# $\mathbb{Z}$-Cyclic Ordered Triplewhist and Directed Triplewhist Tournaments on $p$ elements, where $p \equiv 9 \pmod{16}$ is prime

Ian Anderson and Leigh H. M. Ellison
Department of Mathematics, University of Glasgow
University Gardens, Glasgow G12 8QW, UK

**Abstract**

We show that $\mathbb{Z}$-cyclic ordered triplewhist and directed triplewhist tournaments on $p$ elements exist when $p \equiv 9 \pmod{16}$ is prime.

## 1 Introduction

A whist tournament $Wh(4m+1)$ for $4m+1$ players is a schedule of games (or tables) $(a, b, c, d)$ involving two players $a$, $c$ opposing two other players $b$, $d$ such that

i. the games are arranged into $4m + 1$ rounds each of $m$ games;
ii. each player plays in exactly one game in all but one round;
iii. each player partners every other player exactly once;
iv. each player opposes every other player exactly twice.

We shall be concerned with three refinements of the structure, called triplewhist tournaments, directed whist tournaments and ordered whist tournaments. Call the pairs $\{a, b\}$ and $\{c, d\}$ pairs of *opponents of the first kind*, and call the pairs $\{a, d\}$ and $\{b, c\}$ pairs of *opponents of the second kind*. We further say that $b$ is $a$'s *left hand opponent* and $c$'s *right hand opponent*, and make similar definitions for each of $a$, $c$ and $d$. In addition, we also say that $a$ and $c$ are partners of the first kind while $b$ and $d$ are partners of the second kind. Then a *triplewhist tournament* $TWh(4m+1)$ is a $Wh(4m+1)$ in which every player is an opponent of the first (resp., second) kind exactly once with every other player; a *directed whist tournament* $DWh(4m + 1)$ is a $Wh(4m + 1)$ in which each player is a left (resp., right) hand opponent of every other player exactly once; and an *ordered whist tournament* $OWh(4m + 1)$ is a $Wh(4m + 1)$ in which each player opposes every other player exactly once while being a partner of the first (resp., second) kind. If the players are elements of $\mathbb{Z}_{4m+1}$, and if the *ith* round is obtained from the initial (first) round by adding $i-1$ to each element (mod $4m+1$), then

JCMCC 53 (2005), pp. 39-48

we say that the tournament is $\mathbb{Z}$*-cyclic*. By convention we always take the initial round to be the round from which 0 is absent. The games (tables)

$$(a_1, b_1, c_1, d_1), \dots, (a_m, b_m, c_m, d_m)$$

form the initial round of a $\mathbb{Z}$-cyclic triplewhist tournament if

$$\bigcup_{i=1}^{m} \{a_i, b_i, c_i, d_i\} = \mathbb{Z}_{4m+1} - \{0\}, \tag{A}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - c_i), \pm(b_i - d_i)\} = \mathbb{Z}_{4m+1} - \{0\}, \tag{B}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - b_i), \pm(c_i - d_i)\} = \mathbb{Z}_{4m+1} - \{0\}, \tag{C}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - d_i), \pm(b_i - c_i)\} = \mathbb{Z}_{4m+1} - \{0\}. \tag{D}$$

$(A)$ and $(B)$ show that the partner pairs form a starter [2, p. 136]. Similarly for $(A)$ and $(C)$ with the first kind opponent pairs, and $(A)$ and $(D)$ with the second kind opponent pairs. These games form a $\mathbb{Z}$-cyclic directed whist tournament if, in addition to satisfying $(A)$ and $(B)$,

$$\bigcup_{i=1}^{m} \{(b_i - a_i), (c_i - b_i), (d_i - c_i), (a_i - d_i)\} = \mathbb{Z}_{4m+1} - \{0\}. \tag{E}$$

Alternatively, they form a $\mathbb{Z}$-cyclic ordered whist tournament if, in addition to satisfying $(A)$ and $(B)$,

$$\bigcup_{i=1}^{m} \{(a_i - b_i), (a_i - d_i), (c_i - b_i), (c_i - d_i)\} = \mathbb{Z}_{4m+1} - \{0\}. \tag{F}$$

Abel, Costa and Finizio [1] have dealt with whist tournaments which are simultaneously directed and ordered. We shall be looking at whist tournaments which are simultaneously either both triplewhist and directed tournaments, or triplewhist and ordered tournaments. Such designs will be called *directed triplewhist tournaments* and *ordered triplewhist tournaments* respecively, and will be denoted by $DTWh(v)$ and $OTWh(v)$. First of all though, it should be noted that it is not possible for a $\mathbb{Z}$-cyclic tournament to be triplewhist, ordered and directed at the same time.

