Detour Distance in Graphs

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Abstract

For two vertices u and v in a connected graph G, the detour distance $D(u, v)$ from u to v is defined as the length of a longest $u - v$ path in G. The detour eccentricity $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The detour radius $\text{rad}_D(G)$ of G is the minimum detour eccentricity among the vertices of G, while the detour diameter $\text{diam}_D(G)$ of G is the maximum detour eccentricity among the vertices of G. It is shown that $\text{rad}_D(G) \leq \text{diam}_D(G) \leq 2\text{rad}_D(G)$ for every connected graph G and that every pair a, b of positive integers with $a \leq b \leq 2a$ is realizable as the detour radius and detour diameter of some connected graph. The detour center of G is the subgraph induced by these vertices of G having detour eccentricity $\text{rad}_D(G)$. A connected graph G is detour self-centered if G is its own detour center. The detour periphery of G is the subgraph induced by the vertices of G having detour eccentricity $\text{diam}_D(G)$. It is shown that every graph is the detour center of some connected graph. Detour self-centered graphs are investigated. We present sufficient conditions for a graph to be the detour periphery of some connected graph. Several classes of graphs that are not the detour periphery of any connected graph are determined.

Key Words: distance, detour distance, detour eccentricity.

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1 Introduction

The distance $d(u,v)$ from a vertex u to a vertex v in a connected graph G is the length of a shortest $u-v$ path in G. A $u-v$ *geodesic* is a $u-v$ path of length $d(u,v)$. The *diameter* $\text{diam}(G)$ of G is the largest distance between two vertices in G. Although this is the standard definition of distance between two vertices in a connected graph, it is by no means the only definition that has been given of distance between two vertices. For two vertices u and v in a connected graph G, the *detour distance* $D(u,v)$ from u to v is defined as the length of a longest $u-v$ path in G (see [3, 4, 5, 6, 7, 8, 9, 11, 12, 13]). A $u-v$ path of length $D(u,v)$ is called a $u-v$ *detour.*

For example, in the graph G of Figure 1, $d(u,v)=3$ while $D(u,v)=8$. A $u-v$ detour (indicated by solid lines) is also shown in that figure.

$G:$

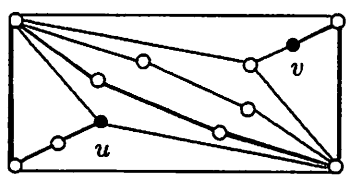

Figure 1: The detour distance between two vertices

As with standard distance, detour distance is also a metric on the vertex set of every connected graph.

Proposition 1.1 *For the detour distance D on a connected graph G, $(V(G),D)$ is a metric space.*

Proof. Let G be a connected graph. Since (1) $D(u,v)\geq 0$, (2) $D(u,v)=0$ if and only if $u=v$, and (3) $D(u,v)=D(v,u)$ for every pair u,v of vertices of G, it remains only to show that detour distance satisfies the triangle inequality. Let u,v, and w be any three vertices of G. Since the inequality $D(u,w)\leq D(u,v)+D(v,w)$ holds if any two of these three vertices are the same vertex, we assume that u,v, and w are distinct. Let P be a $u-w$ detour in G of length $k=D(u,w)$. We consider two cases.

Case 1. *v lies on P.* Let P_1 be the $u-v$ subpath of P and let P_2 be the $v-w$ subpath of P. Suppose that the length of P_1 is s and the length of P_2 is t. So $s+t=k$. Therefore,

$$D(u,w)=k=s+t\leq D(u,v)+D(v,w).$$

Case 2. *v does not lie on P.* Since G is connected, there is a shortest path Q from v to a vertex of P. Suppose that Q is a $v-x$ path. Thus x lies on P but no other vertex of Q lies on P. Let r be the length of Q. Thus $r > 0$. Let the $u-x$ subpath P' of P have length a and the $x-w$ subpath P'' of P have length b. Then $a \geq 0$ and $b \geq 0$. Therefore, $D(u,v) \geq a+r$ and $D(v,w) \geq b+r$. So

$$D(u,w) = k = a+b < (a+r)+(b+r) \leq D(u,v)+D(v,w),$$

and so the triangle inequality holds. ■

For vertices u and v in a connected graph G of order n,

$$0 \leq d(u,v) \leq D(u,v) \leq n-1,$$

where $D(u,v)=0$ if and only if $d(u,v)=0$ if and only if $u=v$, $D(u,v)=1$ if and only if uv is a bridge of G, and $D(u,v)=n-1$ if and only if G contains a Hamiltonian $u-v$ path. Furthermore, $d(u,v)=D(u,v)$ for *every* two vertices u and v of G if and only if G is a tree. It is possible, however, that $d(u,v) = D(u,v)$ for some pairs u,v of distinct vertices in a graph that contains no bridges. For example, if u and v are antipodal vertices (that is, $d(u,v) = \text{diam}(G)$) in the even cycle C_{2k}, $k \geq 2$, then $D(u,v)=d(u,v)=k$. Indeed, even more can be said.

Proposition 1.2 *Let G be a 2-connected graph. If u and v are two vertices of G for which $D(u,v)=d(u,v)$, then u and v are antipodal vertices of G.*

Proof. Assume, to the contrary, that there exists a 2-connected graph G containing two vertices u and v with $D(u,v)=d(u,v)=k$ but u and v are not antipodal vertices of G. Consequently, $2 \leq k < \text{diam}(G)$. This implies that every $u-v$ path of G has length k. We consider two cases.

Case 1. *At least one of u and v is a peripheral vertex of G, say u is a peripheral vertex of G.* Let $x \in V(G)$ such that $d(u,x) = \text{diam}(G)$. Then $x \neq v$. Since G is 2-connected, there exist internally disjoint $x-u$ and $x-v$ paths in G and so there is a $u-v$ path P in G containing x. Since every $u-v$ path in G has length k, it follows that P is a $u-v$ geodesic containing x. However then, $\text{diam}(G) = d(u,x) < d(u,v)$, a contradiction.

