On Face Antimagic Labelings of Plane Graphs

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Abstract

If $G = (V, E, F)$ is a finite connected plane graph on $|V| = p$ vertices, $|E| = q$ edges and $|F| = t$ faces, then G is said to be (a, d)-face antimagic iff there exists a bijection $h : E \to \{1, 2, ..., q\}$ and two positive integers a and d such that the induced mapping $g_h : F \to N$, defined by $g_h(f) = \sum\{h(u, v) : \text{edge } (u, v) \text{ surround the face } f\}$, is injective and has the image set $g_h(F) = \{a, a + d, ..., a + (t - 1)d\}$. We deal with (a, d)-face antimagic labelings for a certain class of plane graphs.

1. Introduction

A graph G consists of a vertex set V and an edge set E with cardinalities p and q, respectively. We only consider graphs without loops and multiple edges. General references for graph-theoretic notions are [10] and [11].

A graph is said to be plane if it is drawn on the Euclidean plane such that edges do not cross each other except at vertices of the graph. We make the convention that all plane graphs considered in this paper possess no vertices of degree one. For a plane graph G, it makes sense to determine its faces, including the unique face of infinite area. Let t be the number of faces of G.

The weight $w(f)$ of a face $f \in F(G)$ under a edge labeling $h : E \to \{1, 2, ..., q\}$ is the sum of the labels of edges surrounding that face.

A connected plane graph $G = (V, E, F)$ is said to be (a, d)-face antimagic if there exist positive integers a and d and bijection $h : E(G) \to \{1, 2, ..., q\}$ such that the induced mapping $g_h : F(G) \to W$ is also a bijection, where $W = \{w(f) : f \in F(G)\} = \{a, a + d, ..., a + (t - 1)d\}$ is the set of weights of faces. If $G = (V, E, F)$ is (a, d)-face antimagic and $h : E(G) \to \{1, 2, ..., q\}$ is a corresponding bijective mapping of G then h is said to be an (a, d)-face antimagic labeling of G.

In [8], Hartsfield and Ringel introduce the concept of an antimagic graph and we find the conjecture in [8] that every connected graph $G = (V, E)$ of order $n = |V| \geq 3$ and size $m = |E| \geq 2$ is antimagic. Since there are no hints that one could be able to prove or disprove this conjecture Bodendiek and Walther started looking for a new edge labeling arising from the antimagic labeling. This search was successful and led to the new concept of an (a, d)-antimagic graph defined in [5,6]. If $G = (V, E)$ is a connected graph of order $p = |V| \geq 3$ and size $q = |E| \geq 2$, then G is said to be (a, d)-antimagic iff there exists a bijection $k : E \to \{1, 2, ..., q\}$ and two positive integers a and d such that the induced mapping $g_k : V(G) \to N$, defined by $g_k(v) = \sum\{k(u, v) : (u, v) \in E(G)\}$, is injective and $g_k(V) = \{a, a + d, ..., a + (p - 1)d\}$ is the set of weights of vertices.

(a, d)-antimagic graphs defined by Bodendiek and Walther we shall call (a, d)-vertex antimagic. (a, d)-vertex antimagic labelings of the special graphs called parachutes are described in [6,7]. In [1] are characterized all (a, d)-vertex antimagic graphs of prisms D_n when n is even and it is shown that if n is odd the prisms D_n are $(\frac{5n+5}{2}, 2)$- vertex antimagic. The prism $D_n, n \geq 3$, is a trivalent graph which can be defined as the cartesian product $P_2 x C_n$ of a path on two vertices with a cycle on n vertices. The prism $D_n, n \geq 3$, is a plane graph and the plane dual graph D_n^* of D_n is the graph of a bipyramid. So, in [1] are characterized all (a, d)-face antimagic graphs of bipyramids D_n^* with even cycles and it is shown that if n is odd the bipyramids D_n^* are $(\frac{5n+5}{2}, 2)$-face antimagic.