**Theorem 1.1** *It is not possible for a $\mathbb{Z}$-cyclic tournament to be triplewhist, ordered and directed simultaneously.*

## Proof

Firstly we will assume that such a tournament does exist. The fact that it is an ordered and directed tournament means that from $(E)$ and $(F)$ we can deduce that

$$\bigcup_{i=1}^{m}\{(a_i - b_i),(c_i - d_i)\} = \bigcup_{i=1}^{m}\{-(a_i - b_i), -(c_i - d_i)\} \tag{G}$$

and that both sides of the equation $(G)$ give the same half of the nonzero elements of $\mathbb{Z}_p$. Since the tournament is also triplewhist, $(C)$ tells us that

$$\bigcup_{i=1}^{m}\{\pm(a_i - b_i), \pm(c_i - d_i)\} = \mathbb{Z}_{4m+1} - \{0\}.$$

This contradicts $(G)$. □

We shall now show that both $\mathbb{Z}$-cyclic $DTWh(v)$ and $\mathbb{Z}$-cyclic $OTWh(v)$ exist for all $v$ whenever $v$ is a prime $p \equiv 9 \pmod{16}$. For $p \equiv 5 \pmod 8$, the directed case has been dealt with in [6] while the ordered case has been dealt with in [4].

The original proof by Anderson, Cohen and Finizio, which dealt with the existence of $\mathbb{Z}$-cyclic $TWh(p)$ with $p = 8n + 5$ prime [3], contained a requirement that certain elements be primitive roots of $\mathbb{Z}_p$. This requirement was shown by Buratti in [7] to be an additional, but not necessary one. The elements in question need only be nonsquare over $\mathbb{Z}_p$, and a less complicated proof is the result. That is also the case when it comes to this work which deals with $p \equiv 9 \pmod{16}$. The theorem of Weil on multiplicative character sums [10, Theorem 5.41] is used in the proof which follows. Here is the statement of Weil's theorem, in which the convention is understood that if $\psi$ is a multiplicative character of $GF(q)$, then $\psi(0) = 0$. Adopting this convention we have $\psi(xy) = \psi(x)\psi(y)$ for all $(x, y) \in GF(q) \times GF(q)$.

**Theorem 1.2** *Let $\psi$ be a character of order $m > 1$ of the finite $GF(q)$. Let $f$ be a polynomial of $GF(q)[x]$ which is not of the form $kg^m$ for some $k \in GF(q)$ and some $g \in GF(q)[x]$. Then we have*

$$\left|\sum_{x\in GF(q)} \psi(f(x))\right| \leq (d-1)\sqrt{q}$$

*where $d$ is the number of distinct roots of $f$ in its splitting field over $GF(q)$.*

*Notation.* Any nonzero element $k$ of $\mathbb{Z}_p$ can be expressed as $\theta^m$ where $\theta$ is a primitive root of $p$. If $b \mid p-1$ and if $m \equiv a \pmod b$, we say that $k \in C_a^b$. Furthermore, if $k \in C_0^2$, we will say that $k = \square$.

## 2 The Existence Theorem

We now take a closer look at two constructions and find the conditions which must be satisfied in order for them to produce a $\mathbb{Z}$-cyclic $DTWh(p)$ and a $\mathbb{Z}$-cyclic $OTWh(p)$ for primes $p \equiv 9 \pmod{16}$.

So let $p = 16t + 9$ be prime, let $x$ be a nonsquare element of $\mathbb{Z}_p$, and let $\theta$ be a primitive root of $p$. We now present two constructions, the first of which is a variation of a construction found in [8, p. 222].