Case 2. *Neither u nor v is a peripheral vertex of G.* Let x and y be two antipodal vertices of G. Thus $\{u,v\} \cap \{x,y\} = \emptyset$. Then $\text{diam}(G) = d(x,y) > d(u,v) = k$. Since G is 2-connected, there exist internally disjoint $x-u$ and $x-v$ paths in G. Hence there is a $u-v$ path P in G containing x. Similarly, there exists a $u-v$ path Q containing y. Since every $u-v$ path in G has length k, the paths P and Q have length k. The $x-v$ subpath of P followed by the $v-u$ path by proceeding along Q in reverse order and then followed by the $u-x$ subpath of P produces a closed walk at x of length $2k$ containing y. Hence there exists an $x-y$ walk of length at most k in G, which implies that $d(x,y) \leq k = d(u,v)$, a contradiction. ■

The converse of Proposition 1.2 is false. For example, every two vertices u and v of $G = K_n$, $n \geq 3$, are antipodal vertices of G and $1 = d(u,v) \neq D(u,v) = n-1$. It is a simple observation that complete graphs are the only graphs G for which there is a constant k such that $d(u,v) = k$ for every two distinct vertices u and v of G. Therefore, the only such constant is $k = 1$. For detour distance, the corresponding result is stated next.

Proposition 1.3 *Let G be a connected graph of order $n \geq 2$. Then there exists an integer k such that $D(u,v) = k$ for every pair u, v of distinct vertices of G if and only if G is Hamiltonian-connected* (*and* $k = n-1$).

Proof. If G is a Hamiltonian-connected graph of order $n \geq 2$, then there exists a $u-v$ Hamiltonian path in G for every pair u, v of distinct vertices of G and $D(u,v) = n-1$. It remains to verify the converse. Assume, to the contrary, that there exists a connected graph G of order $n \geq 2$ such that $D(u,v) = k$ for every pair u, v of distinct vertices of G, but $k < n-1$. Let $uv \in E(G)$. Since $D(u,v) = k$, there exists a $u-v$ detour P of length k in G. Then P together with the edge uv form a cycle C_{k+1} of length $k+1$ in G. Since $n > k+1$ and G is connected, there exists a vertex $x \in V(G) - V(C_{k+1})$ such that x is adjacent to some vertex w in C_{k+1}. Assume that $C_{k+1} : w = v_1, v_2, \cdots, v_{k+1}, v_1 = w$. However then $x, w = v_1, v_2, \cdots, v_{k+1}$ is an $x - v_{k+1}$ path of length $k+1$ and so $D(x, v_{k+1}) \geq k+1$, which is a contradiction. ■

2 Detour Eccentricity, Radius, and Diameter

The *eccentricity* $e(v)$ of a vertex v in a connected graph G is

$$e(v) = \max\{d(v,x) \ : \ x \in V(G)\}.$$

The *radius* of a connected graph G is

$$\text{rad}(G) = \min\{e(v) \ : \ v \in V(G)\};$$

while the *diameter* of G is

$$\text{diam}(G) = \max\{e(v) \ : \ v \in V(G)\}.$$

The detour eccentricity is defined as expected, namely, the *detour eccentricity* $e_D(v)$ of a vertex v in a connected graph G is

$$e_D(v) = \max\{D(v,x) \ : \ x \in V(G)\}.$$

Recall that if u and v are distinct vertices in a connected graph G, then

$$|e(u) - e(v)| \leq d(u, v).$$

In particular, $|e(u) - e(v)| \leq 1$ if u and v are adjacent. There is a corresponding statement for detour distance.

Proposition 2.1 *If u and v are distinct vertices in a connected graph G, then*

$$|e_D(u) - e_D(v)| \leq D(u, v).$$

Proof. We may assume that $e_D(u) \geq e_D(v)$. Let w be a vertex of G such that $D(u, w) = e_D(u)$. Then $e_D(u) = D(u, w) \leq D(u, v) + D(v, w) \leq D(u, v) + e_D(v)$. Thus $|e_D(u) - e_D(v)| \leq D(u, v)$. ■

To see that there are, in fact, connected graphs containing distinct vertices u and v such that $|e_D(u) - e_D(v)| = D(u, v)$, we construct a graph G as follows. For each integer i with $1 \leq i \leq 3$, let F_i be a copy of the complete graph K_n of order $n \geq 2$. Let u_1 and u_2 be two distinct vertices of F_1 and $v_j \in V(F_{j+1})$ for $j = 1, 2$. Let G be the graph obtained from the graphs F_i $(1 \leq i \leq 3)$ by identifying each vertex u_j $(j = 1, 2)$ of F_1 with the vertex v_j of F_{j+1} and labeling the identified vertex by u_j (see Figure 2). Then $e_D(x) = 2(n - 1)$ for all $x \in V(F_1)$ and $e_D(x) = 3(n - 1)$ for all $x \in V(G) - V(F_1)$. Let $v = v_1$ and let u be any vertex in $V(F_2) - \{v_1\}$. Since $e_D(v) = 2n - 2$, $e_D(u) = 3n - 3$, and $D(u, v) = n - 1$, it follows that $|e_D(u) - e_D(v)| = D(u, v)$.

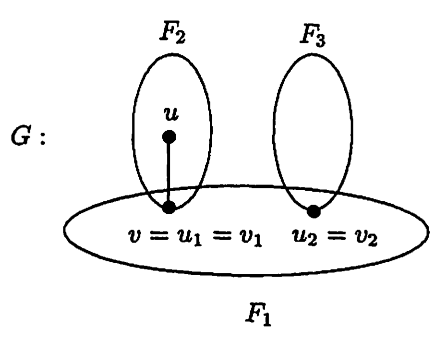

Figure 2: A graph containing vertices u and v with $|e_D(u) - e_D(v)| = D(u, v)$

The *detour radius* $\text{rad}_D(G)$ of a connected graph G is then defined as

$$\text{rad}_D(G) = \min\{e_D(v) \ : \ v \in V(G)\};$$

while the *detour diameter* $\text{diam}_D(G)$ of G is

$$\text{diam}_D(G) = \max\{e_D(v) \ : \ v \in V(G)\}.$$

Since $d(x,y) \leq D(x,y)$ for every two vertices x and y in a connected graph G, it follows that $e(v) \leq e_D(v)$ for every vertex v in a connected graph G. Therefore,

$$\text{rad}(G) \leq \text{rad}_D(G) \quad \text{and} \quad \text{diam}(G) \leq \text{diam}_D(G)$$

for every connected graph G. Because the detour distance between two vertices u and v in a tree is the same as the ordinary distance between u and v, it follows that $\text{rad}(T) = \text{rad}_D(T)$ and $\text{diam}(T) = \text{diam}_D(T)$ for every tree T. In any connected graph, the detour radius and detour diameter are related by the following inequalities.

