Large sets of cycle systems on nine points

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Abstract

An m-cycle system of order v, denoted by $m\mathrm{CS}(v)$, is a decomposition of the complete graph K_v into m-cycles. We discuss two types of large sets of $m\mathrm{CS}(v)$ and construct examples of both types for $(m, v) = (4, 9)$ and one type for $(m, v) = (6, 9)$. These are the first large sets of cycle systems constructed with $m > 3$, apart from the Hamiltonian cycle decompositions given in [2].
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1 Introduction

An *m-cycle system of order v*, denoted by $m\mathrm{CS}(v)$, is a decomposition of the complete graph K_v into m-cycles (i.e. cycles of length m). Necessary conditions for the existence of such a decomposition are that v is odd and that m divides $\binom{v}{2}$. When such a decomposition exists, there also arises the possibility of partitioning the set of all m-cycles on a common set of v points into disjoint $m\mathrm{CS}(v)$s. By "disjoint" we mean that the $m\mathrm{CS}(v)$s have no common cycles. Such a partition is known as a *large set* of $m\mathrm{CS}(v)$s. For

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certain parameter sets (m, v), a variant of this partitioning problem is to seek a set of disjoint mCS(v)s whose m-cycles, when viewed as copies of K_m, cover exactly once all possible copies of K_m on a set of v points. In order to distinguish between these two partitioning problems, we will refer to the first as a *C_m-large set of mCS(v)s* and to the second as a *K_m-large set of mCS(v)s*.

In the case $m = 3$, the complete graph K_m is also an m-cycle and so there is no distinction between the two problems. A 3CS(v) is a Steiner triple system of order v, usually denoted by STS(v), and it is well-known that such systems exist if and only if $v \equiv 1$ or 3 (mod 6). It was shown by Lu and Teirlinck [8, 9, 10, 12] that a large set of STS(v)s exists for all $v \equiv 1$ or 3 (mod 6), apart from $v = 7$. In general, for t-designs of index 1, a partition of the set of all k-tuples taken from a set of v points into t-$(v, k, 1)$ designs is known as a large set and is denoted by LS(t, k, v). Apart from Steiner triple systems, i.e. LS$(2, 3, v)$, very little is known. In fact, for $t \geq 2$ and $k \geq 4$, the only known large sets are LS(2, 4, 13) [3] and LS(2, 4, 16) [11]. The problem of constructing large sets of t-designs of index 1 seems to be difficult.

Returning to the case of m-cycle systems with $m > 3$, the only known result concerning large sets seems to be that given in [2], namely that for every odd $v \geq 3$, there exists a C_v-large set of vCS(v)s. Here $m = v$ and so the systems are actually Hamiltonian cycle decompositions of K_v. The paper [2] also gives results concerning large sets of Hamiltonian path decompositions of K_v. Apart from these Hamiltonian cycle decompositions and the results for Steiner triple systems, nothing else appears to be known concerning large sets of mCS(v)s. Again, the problem of constructing large sets appears to be difficult.

For $v < 9$, all mCS(v)s have $m = 3$ or $m = v$. In the case $v = 9$, as well as 3CS(9)s and 9CS(9)s, we also have 4CS(9)s and 6CS(9)s. So $v = 9$ is the smallest v for which the existence problem for large sets of mCS(v)s is unresolved. In this paper, we give a systematic account of the situation for $v = 9$. In particular we construct C_4- and K_4-large sets of 4CS(9)s and a K_6-large set of 6CS(9)s.

2 Cycle systems of order 9

For $m = 5, 7$ or 8, m does not divide $\binom{9}{2}$ and so no mCS(9) exists. For $m = 3$, as noted above, an mCS(9) is an STS(9) and the C_3- and K_3-large sets problems coincide. It was originally shown by Kirkman [6] that the $\binom{9}{3} = 84$ triples on 9 points may be partitioned into 7 copies of the unique (up to isomorphism) STS(9). Bays [1] proved that there are precisely two nonisomorphic large sets of STS(9)s. For $m = 9$, it is shown in [2] that the