**Construction 1** $\quad (1, x, x^4, -x) \times \theta^{8i+2j},\ 0 \le i \le 2t,\ 0 \le j \le 1$

It can be seen that this is a suitable construction since if 1, $x$, $x^4$ and $-x$ are expressed in terms of $\theta$, we have two square terms whose indices differ by 4 (mod 8), and two nonsquare terms whose indices differ by 4 (mod 8). This means that when they are multiplied by $\theta^{8i+2j}$ for appropriate values of $i$ and $j$, we get all of the nonzero elements of $\mathbb{Z}_p$ as required. First we find the conditions under which this forms a $TWh(p)$. The partner differences are pairs $\pm 2x$, $\pm(x^4 - 1) \times \theta^{8i+2j}$, and so the partner pairs form a starter provided $2x(x^4 - 1) \neq \Box$. Similarly the first kind opponent pairs form a starter provided $x(x-1)(x^3+1) \neq \Box$, and the second kind opponent pairs form a starter provided $x(x+1)(x^3-1) \neq \Box$. We now use the fact that 2 is a square since $p \equiv 9 \pmod{16}$. So Construction 1 yields the initial round tables of a $\mathbb{Z}$-cyclic $TWh(p)$ provided $x^4 - 1 = \Box$, $(x-1)(x^3+1) = \Box$, $(x+1)(x^3-1) = \Box$.

Now we find the conditions under which this also forms a $DWh(p)$. Here, the differences we are interested in are

$$x - 1, \tag{H}$$

$$x^4 - x = x(x^3 - 1) = x(x-1)(x^2 + x + 1), \tag{I}$$

$$-x - x^4 = -x(x^3 + 1) = -x(x+1)(x^2 - x + 1), \tag{J}$$

$$x + 1. \tag{K}$$

We now make the assumption that $x^2 - 1 \neq \Box$. It follows from such an assumption that one of $x - 1$ and $x + 1$ is a square, while the other is nonsquare. Using this information together with the conditions for $TWh(p)$, it can be seen that $(H)$ and $(I)$ are both square (nonsquare), while $(J)$ and $(K)$ are both nonsquare (square). For the construction to work, the indices of the two square (nonsquare) values must also differ by 4 (working mod 8).

Thus, we obtain a $DWh(p)$ using this construction when

$$\frac{x(x-1)(x^2+x+1)}{(x-1)} \in C_4^8 \text{ i.e. } x(x^2+x+1) \in C_4^8$$
$$\text{i.e. } x^5(x^2+x+1) \in C_0^8$$
$$\text{and, } \frac{-x(x+1)(x^2-x+1)}{(x+1)} \in C_4^8 \text{ i.e. } -x(x^2-x+1) \in C_4^8$$
$$\text{i.e. } x(x^2-x+1) \in C_0^8.$$

So, using all of the above information we can say that Construction 1 gives the initial round tables of a $\mathbb{Z}$-cyclic $DTWh(p)$ when $x \neq \Box$, $x^2-1 \neq \Box$, $x^2+1 \neq \Box$, $x^5(x^2+x+1) \in C_0^8$, $x(x^2-x+1) \in C_0^8$.

Now we go back and find the conditions under which Construction 1 gives the initial round tables of what is also an $OWh(p)$. Here, the differences we are interested in are

$$1-x = -(x-1), \tag{L}$$

$$x+1, \tag{M}$$

$$x^4-x = x(x^3-1) = x(x-1)(x^2+x+1), \tag{N}$$

$$x^4+x = x(x^3+1) = x(x+1)(x^2-x+1). \tag{O}$$

Again we assume that $x^2-1 \neq \Box$. Using the same process as above it is seen that $(M)$ and $(O)$ are both square (nonsquare) elements of $\mathbb{Z}_p$ while $(L)$ and $(N)$ are nonsquare (square). Again, the indices of the two square (nonsquare) values must differ by 4 (working mod 8), so it can be seen that we obtain an $OWh(p)$ using this construction when

$$\frac{x(x-1)(x^2+x+1)}{-(x-1)} \in C_4^8 \text{ i.e. } -x(x^2+x+1) \in C_4^8$$
$$\text{i.e. } x(x^2+x+1) \in C_0^8$$
$$\text{and, } \frac{x(x+1)(x^2-x+1)}{(x+1)} \in C_4^8 \text{ i.e. } x(x^2-x+1) \in C_4^8$$
$$\text{i.e. } x^5(x^2-x+1) \in C_0^8.$$

So, using all of the above information we can say that Construction 1 gives us the initial round tables of a $\mathbb{Z}$-cyclic $OTWh(p)$ when $x \neq \Box$, $x^2-1 \neq \Box$, $x^2+1 \neq \Box$, $x^5(x^2-x+1) \in C_0^8$, $x(x^2+x+1) \in C_0^8$.