Theorem 2.2 *For every connected graph G,*

$$\text{rad}_D(G) \leq \text{diam}_D(G) \leq 2\text{rad}_D(G).$$

Proof. The definitions of $\text{rad}_D(G)$ and $\text{diam}_D(G)$ give the inequality

$$\text{rad}_D(G) \leq \text{diam}_D(G).$$

Now let u and v be two vertices of G such that $D(u,v) = \text{diam}_D(G)$ and let w be a vertex of G such that $e_D(w) = \text{rad}_D(G)$. Since detour distance is a metric on $V(G)$, it follows that

$$\text{diam}_D(G) = D(u,v) \leq D(u,w) + D(w,v) \leq 2\text{rad}_D(G),$$

as desired. ∎

The following result provides the detour radius and detour diameter of some familiar graphs.

Proposition 2.3 *Let n, r, and s be integers.*

(*a*) *For $n \geq 2$, $\text{rad}_D(K_n) = \text{diam}_D(K_n) = n - 1$.*

(*b*) *For $n \geq 3$, $\text{rad}_D(C_n) = \text{diam}_D(C_n) = n - 1$.*

(*c*) *For $n \geq 2$, $\text{rad}_D(Q_n) = \text{diam}_D(Q_n) = 2^n - 1$.*

(*d*) *For $2 \leq s \leq t$, $\text{rad}_D(K_{s,t}) = 2s - 1$ and*

$$\text{diam}_D(K_{s,t}) = \begin{cases} 2s-1 & \text{if } s = t \\ 2s & \text{if } s < t. \end{cases}$$

Proposition 2.3(a)-(c) (and (d) for $s = t$) illustrates the fact that for a Hamiltonian graph G of order n, $\text{rad}_D(G) = \text{diam}_D(G) = n - 1$. Every pair a, b of positive integers can be realized as the detour radius and detour diameter, respectively, of some connected graph provided $a \le b \le 2a$.

Theorem 2.4 *For each pair a, b of positive integers with $a \le b \le 2a$, there exists a connected graph G with $\text{rad}_D(G) = a$ and $\text{diam}_D(G) = b$.*

Proof. For $a = b = k \ge 1$, the complete graph K_{k+1} has the desired property. For $a < b \le 2a$, let G be the graph of order $b + 1$ obtained by identifying a vertex v of K_{a+1} and a vertex of K_{b-a+1}. Since $b \le 2a$, it follows that $b - a + 1 \le a + 1$. Thus $e_D(v) = a$. Since there is a Hamiltonian path in G with initial vertex x for every vertex $x \in V(G) - \{v\}$, it follows that $e_D(x) = b$. Hence $\text{rad}_D(G) = a$ and $\text{diam}_D(G) = b$. ∎

For integers a and b with $a < b \le 2a$, each vertex in the graph G in the proof of Theorem 2.4 has detour eccentricity a or b. So unlike standard eccentricity, if k is an integer such that $\text{rad}_D(G) < k < \text{diam}_D(G)$, there may not be a vertex x of G such that $e_D(x) = k$. Next we show that every pair a, b of integers with $1 \le a \le b$ is realizable as the radius (diameter) and detour radius (detour diameter) of some connected graph.

Theorem 2.5 *For every pair a, b of integers with $1 \le a \le b$,*

(*a*) *there is a connected graph F such that*

$$\text{rad}(F) = a \text{ and } \text{rad}_D(F) = b,$$

(*b*) *there is a connected graph H such that*

$$\text{diam}(H) = a \text{ and } \text{diam}_D(H) = b.$$

Proof. If T is a tree, then $\text{rad}(T) = \text{rad}_D(T)$ and $\text{diam}(T) = \text{diam}_D(T)$. Hence $\text{rad}(P_{2a+1}) = \text{rad}_D(P_{2a+1}) = a$ and $\text{diam}(P_{2a+1}) = \text{diam}_D(P_{2a+1}) = a$. So the result is true for $a = b$.

Thus we may assume that $1 \le a < b$. We first verify (a). Let $F_1 : u_1, u_2, \cdots, u_a$ and $F_2 : v_1, v_2, \cdots, v_a$ be two copies of the path P_a of order a and $F_3 = K_{b-a+1}$ be the complete graph of order $b - a + 1$. Let F be the graph obtained from F_i $(1 \le i \le 3)$ by joining every vertex in F_3 to both u_1 and v_1 in F_1 and F_2, respectively. Then $\text{rad}(F) = a$. Since $e_D(v) = b$ if $v \in V(F_3)$ and $e_D(v) > b$ if $v \in V(F) - V(F_3)$, it follows that $\text{rad}_D(F) = b$ and so (a) holds.

To verify (b), let H be the graph obtained from the path $P_a : v_1, v_2, \cdots, v_a$ and the complete graph K_{b-a+1} by joining v_1 to every vertex in

K_{b-a+1}. Then $e(v) = a$ for each $v \in V(K_{b-a+1}) \cup \{v_a\}$ and $e(v) < a$ for each $v \in V(P_a) - \{v_a\}$. Thus $\text{diam}(H) = a$. On the other hand, $e_D(v_a) = (b-a)+a = b$ and $e_D(v) < b$ for all $v \in V(H) - \{v_a\}$. Therefore, $\text{diam}_D(H) = b$ and so (b) holds. ∎

3 Detour Center

The *center* $C(G)$ of a connected graph G is the subgraph of G induced by those vertices of G having eccentricity $\text{rad}(G)$; while the *periphery* $P(G)$ of G is the subgraph of G induced by the vertices of G having eccentricity $\text{diam}(G)$. A vertex v in a connected graph G is called a *detour central vertex* if $e_D(v) = \text{rad}_D(G)$; while the subgraph induced by the detour central vertices of G is the *detour center* $C_D(G)$ of G. A vertex v in a connected graph G is called a *detour peripheral vertex* if $e_D(v) = \text{diam}_D(G)$ and the subgraph induced by the detour peripheral vertices of G is the *detour periphery* $P_D(G)$ of G. The following observation is useful.

Observation 3.1 *No cut-vertex in a connected graph G is a detour peripheral vertex of G.*

Harary and Norman [10] proved, for standard distance in graphs, that the center of every connected graph G lies in a single block of G. This is true for detour distance as well.

Proposition 3.2 *The detour center $C_D(G)$ of every connected graph G lies in a single block of G.*

Proof. Assume, to the contrary, that there is a connected graph G whose detour center does not lie in a single block of G. Then G contains a cut-vertex v such that $G - v$ has components G_1 and G_2, each of which contains vertices of $C_D(G)$. Let u be a vertex of G such that $e_D(v) = D(u, v)$ and let P_1 be a $u - v$ detour in G. At least one of G_1 and G_2, say G_2, contains no vertex of P_1. Let w be a vertex of $C_D(G)$ belonging to G_2 and let P_2 be a $w - v$ path in G. The paths P_1 and P_2 together form a $u - w$ path P_3 in G. Then

$$e_D(w) \geq |V(P_3)| - 1 > |V(P_1)| - 1 = e_D(v),$$

which is a contradiction. ∎

Hedetniemi (see [2]) showed that every graph is the center of some connected graph. We next show that this is true for detour centers as well.