The antiprism $Q_n, n \geq 3$, is a regular graph of degree $r = 4$ (Archimedean convex polytope) and for $n = 3$ it is octahedron. The papers [3,9] describes $(6n + 3, 2)$-vertex antimagic labelings and $(4n + 4, 4)$-vertex antimagic labelings for the antiprisms Q_n, $n \geq 3$. The graph of a quasibipyramid Q_n^* is the plane dual graph of the plane graph Q_n. That means that if $n \geq 3$, then the plane graph of quasibipyramid Q_n^* has $(6n + 3, 2)$-face antimagic labeling and $(4n + 4, 4)$- face antimagic labeling.

2. Construction of plane graph Q_n

Let $I = \{1, 2, 3, ..., n\}$ be index set and Q_n be the graph of an antiprism. The antiprism $Q_n, n \geq 3$, is the plane regular graph. Let us denote the vertex set of Q_n by $V(Q_n) = \{y_1, y_2, ..., y_n, z_1, z_2, ..., z_n\}$ and edge set by $E(Q_n) = \{y_i y_{i+1} : i \in I\} \cup \{z_i z_{i+1} : i \in I\} \cup \{y_i z_i : i \in I\} \cup \{y_i z_{i+1} : i \in I\}$. We make the convention that $y_{n+1} = y_1$ and $z_{n+1} = z_1$ to simplify later notations. The face set $F(Q_n)$ contains $2n$ 3-sided faces and two n-sided faces (internal and external). We insert exactly one vertex x (t) into the internal (external) n-sided face of Q_n and consider the graph $\mathbb{Q}_n$ with the vertex set $V(\mathbb{Q}_n) = V(Q_n) \cup \{x, t\}$ and the edge set $E(\mathbb{Q}_n) = E(Q_n) \cup \{xy_i : i \in I\} \cup \{z_i t : i \in I\}$. The $\mathbb{Q}_n$ is the plane graph consisting of 3-sided faces. Let its vertices be labeled as in Figure 1.

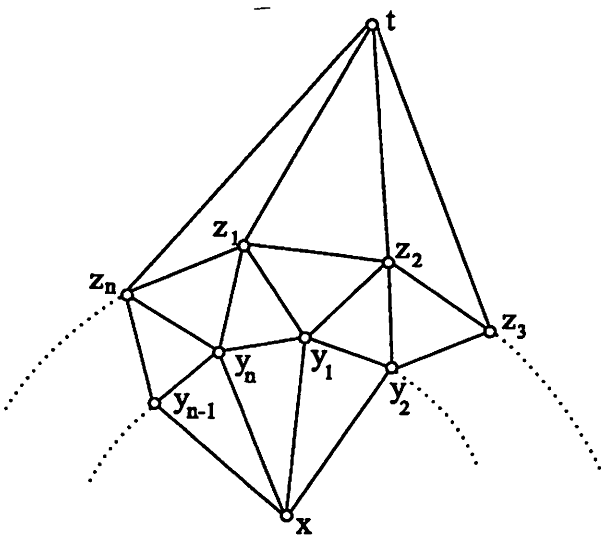

Figure 1

In paper [4] is defined a plane graph Q_n^m, $n \geq 3$, $m \geq 1$, and described $(a, 1)$-face antimagic labelings of Q_n^m for $n \geq 3$ and $m \geq 3$. $(\frac{21n}{2} + 2, 1)$-face antimagic labeling for the plane graph Q_n^m, if $m = 2$ and n is even, can be found in [2].

If $m = 1$ then the plane graph Q_n^m is the considered graph $\mathbb{Q}_n$. In this paper we construct $(a, 1)$-face antimagic labeling of $\mathbb{Q}_n$ which complete the results in [4].