**Construction 2** $(1, x^3, x^4, -x^3) \times \theta^{8i+2j}$, $0 \le i \le 2t$, $0 \le j \le 1$

It can be seen that this is a suitable construction since if 1, $x^3$, $x^4$ and $-x^3$ are expressed in terms of $\theta$, we have two square terms whose indices differ by 4 (mod 8), and two nonsquare terms whose indices differ by 4 (mod 8). This means that when they are multiplied by $\theta^{8i+2j}$ for appropriate values of $i$ and $j$, we get all of the nonzero elements of $\mathbb{Z}_p$ as required. These are the initial round tables of a $\mathbb{Z}$-cyclic $TWh(p)$ provided $x^4 - 1 = \square$, $(x-1)(x^3+1) = \square$, $(x+1)(x^3-1) = \square$. Combining this with the assumption that $x^2 - 1 \neq \square$ we see that Construction 2 gives the initial round tables of a $\mathbb{Z}$-cyclic $DTWh(p)$ when $x \neq \square$, $x^2-1 \neq \square$, $x^2+1 \neq \square$, $x^5(x^2-x+1) \in C_0^8$, $x(x^2+x+1) \in C_0^8$. It is also the case that it gives a $\mathbb{Z}$-cyclic $OTWh(p)$ when $x \neq \square$, $x^2-1 \neq \square$, $x^2+1 \neq \square$, $x^5(x^2+x+1) \in C_0^8$, $x(x^2-x+1) \in C_0^8$.

**Theorem 2.1** *Let $p = 16t+9$ be prime. If there exists a nonsquare element $x$ of $\mathbb{Z}_p$ such that $x^2 \pm 1$ are both nonsquares and either*
*$x(x^2+x+1) \in C_0^8$ and $x^5(x^2-x+1) \in C_0^8$, or*
*$x(x^2-x+1) \in C_0^8$ and $x^5(x^2+x+1) \in C_0^8$,*
*then a $\mathbb{Z}$-cyclic $DTWh(p)$ ($\mathbb{Z}$-cyclic $OTWh(p)$) exists.*

**Proof**
Suppose there exists such a nonsquare $x$. If $x(x^2-x+1) \in C_0^8$ and $x^5(x^2+x+1) \in C_0^8$ then use Construction 1 (Construction 2). If $x(x^2+x+1) \in C_0^8$ and $x^5(x^2-x+1) \in C_0^8$ then use Construction 2 (Construction 1). □

It therefore remains to show that a nonsquare $x$ satisfying the conditions of Theorem 2.1 can be obtained.

It can be seen that this task is assisted by the fact that working with Construction 1 and Construction 2 results in the same conditions being required in order to show that a suitable $x$ exists in order for a directed triplewhist and an ordered triplewhist tournament to be built. Thus, if $x \neq \square$, $x^2 \pm 1 \neq \square$, $x(x^2+x+1) \in C_0^4$ and $x^2(x^2+x+1)(x^2-x+1) \in C_4^8$ then a suitable value of $x$ exists such that a $\mathbb{Z}$-cyclic $DTWh(p)$ and a $\mathbb{Z}$-cyclic $OTWh(p)$ can be constructed using either Construction 1 or Construction 2.

Let $\lambda$ denote the quadratic character mod $p$, so that

$$\lambda(y) = \begin{cases} 1 & \text{if } y \in C_0^2; \\ -1 & \text{if } y \in C_1^2. \end{cases}$$

Let $\psi$ be the character of order 4 exactly which is defined by

$$\psi(y) = \begin{cases} 1 & \text{if } y \in C_0^4; \\ -1 & \text{if } y \in C_2^4; \\ i & \text{if } y \in C_1^4; \\ -i & \text{if } y \in C_3^4. \end{cases}$$

Let $\chi$ be the character of order 8 exactly which is defined by

$$\chi(y) = \omega^j \text{ if } y \in C_j^8,$$

where $\omega = e^{\frac{\pi i}{4}}$.

It follows from these definitions that

$$1 - \lambda(y) = \begin{cases} 2 & \text{if } y \in C_1^2; \\ 0 & \text{otherwise,} \end{cases}$$

$$1 + \psi(y) + \psi(y^2) + \psi(y^3) = \begin{cases} 4 & \text{if } y \in C_0^4; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$1 + \chi(y) + ... + \chi(y^7) = \begin{cases} 8 & \text{if } y \in C_0^8; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we let $f(x) = x(x^2+x+1)$, $g(x) = x^6(x^2+x+1)(x^2-x+1)$ and

$$S = \sum_{x \in GF(p)} (1-\lambda(x))(1-\lambda(x^2-1))(1-\lambda(x^2+1)) \times$$
$$(1+\psi(f(x)) + ... + \psi(f^3(x)))(1+\chi(g(x)) + ... + \chi(g^7(x))).$$