Theorem 3.3 *Every graph is the detour center of some connected graph.*

Proof. Let G be a graph of order n and let $H = G + \overline{K}_{n+1}$ be the join of G and $\overline{K}_{n+1}$. Since $e_D(v) = 2n - 1$ if $v \in V(G)$ and $e_D(v) = 2n$ if $v \in V(\overline{K}_{n+1})$, it follows that G is the detour center of H. ∎

A connected graph G is called *detour self-centered* if

$$\text{rad}_D(G) = \text{diam}_D(G),$$

that is, if G is its own detour center. For example, if $G = K_n$ or $G = C_n$, then $\text{rad}_D(G) = \text{diam}_D(G) = n - 1$ and so G is detour self-centered. We made the following observation earlier.

Observation 3.4 *If G is a Hamiltonian graph of order n, then G is detour self-centered having* $\text{rad}_D(G) = \text{diam}_D(G) = n - 1$.

A graph need not be Hamiltonian to be detour self-centered, however. For example, the Petersen graph is a non-Hamiltonian detour self-centered graph. By Observation 3.1, we do have the following, however.

Lemma 3.5 *If G is a detour self-centered graph of order* 3 *or more, then G is* 2*-connected.*

The length of a longest cycle in a connected graph is called the *circumference* of G and is denoted by $\text{cir}(G)$. If G is a tree, then we write $\text{cir}(G) = 0$. If G is not a tree, then $\text{cir}(G) \geq 3$.

Lemma 3.6 *If G is a connected non-Hamiltonian graph, then*

$$\text{diam}_D(G) \geq \text{cir}(G).$$

Proof. The result is certainly true if G is a tree. Thus, we may assume that G is not a tree and let $C : v_1, v_2, \cdots, v_k, v_1$ be a longest cycle in G, where $\text{cir}(G) = k$. Since G is not Hamiltonian and G is connected, there exists $v \in V(G) - V(C)$ such that v is adjacent to some vertex on C, say $vv_1 \in E(G)$. Then $v, v_1, v_2, \cdots, v_k$ is a $v - v_k$ path of length k and so $e_D(v) \geq k$. ∎

Next, we show that the detour eccentricity of a vertex in a detour self-centered graph of sufficiently large order cannot be extremely small.

Theorem 3.7 *Let G be a connected graph of order* 6 *or more. If G is detour self-centered, then $e_D(v) \geq 5$ for every vertex v in G.*

Proof. Assume, to the contrary, that there is a detour self-centered graph G of order $n \geq 6$ for which $e_D(v) = k \leq 4$ for every vertex v in G. By Lemma 3.5, G is 2-connected and so contains cycles. Moreover, G is not

Hamiltonian by Observation 3.4. By Lemma 3.6, $\mathrm{diam}_D(G) \geq \mathrm{cir}(G) \geq 3$. Therefore, $k = 3$ or $k = 4$. We consider these two cases.

Case 1. $k = 3$. Let $P : u = v_0, v_1, v_2, v_3 = v$ be a $u - v$ detour in G. Since the order of G is at least 6 and G is 2-connected, there exists $x \in V(G)$ such that x is adjacent to some vertex of P. Since P is a longest path in D, it follows that (1) x is adjacent neither u nor v and (2) x is not adjacent to both v_1 and v_2. Thus x is adjacent to exactly one vertex in P and this vertex is either v_1 or v_2, say x is adjacent to v_1. By Lemma 3.5, x must be adjacent to a vertex y that is not on P. However then y, x, v_1, v_2, v is a path of length 4, which is a contradiction.

Case 2. $k = 4$. Let $P : u = v_0, v_1, v_2, v_3, v_4 = v$ be a $u - v$ detour in G and let

$$X = \{x \in V(G) : \ x \text{ is adjacent to some vertex on } P\}.$$

Then $X \neq \emptyset$. Since P is a longest path in G, no vertex in X is adjacent to u or to v and no vertex of X is adjacent to consecutive vertices on P. Thus each vertex in X is adjacent to at least one of v_1, v_2, v_3. If no vertex in X is adjacent to v_1 or v_3, then every vertex in X is adjacent to exactly one vertex of P, namely v_2. However then, v_2 is a cut-vertex, which is impossible by Lemma 3.5. Therefore, at least one vertex $x \in X$ is adjacent to v_1 or v_3, say the former. Then x is not adjacent to v_2. If x is also not adjacent to v_3, then x must be adjacent to some vertex $y \notin V(P)$ by Lemma 3.5. However then $y, x, v_1, v_2, v_3, v_4 = u$ is a path of length 5, a contradiction. Thus we may assume that $xv_3 \in E(G)$. Note that u is not adjacent to any vertex in $V(G) - V(P)$. Moreover, u is not adjacent to v by Lemma 3.6. It then follows by Lemma 3.5 that u is adjacent to v_2 or v_3.

If $uv_2 \in E(G)$, then v, v_3, x, v_1, u, v_2 is a path of length 5, which is a contradiction. Therefore, $uv_3 \in E(G)$. Similarly, $vv_1 \in E(G)$. Thus G contains $K_{2,4}$ as an induced subgraph. Let k be the largest positive integer such that $K_{2,k}$ is an induced subgraph of G. Since G is detour self-centered and $K_{2,n-2}$ is not, $G \neq K_{2,n-2}$ and so $k < n - 2$. Note that no vertex in $V(G) - V(K_{2,k})$ can be adjacent to a vertex of degree 2 in $K_{2,k}$. Thus there exists $y \in V(G) - V(K_{2,k})$ such that y is adjacent one of the two vertices of degree k in $K_{2,k}$. Since k is the largest positive integer such that $K_{2,k}$ is an induced subgraph of G, it follows that y is not adjacent to the other vertex of degree k in $K_{2,k}$. By Lemma 3.5, y must be adjacent to some vertex z in G that is not in $K_{2,k}$. However then $e_D(z) > k$, which is a contradiction. ∎

We anticipate that there is a considerably stronger result than Theorem 3.7, however.