8!/2 9-cycles on 9 points may be partitioned into 7! copies of a Hamiltonian cycle decomposition of K_9. In fact this decomposition is given by the following four 9-cycles

$$(\infty, 0+i, 7+i, 1+i, 6+i, 2+i, 5+i, 3+i, 4+i) \pmod 8 \quad \text{for } i = 0, 1, 2, 3,$$

and a C_9-large set of 9CS(9)s is obtained from this by applying all permutations of the 9 points which fix ∞ and 0. Clearly, no K_9-large set is feasible because any single 9-cycle, when viewed as a K_9, covers the complete graph on 9 points, and therefore any individual 9CS(9) already covers (in this sense) K_9 fourfold. Excluding the trivial values $m = 1$ and $m = 2$, the remaining values to be investigated are $m = 4$ and $m = 6$.

3 Construction of C_4-large sets of 4CS(9)s

The construction is based on the unique cyclic 4CS(9)= C, say, which may be represented on the points $0, 1, \ldots, 8$ by developing the starter $(0, 1, 5, 3)$ cyclically modulo 9 [4]. However, the point set for our large sets will be taken as $V = \{0, 1, \ldots, 6, A, B\} = Z_7 \cup \{A, B\}$, and we denote by G the cyclic group of order 7 with generator $g = (0\ 1\ 2\ 3\ 4\ 5\ 6)(A)(B)$.

The initial step in the construction is to find a copy, say C_1, of C on V whose nine 4-cycles lie in nine distinct G-orbits. This may be visualised by placing the points of V in some permuted order $(x_0, x_1, \ldots, x_8)$ at the vertices of a 9-gon, then taking a 4-cycle with edge "lengths" 1, 4, 2 and 3 corresponding to the cyclic starter given above, and rotating this through its nine possible positions to generate C_1, and finally checking that the resulting nine 4-cycles lie in distinct G-orbits (see Figure 1).

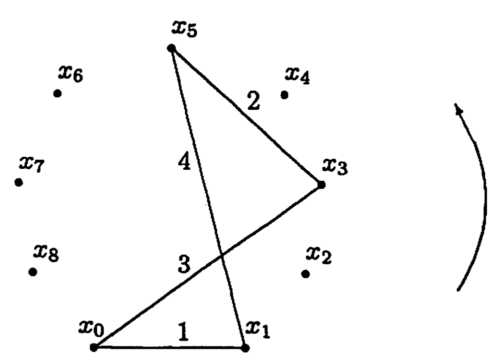

Figure 1: Generating a copy of a cyclic 4CS(9).

If a permutation p of the points of V does indeed lead to nine distinct G-orbits from the 4-cycles of the associated system C_1, then we say that p

is *suitable*. We may then form six further copies of C_1, say $C_2, C_3, \ldots, C_7$, as images of C_1 under g^i for $i = 1, 2, \ldots, 6$ respectively. In such a case, the seven systems C_i will form seven disjoint sets of 4CS(9)s, together making a total of 63 distinct 4-cycles.

The second step in the construction is to note that there are 54 distinct G-orbits of 4-cycles and so, if a collection of six suitable permutations can be found which cover all of these, then the corresponding collection of 4-cycles will form a large set of $6 \times 7 = 42$ copies of C covering all $6 \times 63 = 378$ 4-cycles on the nine points of V.

A small simplification in the search for suitable permutations p is that we may assume that $p(0) = 0$. This may be seen by considering the rotational symmetry of Figure 1. Computer enumeration gives precisely 3348 suitable permutations having $p(0) = 0$, and an exhaustive search then gives 76 distinct solutions to the C_4-large sets problem, each of which is obtained from six such permutations. It remains to determine possible isomorphisms between these solutions, and the automorphism group of each solution. This is done as follows.

Given a C_4-large set $\mathcal{L}$ corresponding to six suitable permutations $p_1, p_2, \ldots, p_6$, there will be six sets each of seven disjoint copies of C which we will denote by $\{C_1, C_2, \ldots, C_7\}, \{C_8, C_9, \ldots, C_{14}\}, \ldots, \{C_{36}, C_{37}, \ldots, C_{42}\}$. These 42 systems cover all $42 \times 9 = 378$ 4-cycles on the points of V; we may write $\mathcal{L} = \{C_1, C_2, \ldots, C_{42}\}$.