Substituting in the fact that $\lambda(x) = \psi(x^2) = \chi(x^4)$ it is seen that

$$S = \sum_{x \in GF(p)} (1-\chi(x^4))(1-\chi(x^2-1)^4)(1-\chi(x^2+1)^4) \times$$
$$(1+\chi(f^2(x)) + ... + \chi(f^6(x)))(1+\chi(g(x)) + ... + \chi(g^7(x))).$$

After multiplying this out and making the appropriate substitutions (using Theorem 1.2), it can be seen that,

$$S \geq p - 1453\sqrt{p}.$$

It is also clearly the case that,

$$S = 256|A|,$$

where elements in $A$ are of the form given in Theorem 2.1.

Thus,

$$S = 256|A| \geq p - 1453\sqrt{p} > 0$$
$$\text{if } p \geq 1453\sqrt{p}$$
$$\text{i.e. if } \sqrt{p} \geq 1453$$
$$\text{i.e. if } p > 2,111,209.$$

It was then checked by computer that appropriate values of $x$ existed for all primes $p < 2,111,209$ where $p \equiv 9 \pmod{16}$, excluding $p = 41$, 73, 89, 137, 233, 281, 313, 521, 569, 617, 809, 1097, 2729, 2953, 3001. Here, we list $(p, x_p)$ where $p$ is the prime and $x_p$ is a suitable value of $x$ for that prime for all relevant primes $p < 10,000$.

(409, 79), (457, 10), (601, 142), (761, 142), (857, 268), (937, 132), (953, 344), (1033, 103), (1049, 82), (1129, 119), (1193, 27), (1289, 208), (1321, 76), (1433, 605), (1481, 29), (1609, 479), (1657, 164), (1721, 12), (1753, 89), (1801, 130), (1913, 171), (1993, 542), (2089, 194), (2137, 157), (2153, 888), (2281, 402), (2297, 520), (2377, 454), (2393, 259), (2441, 411), (2473, 812), (2521, 34), (2617, 19), (2633, 5), (2713, 163), (2777, 659), (2857, 106), (2969, 505), (3049, 153), (3209, 383), (3257, 68), (3433, 14), (3449, 350), (3529, 115), (3593, 84), (3673, 115), (3769, 1344), (3833, 38), (3881, 24), (3929, 375), (4057, 111), (4073, 871), (4153, 71), (4201, 130), (4217, 329), (4297, 193), (4409, 709), (4441, 467), (4457, 377), (4649, 102), (4729, 218), (4793, 218), (4889, 1197), (4937, 21), (4969, 21), (5081, 164), (5113, 79), (5209, 218), (5273, 606), (5417, 56), (5449, 570), (5641, 616), (5657, 496), (5689, 190), (5737, 173), (5801, 165), (5849, 24), (5881, 351), (5897, 104), (6073, 60), (6089, 381), (6121, 29), (6217, 383), (6329, 651), (6361, 272), (6473, 173), (6521, 124), (6553, 88), (6569, 953), (6761, 243), (6793, 181), (6841, 164), (6857, 111), (7001, 108), (7129, 332), (7177, 805), (7193, 39), (7321, 138), (7369, 127), (7417, 335), (7433, 10), (7481, 295), (7529, 528), (7561, 52), (7577, 117), (7673, 996), (7753, 519), (7817, 113), (7993, 114), (8009, 328), (8089, 61), (8233, 245), (8297, 443), (8329, 88), (8377, 233), (8521, 109), (8537, 258), (8681, 29), (8713, 59), (8761, 344), (8969, 583), (9001, 29), (9049, 656), (9161, 473), (9209, 54), (9241, 174), (9257, 57), (9337, 207), (9433, 77), (9497, 888), (9689, 803), (9721, 688), (9769, 426), (9817, 515), (9833, 14), (9929, 603)

It has been already been shown in [5] that for all primes $p \equiv 1 \pmod 4$, $29 \leq p \leq 10,000$, $p \neq 97$, 193, 257, 449, 641, 769, 1153, 1409, 7681, there exist $\mathbb{Z}$-cyclic $DTWh(p)$. This takes care of the 15 values for which a suitable value of $x$ was not found by the computer (as listed above).

Thus the following theorem is established.