Conjecture 3.8 *If G is a detour self-centered graph of order n, then $e_D(v) = n - 1$ for every vertex v of G.*

4 Detour Periphery

Bielak and Syslo [1] showed that a nontrivial graph G is the periphery of some connected graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1. This suggests a natural question: Which graphs are the detour periphery of some connected graph? There is one obvious class of graphs with this property.

Observation 4.1 *If G is a detour self-centered graph, then G is its own detour periphery.*

Of course, detour self-centered graphs are connected. This suggests another question.

Problem 4.2 *Is there an example of a connected graph G that is not detour self-centered such that G is the detour periphery of some graph?*

For connected graphs having radius 1, the answer to the question in Problem 4.2 is no.

Theorem 4.3 *A connected graph G of order $n \geq 3$ and radius 1 is the detour periphery of some connected graph if and only if G is Hamiltonian.*

Proof. If G is Hamiltonian, then G is its own detour periphery by Observation 3.4. For the converse, assume, to the contrary, that there exists a connected graph G of order $n \geq 3$ and radius 1 that is not Hamiltonian such that G is the detour periphery of some connected graph H. Let v be a vertex in G such that $e(v) = 1$. Since v is a detour peripheral vertex of H, it follows that $D(v, w) = \text{diam}_D(H)$ for some $w \in V(G)$. Let P be a $v - w$ detour in H. Since v is adjacent to every vertex in G, it follows that P contains all vertices of G. However then, P together with the edge vw forms a cycle C in H and every vertex of C is then a detour peripheral vertex of H. So C is a subgraph of $P_D(H)$. Since no vertex of $H - V(G)$ is a detour peripheral vertex of H, it follows that $V(C) = V(G)$ and so C is a Hamiltonian cycle of G. This, however, contradicts the fact that G is not Hamiltonian. ■

Of course, if the detour eccentricity of every vertex in a graph G of order n is $n - 1$, then G is detour self-centered. Then every vertex of G is the initial vertex of a Hamiltonian path in G. A graph G is called *vertex-traceable* if every vertex of G is the initial vertex of a Hamiltonian path of G. Although every Hamiltonian graph is vertex-traceable, the converse is not true. For example, the Petersen graph is vertex-traceable.

Theorem 4.4 *If G is a graph in which every component of G is vertex-traceable, then G is the detour periphery of some connected graph.*

Proof. Let G be a graph with components $G_1, G_2, \cdots, G_k$, each of which is vertex-traceable. If G is connected, then G is detour self-centered and so $P_D(G) = G$. Hence we may assume that G is disconnected. We construct a connected graph H such that $P_D(H) = G$. Let $K_{1,k}$ be a star with $V(K_{1,k}) = \{u, u_1, u_2, \cdots, u_k\}$, where u is the central vertex of $K_{1,k}$. We first construct a graph F from G and $K_{1,k}$ by joining u_i to all vertices of G_i for $1 \leq i \leq k$. Then the graph H is obtained from F by subdividing the edges uu_i $(1 \leq i \leq k)$ in such a way that the k components of $H - u$ have the same order. The graph H is shown in Figure 3 for $k = 3$.

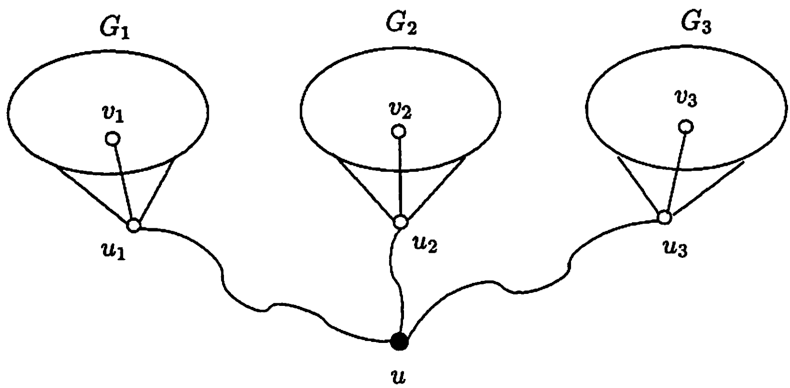

Figure 3: A graph H in the proof of Theorem 4.4 for $k = 3$

Since each vertex in $V(H) - V(G)$ is a cut-vertex of H, it follows by Observation 3.1 that no vertex in $V(H) - V(G)$ is a detour peripheral vertex of H. On the other hand, suppose that each component of $H - u$ has order n. Then $e_D(x) = 2n$ for each $x \in V(G)$ and so every vertex in G is a detour peripheral vertex of H. Since G is an induced subgraph of H, it follows that $P_D(H) = G$. ∎

We know of no counterexample to the converse of Theorem 4.4. For graphs G of small order, however, the condition presented in Theorem 4.4 is both necessary and sufficient for G to be the detour periphery of some graph.

Proposition 4.5 *A graph G of order n, where $2 \leq n \leq 4$, is the detour periphery of some connected graph if and only if every component of G is vertex-traceable.*

By Theorem 4.3, no star of order 3 or more is the the detour periphery of a connected graph. We now show that no double star is the detour periphery of any connected graph either. In order to do this, we first present a lemma.

Lemma 4.6 *Let H be a connected graph of order at least 3. If u and v are adjacent vertices of H with $D(u,v) = \text{diam}_D(H)$, then the detour periphery of H contains a cycle of order $1 + \text{diam}_D(H)$.*

Proof. A $u - v$ detour P together with the edge uv forms a cycle of order $1 + \text{diam}_D(H)$ in H. Since every vertex of C is a detour peripheral vertex of H, it follows that C is a subgraph of $P_D(H)$. ∎

Proposition 4.7 *No double star is the detour periphery of any connected graph.*

Proof. Let $S_{a,b}$ be a double star with central vertices u and v such that $\deg u = a + 1$ and $\deg v = b + 1$, where $a \geq b \geq 1$. Suppose that $N(u) = \{u_1, u_2, \cdots, u_a, v\}$ and $N(v) = \{v_1, v_2, \cdots, v_b, u\}$. The graph $S_{a,b}$ is shown in Figure 4.