Now suppose that the permutation p_1 assigns the elements of V in the cyclic order $(x_0, x_1, \ldots, x_8)$ to the nine vertices of Figure 1, so that C_1 contains (amongst others) the 4-cycles $(x_0, x_1, x_5, x_3), (x_0, x_5, x_4, x_7), (x_0, x_4, x_2, x_8)$ and (x_1, x_2, x_6, x_4). Note that these four 4-cycles span all nine points of V. Next consider a C_4-large set $\mathcal{L}' = \{C'_1, C'_2, \ldots, C'_{42}\}$ which is formed under G in a similar manner to $\mathcal{L}$ by permutations $p'_1, p'_2, \ldots, p'_6$. If there is an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$ then, without loss of generality, by considering the action of G, we may assume that there is a mapping taking the four 4-cycles of C_1 described above to four 4-cycles in one of the six systems $C'_1, C'_8, C'_{15}, C'_{22}, C'_{29}, C'_{36}$. To check whether such a mapping exists, and if it does to determine it completely, we must take each of these six systems, and assign one of its nine 4-cycles as the potential image of (x_0, x_1, x_5, x_3), an assignment which may be made in eight distinct ways corresponding to the four vertices and two directions of this cycle. Having thus selected potential images for x_0, x_1, x_5 and x_3, the remaining images are determined (if possible) by considering the other three specified cycles of C_1.

Thus to determine isomorphisms, if any, from $\mathcal{L}$ to $\mathcal{L}'$ we need consider at most $6 \times 9 \times 8$ permutations q of the points of V, and for each of these to check whether it preserves the structure of $\mathcal{L}$, i.e. whether $q(C_i) \in \mathcal{L}'$ for $i = 1, 2, \ldots, 42$. In the case $\mathcal{L} = \mathcal{L}'$, the same procedure will determine automorphisms.

The result of this analysis is that the 76 large sets partition into 11 isomorphism classes. Representatives of the 11 classes are given below by specifying in each case the six cyclic orders in which the points of V are assigned to the vertices of Figure 1. Of these, four have just G as their full automorphism group. The remaining seven have the group $H = \langle z \rightarrow a^2z + b,\ a, b, z \in Z_7,\ a \neq 0;\ A \rightarrow A;\ B \rightarrow B \rangle$ as their full automorphism group.

Number and automorphism group		six cyclic orders of vertices					
1.	G	01235B6A4	01463BA52	016AB2543	0165324BA	01A6B5243	016245A3B
2.	G	0124B563A	01523A6B4	042A65B31	01B2436A5	01463AB52	01325BA64
3.	G	012A5346B	032B5416A	03BA26541	0215463AB	031652AB4	01B426A53
4.	G	012A6354B	014362AB5	016B52A43	012456B3A	0241B6A35	0265BA314
5.	H	012A63B45	0152A46B3	0153A4B62	01B35246A	014A6B325	021A5B643
6.	H	012A6B354	014263A5B	0156432AB	024A5B631	013642AB5	026514AB3
7.	H	012AB3645	0152AB643	024A615B3	012A346B5	0152A634B	024AB6513
8.	H	012AB4365	043A265B1	024AB1653	0152AB364	016A453B2	025A136B4
9.	H	012B4A653	0265431BA	016425BA3	0263B4A51	0136254BA	0135B2A64
10.	H	014AB3652	021A6B543	0153A6B42	02153AB64	014A3B625	01465AB32
11.	H	01B43265A	016A35B42	015A364B2	025A63B14	01A436B25	02A165B43

4 Construction of K_4-large sets of 4CS(9)s

The method is similar to that of the previous section. However, each 4-cycle gives a K_4 and the aim is to cover each copy of K_4 on the nine points precisely once. There are $\binom{9}{4} = 126$ such copies of K_4 and so we seek sets of two suitable permutations. The meaning of "suitable" here is more restrictive than previously; we now require that the nine 4-cycle orbits under G are not only distinct, but that they give nine distinct K_4 orbits under G. Thus each suitable permutation gives 63 distinct copies of K_4, and again we may assume that each suitable permutation has $p(0) = 0$. Computer enumeration gives precisely 816 suitable permutations having $p(0) = 0$, and an exhaustive search then gives 96 distinct solutions to the K_4-large sets problem, each of which is obtained from two such permutations.