3. Necessary conditions

Assume that a bijection $h : E(\mathbb{Q}_n) \to \{1, 2, ..., q\}$ is (a, d)-face antimagic labeling of $\mathbb{Q}_n$, $p = 2n + 2$, $q = 6n$ and $t = 4n$. Clearly, the sum of weights in the set $W = \{w(f) : f \in F(\mathbb{Q}_n)\} = \{a, a + d, ..., a + (4n - 1)d\}$ is

$$\sum_{f \in F(\mathbb{Q}_n)} w(f) = 2n(2a + d(4n - 1)) \qquad (1)$$

The sum of all the edge labels used to calculate the weights of faces is equal to

$$2 \sum_{e \in E(Q_n)} h(e) = 6n(6n+1) \tag{2}$$

Thus the following equation holds

$$2 \sum_{e \in E(Q_n)} h(e) = \sum_{f \in F(Q_n)} w(f) \tag{3}$$

which is obviously equivalent to the equation

$$3(6n+1) = 2a + d(4n-1) \tag{4}$$

The equation (4) is a linear Diophantine equation.

If $a = 6$ is the minimal value of weight which can be assigned to a 3-sided face then we get that $0 < d \leq 4$ and from (4) we can see that d is odd. This implies that the Diophantine equation (4) has exactly the two different solutions $(a, d) = (7n+2, 1)$ or $(a, d) = (3n+3, 3)$, respectively

4. $(7n+2, 1)$-FACE ANTIMAGIC LABELING

Theorem. *For $n \geq 3$, the plane graph Q_n has a $(7n+2, 1)$-face antimagic labeling.*

Proof. We construct an edge labeling h of Q_n in the following way.

$$h(xy_i) = 4n + 1 - i,$$

$$h(z_i t) = (4n + 2 + i)\lambda(i, n-2) + (3n + 2 + i)\lambda(n-1, i),$$

$$h(y_i y_{i+1}) = 2n + i,$$

$$h(z_i z_{i+1}) = (2n - 1 - i)\lambda(i, n-2) + (3n - 1 - i)\lambda(n-1, i),$$

$$h(y_i z_i) = i,$$

$$h(y_i z_{i+1}) = 6n + 1 - i,$$

for $i \in I$, where

$$\lambda(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{if } \alpha > \beta. \end{cases}$$

It is easy to verify that the labeling h uses each integer $1, 2, ..., 6n$ exactly once. Let us denote the weights of 3-sided faces of Q_n (under the edge labeling h) by

$$\begin{aligned} w_h^1 &= h(xy_i) + h(y_i y_{i+1}) + h(xy_{i+1}), \\ w_h^2 &= h(y_i y_{i+1}) + h(y_i z_{i+1}) + h(y_{i+1} z_{i+1}), \\ w_h^3 &= h(y_i z_i) + h(z_i z_{i+1}) + h(y_i z_{i+1}) \qquad \text{and} \\ w_h^4 &= h(z_i t) + h(z_i z_{i+1}) + h(z_{i+1} t). \end{aligned}$$

Under the labeling h the weights of all 3-sided faces constitute sets

$$\begin{aligned} W_1 &= \{w_h^1 : i \in I\} = \{10n + 2 - i : i \in I\}, \\ W_2 &= \{w_h^2 : i \in I\} = \{8n + 1 + i : i \in I\}, \\ W_3 &= \{w_h^3 : i \in I\} = \{8n + 2 - i : i \in I\} \qquad \text{and} \\ W_4 &= \{w_h^4 : i \in I\} = \{10n + 1 + i : i \in I\}. \end{aligned}$$

We can see that the set of weights of faces

$$W = \bigcup_{i=1}^{4} W_i = \{a, a + d, a + 2d, \dots, a + (4n - 1)d\},$$

where $a = 7n + 2$ and $d = 1$.

This proves that the edge labeling h is $(7n + 2, 1)$-face antimagic. □

Concluding this paper, let us pose the following conjecture:

Conjecture. *For $n \geq 3$ the plane graph Q_n is $(3n + 3, 3)$-face antimagic.*

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