$S_{a,b}$:

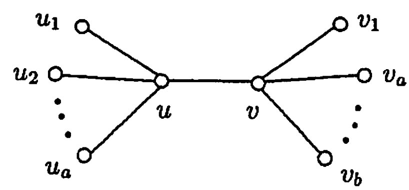

Figure 4: The double star $S_{a,b}$

Assume, to the contrary, that $S_{a,b}$ is the detour periphery of some connected graph H. By Lemma 4.6, $D(u,v) \neq \text{diam}_D(H)$ for otherwise, $P_D(H)$ contains a cycle. Hence $D(u, v_i) = \text{diam}_D(H)$ for some vertex v_i with $1 \leq i \leq b$, say $D(u, v_1) = \text{diam}_D(H)$. Let P be a $u - v_1$ detour in H. Then $N(u) \subseteq V(P)$. Suppose that s is an ordering of the vertices of $N(u)$ that appears on P. Assume, without loss of generality, that s is one of the three sequences:

$$s_1 : u_1, u_2, \cdots, u_p, v, u_{p+1}, u_{p+2}, \cdots, u_a \ (1 \leq p < a),$$
$$s_2 : v, u_1, u_2, \cdots, u_a,$$
$$s_3 : u_1, u_2, \cdots, u_a, v.$$

We consider these three cases. Let $k = \text{diam}_D(H)$ and $V = \{v, v_1, v_2, \cdots, v_b\}$.

Case 1. $s = s_1$. Suppose that

$$P : u = x_0, \cdots, u_p, \cdots, v, \cdots, x_{i-1}, x_i = u_{p+1}, \cdots, x_k = v_1.$$

Since $x_{i-1}u_{p+1} \in E(H)$ and $yu_{p+1} \notin E(G)$ for each $y \in V$, it follows that $x_{i-1} \notin V(S_{a,b})$. However then

$$P' : x_{i-1}, x_{i-2}, \cdots, x_0 = u, x_i = u_{p+1}, x_{i+1}, \cdots, x_k = v_1.$$

is an $x_{i-1} - v_1$ path of length k. This implies that x_{i-1} is a detour peripheral vertex of H, a contradiction.

Case 2. $s = s_2$. Suppose that

$$P : u = x_0, \cdots, v, \cdots, x_{i-1}, x_i = u_1, x_{i+1}, \cdots, x_k = v_1.$$

Since $x_{i-1}u_1 \in E(H)$ and $yu_1 \notin E(G)$ for each $y \in V$, it follows that $x_{i-1} \notin V(S_{a,b})$. However then

$$P' : x_{i-1}, x_{i-2}, \cdots, x_0 = u, x_i = u_1, x_{i+1}, \cdots, x_k = v_1.$$

is an $x_{i-1} - v_1$ path of length k. This implies that x_{i-1} is a detour peripheral vertex of H, a contradiction.

Case 3. $s = s_3$. Suppose that

$$P : u = x_0, x_1, \cdots, x_{i-1}, x_i = u_a, x_{i+1}, \cdots, x_{j-1}, x_j = v, x_{j+1}, \cdots, x_k = v_1.$$

If $a = 1$, then $b = 1$ and $S_{a,b} = S_{1,1} = P_4 : u_1, u, v, v_1$ is a path of order 4. Since $u_1v \notin E(H)$, it follows that $x_{j-1} \neq u_1$ and so $x_{j-1} \notin V(S_{a,b})$. However then

$$P' : x_{j-1}, x_{j-2}, \cdots, x_0 = u, x_j = v, x_{j+1}, \cdots, x_k = v_1.$$

is an $x_{j-1} - v_1$ path of length k. This implies that x_{j-1} is a detour peripheral vertex of H, a contradiction. If $a \geq 2$, then $x_{i-1} \notin V(S_{a,b})$ since $x_{i-1}u_a \in E(H)$. However then,

$$P' : x_{i-1}, x_{i-2}, \cdots, x_0 = u, x_i = u_a, x_{i+1}, \cdots, x_k = v_1.$$

is an $x_{i-1} - v_1$ path of length k. This implies that x_{i-1} is a detour peripheral vertex of H, a contradiction. ■

As a consequence of Proposition 4.7, P_4 is not the detour periphery of any graph. This can be extended to P_5.

Proposition 4.8 *The path P_5 is not the detour periphery of any connected graph.*

Proof. Assume, to the contrary, that $P_5 : u, v, w, x, y$ is the detour periphery of some connected graph H. By Lemma 4.6, $D(w, u) = \text{diam}_D(H)$ or $D(w, y) = \text{diam}_D(H)$, say the former. Let $\text{diam}_D(H) = k$ and

$$P : w = v_0, v_1, v_2, \cdots, v_k = u$$

be a $w - u$ detour in H. Then $v, x \in V(P)$. We consider two cases.

Case 1. $y \notin V(P)$. Suppose that $\{v, x\} = \{v_i, v_j\}$, where $0 < i < j < k$. Since v and x are not adjacent in P_5, it follows that $j \geq i + 2$ and so $v_{j-1} \notin V(P_5)$. Since

$$v_{j-1}, v_{j-2}, \cdots, v_0 = w, v_j, v_{j+1}, \cdots, v_k = u,$$

is a $v_{j-1} - u$ path of length k. This implies that v_{j-1} is a peripheral vertex of H, a contradiction.

Case 2. $y \in V(P)$. Suppose that $\{v, x, y\} = \{v_r, v_s, v_t\}$, where $0 < r < s < t < k$. If $y = v_t$, then $\{v, x\} = \{v_r, v_s\}$. An argument similar to that in Case 1 shows that $v_{s-1} \notin V(P_5)$ and v_{s-1} is a peripheral vertex of H, a contradiction. If $y = v_r$, then then $\{v, x\} = \{v_s, v_t\}$. Similarly, $v_{t-1} \notin V(P_5)$ and v_{t-1} is a peripheral vertex of H, a contradiction. Therefore, $y = v_s$. If $v = v_t$, then $v_{t-1} \neq x, y$ since v is not adjacent to y in P_5. Thus $v_{t-1} \notin V(P_5)$ and v_{t-1} is a peripheral vertex of H, a contradiction. Hence $v = v_r$, $y = v_s$, and $x = v_t$. Since v is not adjacent to y in P_5, it follows that $v_{r+1} \notin V(P)$. However then,

$$v_{r+1}, v_{r+2}, \cdots, v_k = u, v_r = v, v_{r-1}, \cdots, v_0 = w,$$

is a $v_{r+1} - w$ path of length k. This implies that v_{r+1} is a peripheral vertex of H, a contradiction. ■

We now show that another class of trees cannot be the detour periphery of any graph.