**Theorem 2.2** *A $\mathbb{Z}$-cyclic $DTWh(p)$ exists for all primes $p \equiv 9 \pmod{16}$.*

In order to have proved a similar theorem for $\mathbb{Z}$-cyclic $OTWh(p)$ where $p \equiv 9 \pmod{16}$ is prime, all that is left to do is show that a $\mathbb{Z}$-cyclic $OTWh(p)$ exists for each of the 15 values listed above. To assist in this task, the following construction (as given in [8, p. 224]) is used.

**Construction 3** Let $p = 2^k t + 1$ be prime, $k \geq 3$, $t$ odd and let $\theta$ be a primitive root of $p$. For the purposes of working with this construction, we write $d = 2^k$, $n = 2^{k-2}$ and $a \equiv 2^{k-1} - 1 \pmod{d}$. Then we consider the following initial round games

$$(1, \theta, -\theta, \theta^{a+1}) \times \theta^{2i+dj},\ 0 \leq i \leq n-1,\ 0 \leq j \leq t-1$$

For 13 of the 15 values that we are interested in, it is now possible to list $(p, \theta, a)$ where $p$ is the prime, $\theta$ is a suitable primitive root and $a$ is a suitable value which, when substituted into Construction 3, give the initial round games of a $\mathbb{Z}$-cyclic $OTWh(p)$.

$$(89,35,51),\ (137,45,11),\ (233,138,11),\ (281,178,11),\ (313,268,19),$$
$$(521,85,3),\ (569,96,3),\ (617,337,3),\ (809,396,3),\ (1097,382,3),$$
$$(2729,27,3),\ (2953,1264,3),\ (3001,1305,3)$$

Thus it has now been shown that a $\mathbb{Z}$-cyclic $OTWh(p)$ exists for all $p \equiv 9 \pmod{16}$ where $p$ is prime, with the possible exception of $p = 41$ and $p = 73$.

Using two constructions given in [9], the following initial round games were found which generate $\mathbb{Z}$-cyclic $OTWh(p)$ for $p = 41$ and 73.

*Example 2.1.* A $\mathbb{Z}$-cyclic $OTWh(41)$ is given by the initial round $\{\{1,27,40,28\},\{20,30,39,19\}\} \times \{1, 26^8, ..., 26^{32}\}$.

*Example 2.2.* A $\mathbb{Z}$-cyclic $OTWh(73)$ is given by the initial round $\{\{1,18,59,43\},\{50,6,30,5\}\} \times \{1, 59^8, ..., 59^{64}\}$.

Thus the following theorem is established.

**Theorem 2.3** *A $\mathbb{Z}$-cyclic $OTWh(p)$ exists for all primes $p \equiv 9 \pmod{16}$.*

## Acknowledgments

We would like to thank Charles King for his assistance in using the Magma computational algebra system.

# References


[1] R. J. R. Abel, S. Costa, and N. J. Finizio. Directed-ordered whist tournaments - existence results and some new classes of $\mathbb{Z}$-cyclic solutions, preprint.

[2] I. Anderson. A hundred years of whist tournaments. *J. Combin. Math. Combin. Comput.* 19, (1995) 129-150.

[3] I. Anderson, S. D. Cohen, and N. J. Finizio. An existence theorem for cyclic triplewhist tournaments. *Discrete Math.* 138, (1995), 31-41.

[4] I. Anderson, L. H. M. Ellison. $\mathbb{Z}$-cyclic ordered triplewhist tournaments on $p$ elements, where $p \equiv 5 \pmod 8$. *Discrete Math.*, to appear.

[5] I. Anderson and N. J. Finizio. On the construction of directed triplewhist tournaments. *J. Combin. Math. Combin. Comput.* 35, (2000), 107-115.

[6] I. Anderson and N. J. Finizio. Triplewhist tournaments that are also Mendelsohn designs. *J. Combin. Des.* 5, (1997), 397-406.

[7] M. Buratti. Existence of $\mathbb{Z}$-cyclic triplewhist tournaments for a prime number of players. *J. Combin. Theory Ser. A* 90, (2000), 315-325.

[8] Y. S. Liaw. Construction of $\mathbb{Z}$-cyclic triplewhist tournaments. *J. Combin. Des.* 4, (1996), 219-233.

[9] Y. S. Liaw. Some constructions of combinatorial designs. PhD Thesis, University of Glasgow, 1994.

[10] R. Lidl and H. Niederreiter. 'Finite Fields.' *Encyclopaedia of Mathematics*, Volume 20, Cambridge University Press, Cambridge, UK, 1983.