The isomorphism classes of these 96 solutions may be established in a similar manner to those of the C_4-large sets problem in the previous section. The automorphism groups are also determined similarly. The result of this analysis is that the 96 K_4-large sets partition into eight isomorphism classes. Representatives of the eight classes are given below by specifying in each case the two cyclic orders in which the points of V are assigned to the vertices of Figure 1. All eight have G as their full automorphism group.

Number and automorphism group		two cyclic orders of vertices	
1.	G	01235A6B4	04BA16532
2.	G	01246BA53	021B654A3
3.	G	0124BA635	015326B4A
4.	G	012546AB3	016B3A524
5.	G	0126A53B4	016B3A524
6.	G	012A654B3	016AB4253
7.	G	01325B6A4	01B24A365
8.	G	01346AB25	015326B4A

5 Construction of a K_6-large set of 6CS(9)s

The initial step is to take a large set of seven STS(9)s covering all $\binom{9}{3} = 84$ triples on a set of nine points, see for example [4], page 267. Each STS(9) is isomorphic to the following system which may be decomposed into four *parallel classes* each consisting of three triples covering all nine points. These parallel classes are shown as the four columns in Table 1.

$$\begin{array}{cccc} \{0,1,2\} & \{0,3,6\} & \{0,4,8\} & \{0,5,7\} \\ \{3,4,5\} & \{1,4,7\} & \{1,5,6\} & \{1,3,8\} \\ \{6,7,8\} & \{2,5,8\} & \{2,3,7\} & \{2,4,6\} \end{array}$$

Table 1: STS(9) in four parallel classes.

For each of the seven STS(9)s forming the large set, combine the parallel classes in pairs, say the first with the second and the third with the fourth, giving 14 sets each of six triples. Next take the complement of each triple so that we now have 14 sets, each of six 6-tuples. Regarded as copies of K_6, these 84 6-tuples are distinct because the original 84 triples were distinct. A K_6-large set of 6CS(9)s will result if each 6-tuple can be replaced by a 6-cycle on the same six points in such a manner that each of the 14 sets of six 6-tuples is transformed into a 6CS(9). To do this it is sufficient to show how the replacement may be effected for the two sets of six 6-tuples arising from the STS(9) of Table 1. This is because the STS(9) is unique up to isomorphism. In fact, the automorphism group of the STS(9) is sufficiently rich to ensure that any pair of parallel classes is isomorphic to the remaining pair. Consequently, we only need to demonstrate the replacement operation for the set of six 6-tuples formed from the first two columns of Table 1. This may be achieved by the following 6CS(9):

$$\begin{array}{ccc} (0,3,2,5,4,1), & (3,6,5,8,7,4), & (6,0,8,2,1,7), \\ (0,4,6,1,3,7), & (1,5,7,2,4,8), & (2,6,8,3,5,0). \end{array}$$

Thus we form a K_6-large set of 6CS(9)s.

The success of this method naturally raises the question of whether a similar approach might lead to the construction of further K_4-large sets of 4CS(9)s. Unfortunately, the answer is in the negative. There are two known large sets of 2-(9,4,3) designs, [7]. However, in neither case can the seven designs in the large set each be partitioned into two collections of nine 4-tuples which can be replaced by 4-cycles on the same points to form a 4CS(9).

6 Concerning C_6-large sets of 6CS(9)s

We have tried a number of approaches both theoretical and computational. There are 640 nonisomorphic 6CS(9)s and these are given in [5]. The largest automorphism group of any of these is of order 36, but an exhaustive computer search showed that there is no C_6-large set formed from copies of the unique system having an automorphism group of this order. Searches using copies of a single system having an automorphism group of order 6 (the next largest) and copies of a single system having an automorphism group of order 1 were also tried. However the search spaces were too large to complete an exhaustive search. Randomised packing of 6-cycles into disjoint 6CS(9)s achieved a maximum total of 780 out of the 840 systems required for a C_6-large set, but the search space here was even larger and, perhaps consequently, we were unable to improve on this. Nevertheless, that such a total can be achieved strongly suggests that a C_6-large set of 6CS(9)s does exist.

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