Theorem 4.9 *If T is a tree of order $n \geq 3$ with $\Delta(T) \geq n/2$, then T is not the detour periphery of any connected graph.*

Proof. Assume, to the contrary, that there is a tree T of order $n \geq 3$ with $\Delta(T) \geq n/2$ such that T is the detour periphery of some connected graph H. Let $v \in V(T)$ such that $\deg v = \Delta(T) = k \geq n/2$. Then $n \leq 2k$. Let $N(v) = \{v_1, v_2, \cdots, v_k\}$. By Lemma 4.6, $D(v, v_i) \neq \text{diam}_D(H)$ for each v_i, for otherwise, $P_D(H)$ contains a cycle. It follows that $D(v, u) = \text{diam}_D(H) = d$ for some $u \in V(T) - N[v]$. Let P be a $v - u$ detour in H. Then $N[v] \subseteq V(P)$. Assume, without loss of generality, that

$$P : v = x_0, \cdots, x_{i_1 - 1}, x_{i_1} = v_1, \cdots, x_{i_2 - 1}, x_{i_2} = v_2, \cdots,$$
$$x_{i_k - 1}, x_{i_k} = v_k, \cdots, x_d = u$$

where $i_1 < i_2 < \cdots < i_k$. Since T is a tree, $N(v)$ is an independent set of vertices in H. Thus $i_j - i_{j-1} \geq 2$ for $2 \leq j \leq k$. Let

$$W = \{x_{i_j - 1} : \ 2 \leq j \leq k\}.$$

Then $|W| = k-1$ and $W \subseteq V(H) - (N(v) \cup \{u,v\})$. Since

$$\begin{aligned}|V(T) - (N(v) \cup \{u,v\})| &= n-(k+2) \le 2k-(k+2) \\ &= k-2 < |W|,\end{aligned}$$

there exists $w \in W$ such that $w \notin V(T)$, say $w = x_{i_j-1}$, where $2 \le j \le k$. However then,

$$P' : x_{i_j-1}, x_{i_j-2}, \cdots, v, x_{i_j} = v_j, x_{i_j+1}, \cdots, u$$

is an $x_{i_j-1} - u$ path of length d and so x_{i_j-1} is a detour peripheral vertex of H, a contradiction. ∎

For a tree T of order $n \ge 3$, let

$$S_T = \{v \in V(T) : \deg v \ge 2\} \text{ and } \sigma_T = \sum_{v \in S_T} (\deg v - 2).$$

Lemma 4.10 *Let T be a tree of order $n \ge 3$. Then*

$$\operatorname{diam}(T) \le n - \sigma_T - 1.$$

Proof. We proceed by induction, the result being true if $n = 3$. Assume that the inequality holds for all trees of order $n-1 \ge 3$. Let T be a tree of order n, let v be a peripheral vertex of T, and let $T' = T - v$. Then $\operatorname{diam}(T) \le \operatorname{diam}(T') + 1$ and by the induction hypothesis,

$$\operatorname{diam}(T') \le (n-2) - \sigma_{T'} = n - \sigma_{T'} - 2.$$

Observe that either $\sigma_T = \sigma_{T'}$ or $\sigma_T = \sigma_{T'} + 1$, according to whether v is adjacent to a vertex of degree 2 in T or adjacent to a vertex of degree 3 or more in T.

If $\sigma_T = \sigma_{T'}$, then

$$\operatorname{diam}(T) \le \operatorname{diam}(T') + 1 \le n - \sigma_{T'} - 2 + 1 = n - \sigma_T - 1.$$

So we may assume that $\sigma_T = \sigma_{T'} + 1$ and so v is adjacent to a vertex v' of degree 3 or more. We show, in this case, that $\operatorname{diam}(T) = \operatorname{diam}(T')$. Let P be a $u - v$ path of length $\operatorname{diam} T$ in T. Let v'' be a vertex of T that is adjacent to v' and is not on P. Then the path obtained from P by replacing v by v'' is a $u - v''$ path of length $\operatorname{diam}(T)$. Hence $\operatorname{diam}(T) = \operatorname{diam}(T')$. Therefore,

$$\begin{aligned}\operatorname{diam}(T) &= \operatorname{diam}(T') \le n - \sigma_{T'} - 2 \\ &= n - (\sigma_{T'} + 1) - 1 = n - \sigma_T - 1,\end{aligned}$$

as desired. ∎

Using an argument similar to that employed in the proof of Lemma 4.10, we have the following.

Lemma 4.11 *Let T' be a tree of order n' and diameter d' containing exactly one vertex v with $\deg_{T'} v \geq 3$. If T is a tree of order n containing T' as a subtree, then*

$$\operatorname{diam}(T) \leq n - n' + d' + \deg_{T'} v - 2 - \sigma_T.$$

Theorem 4.12 *Let T be a tree of order $n \geq 3$. If*

$$\Delta(T) + \operatorname{diam}(T) + \sigma_T \geq n + 3,$$

then T is not the detour periphery of any connected graph.

Proof. Assume, to the contrary, that there exists a tree T of order $n \geq 3$ for which

$$\Delta(T) + \operatorname{diam}(T) + \sigma_T \geq n + 3$$

and a connected graph H such that $P_D(H) = T$. Thus T is not a path and so $\Delta(T) \geq 3$.

Let v be a vertex of T such that

$$\deg_T v = \Delta(T) = k \geq n + 3 - \operatorname{diam}(T) - \sigma_T.$$

Let $N(v) = \{v_1, v_2, \cdots, v_k\}$. Observe that $D_H(v, v_i) \neq \operatorname{diam}_D(H)$ for each v_i, for otherwise, by Lemma 4.6, $P_D(H)$ contains a cycle. Consequently, $D_H(v, u) = \operatorname{diam}_D(H) = d$ for some $u \in V(T) - N[v]$. Let P be a $v - u$ detour in H. Since the length of P is $d = \operatorname{diam}_D(H)$, it follows that $N[v] \subseteq V(P)$. We may assume that the vertices of $N(v)$ are labeled so that

$$\begin{aligned} P : v = x_0, \cdots, x_{i_1 - 1}, x_{i_1} = v_1, \cdots, x_{i_2 - 1}, x_{i_2} = v_2, \cdots, \\ x_{i_k - 1}, x_{i_k} = v_k, \cdots, x_d = u, \end{aligned}$$

where then $i_1 < i_2 < \cdots < i_k$. Since T is a tree, $N(v)$ is an independent set of vertices in T and therefore in H. Thus $i_j - i_{j-1} \geq 2$ for $2 \leq j \leq k$. Let

$$W = \{x_{i_j - 1} : \ 2 \leq j \leq k\}.$$

Then $|W| = k - 1$ and $W \cap (N(v) \cup \{u, v\}) = \emptyset$. We now consider two cases.

