\ 4C,\ 4E,\ 6A,\ 6F,\ 7A,B,\ 12A,\ 14A,B,\ 21A,B.$$

From table 7 we see that these 16 classes are indeed a selection from the 33 classes represented by elements belonging to a $\Gamma\mathrm{L}(3,4)$ subgroup of GL(6,2).

Table 7. $SL(6/d, 2^d)$, $GL(6/d, 2^d)$ and $\Gamma L(6/d, 2^d)$ classes inside $GL(6,2)$, $d = 2,3$

GL(6,2) class	Representative	Cycle type	SL(3,4)	GL(3,4)	ΓL(3,4)	SL(2,8)	GL(2,8)	ΓL(2,8)
1A	I_6	1^{63}	1	1+0	1+0	1	1+0	1+0
2B	$I_2 \oplus J_2 \oplus J_2$	$2^{24}\,1^{15}$	1	1+0	1+0	—	—	—
2C	$J_2 \oplus J_2 \oplus J_2$	$2^{28}\,1^{7}$	—	—	0+1	1	1+0	1+0
3A	$S_2 \oplus S_2 \oplus S_2$	3^{21}	2	2+2	2+0	1	1+0	1+0
3B	$I_4 \oplus S_2$	$3^{16}\,1^{15}$	—	0+2	1+0	—	—	—
3C	$I_2 \oplus S_2 \oplus S_2$	$3^{20}\,1^{3}$	1	1+2	2+0	—	—	0+2
4C	$J_3 \oplus J_3$	$4^{12}\,2^{6}\,1^{3}$	3	1+0	1+0	—	—	—
4E	$J_2 \oplus J_4$	$4^{12}\,2^{6}\,1^{3}$	—	—	0+1	—	—	—
5A	$I_2 \oplus (S_4)^3$	$5^{12}\,1^{3}$	2	2+0	1+0	—	—	—
6A	$S_2 \oplus (J_2 \otimes S_2)$	$6^{8}\,3^{5}$	2	2+2	2+0	—	—	—
6D	$J_2 \oplus J_2 \oplus S_2$	$6^{6}\,3^{4}\,2^{6}\,1^{3}$	—	0+2	1+0	—	—	—
6E	$I_2 \oplus (J_2 \otimes S_2)$	$6^{8}\,3^{4}\,1^{3}$	—	0+2	1+0	—	—	—
6F	$J_2 \oplus (J_2 \otimes S_2)$	$6^{9}\,3^{2}\,2^{1}\,1^{1}$	—	—	0+1	—	—	0+2
7A,B	$S_3 \oplus S_3,\ S_3^{-1} \oplus S_3^{-1}$	7^{9}	1,1	$(1+0)^2$	$(1+0)^2$	—	$(0+6)^2$	$(2+0)^2$
7C,D	$I_3 \oplus S_3,\ I_3 \oplus S_3^{-1}$	$7^{8}\,1^{7}$	—	—	—	—	$(0+3)^2$	$(1+0)^2$
7E	$S_3 \oplus S_3^{-1}$	7^{9}	—	—	—	3	3+6	3+0
8A	$I_1 \oplus J_5$	$8^{4}\,4^{6}\,2^{2}\,1^{3}$	—	—	0+1	—	—	—
9A	$(S_6)^7$	9^{7}	—	0+2	1+0	3	3+0	1+2
12A	$J_3 \otimes S_2$	$12^{4}\,6^{2}\,3^{1}$	6	2+0	1+0	—	—	—
14A,B	$J_2 \otimes S_3,\ J_2 \otimes S_3^{-1}$	$14^{4}\,7^{1}$	—	—	$(0+1)^2$		$(0+3)^2$	$(1+0)^2$
15A,B	$I_2 \oplus S_4,\ I_2 \oplus S_4^{-1}$	$15^{4}\,1^{3}$	—	$(0+2)^2$	$(1+0)^2$	—	—	—
15C	$S_2 \oplus (S_4)^3$	$15^{3}\,5^{3}\,3^{1}$	—	0+4	2+0	—	—	—
15D,E	$S_2 \oplus S_4,\ S_2 \oplus S_4^{-1}$	$15^{4}\,3^{1}$	2, 2	$(2+2)^2$	$(2+0)^2$	—	—	—
21A,B	$(S_6)^3, (S_6)^{-3}$	21^{3}	2, 2	$(2+0)^2$	$(1+0)^2$	—	$(0+3)^2$	$(1+0)^2$
63A...F	$S_6,\ \ldots\ S_6^{-11}$	63^{1}	—	$(0+2)^6$	$(1+0)^6$	—	$(0+3)^6$	$(1+0)^6$
Total number of classes			28	60	39	9	63	29

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