Case 1. *There exists $w \in W$ such that $w \notin V(T)$, say $w = x_{i_j - 1}$, where* $2 \leq j \leq k$. Then

$$P' : x_{i_j - 1}, x_{i_j - 2}, \cdots, v, x_{i_j} = v_j, x_{i_j + 1}, \cdots, u$$

is an $x_{i_j - 1} - u$ path of length d, which implies that $x_{i_j - 1}$ is a detour peripheral vertex of H. This, however, produces a contradiction since $x_{i_j - 1} \notin V(T)$.

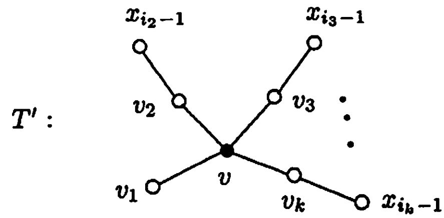

Figure 5: The subtree T' in Case 2

Case 2. Every vertex of W belongs to T. Hence T contains a subtree T' of order $2k$ and diameter 4 shown in Figure 5.

By Lemma 4.11,

$$\mathrm{diam}(T) \leq (n-2k) + 4 + (k - 2 - \sigma_T) = n - k - \sigma_T + 2.$$

Since $\mathrm{diam}(T) \geq n - k - \sigma_T + 3$, a contradiction is produced. ∎

A *caterpillar* is a tree the removal of whose end-vertices produces a path. In the case of caterpillars, Theorem 4.12 has the following consequence.

Corollary 4.13 *If T is a caterpillar of order $n \geq 3\,\mathrm{diam}(T) - 1$, then T is not the detour periphery of any connected graph.*

Theorem 4.12 also has the following corollary.

Corollary 4.14 *Let T be a tree of order $n \geq 3$. If*

$$2\Delta(T) + \mathrm{diam}(T) \geq n + 5,$$

then T is not the detour periphery of any connected graph.

A result that is nearly analogous to Theorem 4.9 holds for graphs that are not necessarily trees.

Proposition 4.15 *Let G be a graph of order $n \geq 3$. If G contains a vertex u such that $\deg u \geq (n+1)/2$ and the neighborhood $N(u)$ of u is an independent set in G, then G is not the detour periphery of any connected graph.*

Proof. Assume, to the contrary, that G is the detour periphery of some connected graph H. Suppose that $\deg u = k$, where $N_G(u) = \{v_1, v_2, \cdots, v_k\}$. Then $n \leq 2k - 1$. Let $v \in V(G)$ such that $D(u,v) = \mathrm{diam}_D(H) = d$. Let P be a $u - v$ detour in H. Then $N_H(u) \subseteq V(P)$. Assume, without loss of generality, that

$$P : u = x_0, \cdots, x_{i_1-1}, x_{i_1} = v_1, \cdots, x_{i_2-1}, x_{i_2} = v_2, \cdots,$$
$$x_{i_k-1}, x_{i_k} = v_k, \cdots, x_d = v.$$

Since $N_G(u)$ is an independent set of vertices in G as well as in H, it follows that $i_j - i_{j-1} \geq 2$ for $2 \leq j \leq k$. Let $W = \{x_{i_j-1} : 2 \leq j \leq k\}$. Then $W \cap N_G(u) = \emptyset$, $|W| = k-1$, and $W \subseteq V(H) - (N_G(u) \cup \{u, v\})$. Since $|N_G(u) \cup \{u,v\}| \geq k+1$, it follows that

$$\begin{aligned} |V(G) - (N_G(u) \cup \{u,v\})| &\leq n - (k+1) \leq (2k-1) - (k+1) \\ &= k - 2 < |W|. \end{aligned}$$

Hence there exists $w \in W$ such that $w \notin V(G)$, which implies that $w \in V(H) - V(G)$. If $uv \in E(G)$, then the path P together with uv forms a cycle of length $d+1$ that contains w. This implies that w lies on a cycle of order $d+1$. Hence $e_D(w) = d$ and so w is a detour peripheral vertex of H, a contradiction. Thus, $uv \notin E(G)$ and so $v_k \neq v$. Then $w = x_{i_j-1}$ for some j, where $2 \leq j \leq k$. However then,

$$P' : x_{i_j-1}, x_{i_j-2}, \cdots, u, x_{i_j} = v_j, x_{i_j+1}, \cdots, v$$

is an $x_{i_j-1} - v$ path of length d, which implies that $w = x_{i_j-1}$ is a detour peripheral vertex of H, a contradiction. ∎

Corollary 4.16 *If G is a bipartite graph of order $n \geq 3$ with $\Delta(G) \geq (n+1)/2$, then G is not the detour periphery of any connected graph. In particular, if $G = K_{s,t}$, where $s \neq t$, then G is not the detour periphery of any connected graph.*

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References

[1] H. Bielak and M.M. Syslo, Peripheral vertices in graphs. *Studia Sci. Math. Hungar.* **18** (1983) 269-75.

[2] F. Buckley, Z. Miller, and P. J. Slater, On graphs containing a given graph as center. *J. Graph Theory* 5 (1981) 427-434.

[3] G. Chartrand, G. L. Johns, and P. Zhang, The detour number of a graph. *Util. Math.* **64** (2003) 97-113.

[4] G. Chartrand, G. L. Johns, and P. Zhang, On the detour number and geodetic number of a graph. *Ars Combin.* To appear.

[5] G. Chartrand, T. W. Haynes, M. A. Henning, and P. Zhang, Detour domination in graphs. *Ars Combin.* To appear.

[6] G. Chartrand, T. W. Haynes, M. A. Henning, and P. Zhang, Hamiltonian domination in graphs. *Util. Math.* To appear.

[7] G. Chartrand, L. Nebeský, and P. Zhang, Greedy F-colorings of graphs. *Discrete Math.* **272** (2003) 37-46.

[8] G. Chartrand, L. Nebeský, and P. Zhang, Hamiltonian colorings of graphs. Preprint.

[9] G. Chartrand, L. Nebeský, and P. Zhang, On hamiltonian colorings of graphs. Preprint.

[10] F. Harary and R. Z. Norman, The dissimilarity characteristic of Husimi trees. *Ann. of Math.* **58** (1953) 134-141.

[11] S. F. Kapoor, H. V. Kronk, and D. R. Lick, On detours in graphs. *Canad. Math. Bull.* **11** (1968) 195-201.

[12] L. Nebeský, Hamiltonian colorings of graphs without long cycles. *Math. Bohem.* **128** (2003) 263-275.

[13] L. Nebeský, The hamiltonian chromatic number of a connected graph without large hamiltonian-connected subgraphs. *Czech. Math. J.* Preprint.