The Effects of Vertex and Edge Deletion on Graphs

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Abstract

Many different approaches exist in studying graphs with high connectivity and small diameter. We consider the effect of deleting vertices and edges from a graph while maintaining a small diameter. The following property is introduced: A graph G has property $B_{d,i,j}$ if and only if after the removal of at most i vertices and at most j edges, the resulting graph has diameter at most d and is not the trivial graph on one vertex. The central theme of this paper is to investigate the structure of graphs that have property $B_{d,i,j}$ and to investigate the structure that is needed to imply that a graph has property $B_{d,i,j}$. Lower bounds on minimum degree and connectivity that imply property $B_{d,i,j}$ for specific values of d are found. These bounds are also shown to be sharp in all but one case.

1 Introduction and Preliminaries

In any communication network, it is imperative that any two nodes are able to communicate with each other. Thus, the underlying graph G of the network is connected. For reliability or routing purposes, we wish to have multiple independent paths (vertex or edge disjoint) between every pair of vertices in the graph. This implies that the graph has high connectivity. Menger's Theorem [11] answers the questions dealing with the existence of such independent paths. In addition to high connectivity, it is also

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important to keep these independent paths short in length due to speed considerations. So, we also require that the diameter of the graph be small. Note that Menger's Theorem does not help us here in that it does not tell us anything about the lengths of our paths. Many different approaches exist concerning the study of graphs with small diameter. We mention only a few here. Readers who wish to view a more comprehensive overview of graph theoretic problems related to the design of communication networks are encouraged to consult Bermond, Homobono, and Peyrat [1]. Chung and Garey [5] studied the effects of deleting k edges from a $(k+1)$ edge-connected graph G with respect to the diameter of the altered graph. They found upper bounds on the diameter of the altered graph as a function of k and the diameter of G. They also found an upper bound for the analogous vertex deletion problem.

Now, we suppose processors or links fail or are sabotaged. One approach to ensure a correct message is received quickly is to require that there are many short paths between each pair of vertices in the original graph. This motivated the concept of a *Menger Path System* which was introduced by Ordman [12] in 1987. For positive integers d and m, let $P_{d,m}$ denote the graph property that between each pair of vertices of G, there are at least m internally vertex disjoint paths of length at most d. Faudree, Jacobson, Ordman, Schelp, and Tuza [8] investigated minimal conditions involving various combinations of the connectivity, minimum degree, edge density, and size of G that are sufficient to insure that property $P_{d,m}$ is satisfied. Lower bounds for other Hamiltonian-type conditions that imply that G has property $P_{d,m}$ were investigated by Faudree, Gould, and Schelp [7]; Faudree, Gould, and Lesniak [6]; and Faudree and Tuza [9]. The computer science community is particularly interested in the fault tolerance (node failure) and maximum transmission delay of networks. Menger Path Systems have been applied in this context and are typically referred to as *containers* of width m and length d. Hsu [10] compiled a survey of connectivity and diameter results as they relate to computer networks. Included in this survey are results involving containers of width m and length d.

Requiring at least m paths of length at most d between each pair of vertices in G creates more structure in G than what may be needed. A weaker graph property was then considered. A graph G has property $D_{d,m}$ if and only if after the deletion of any set of $m-1$ vertices, the remaining graph has diameter at most d. It is easy to see that $P_{d,m}$ implies $D_{d,m}$. An edge analogue to property $D_{d,m}$ also exists. Questions related to $D_{d,m}$ and its edge analogue have been studied by Bond and Peyrat [2]; Chung and Garey [5]; Schoone, Bodlaender, and van Leeuwen [13]; and Chung [4]. Up to this point, researchers have studied the diameter of graphs after the deletion of only vertices or edges. In this paper, we consider the effects of deleting both vertices and edges. We say a graph G has property $B_{d,i,j}$

if and only if after the deletion of at most i vertices and at most j edges from G, the remaining graph has diameter at most d and is not the trivial graph on one vertex. We examine lower bounds for minimum degree and connectivity to imply that a graph G has property $B_{d,i,j}$ for specific values of d. All but one of these bounds are also shown to be sharp.

Throughout this paper, we only consider finite graphs without loops or multiple edges. The notation generally follows that found in [3]. We use $P_i(m_1, m_2, \ldots, m_i)$ to denote the blown-up path $C_{m_1} + C_{m_2} + \cdots + C_{m_i}$. A blown-up cycle is defined analogously and is denoted $C_i(m_1, m_2, \ldots, m_i)$. From this point on, when we say vertex (edge) disjoint x-y paths, we mean internally vertex (edge) disjoint x-y paths.

2 Main Results

This section focuses on finding lower bounds on $\delta(G)$ and $\kappa(G)$ that insure a graph G has property $B_{d,i,j}$. Throughout this section, we assume that n, the order of a graph G, is sufficiently large. Before we begin our formal discussion, we make some observations about a graph G that has property $B_{d,i,j}$ in relation to $\delta(G)$ and $\kappa(G)$. For all values of d, i, and j, the existence of property $B_{d,i,j}$ implies that $\delta(G) \geq i + j + 1$; otherwise, we delete vertices adjacent to and edges incident to a vertex of minimum degree so as to disconnect the graph or make it trivial.

For the lower bound of $\kappa(G)$ for a graph G with property $B_{d,i,j}$, the value of d is influencial. We consider the separate cases in which $d = 2$, $d = 3$, and $d \geq 4$ as individual propositions.

Proposition 1. *If G is a graph with property $B_{2,i,j}$, then $\kappa(G) \geq i+j+1$.*

Proof: Suppose that G is a graph with property $B_{2,i,j}$, and assume that $\kappa(G) = m \leq i+j$. Since $\delta(G) \geq i+j+1$, $|V(G)| \geq m+2$. If $m \leq i$, we easily obtain a contradiction. Thus, we assume $m > i$, and let $\{v_1, \ldots, v_i, \ldots, v_m\}$ be a set of m vertices in G whose removal disconnects G. We choose the vertices x and y from different components in $G - \{v_1, \ldots, v_i, \ldots, v_m\}$. In G, we delete the vertices $v_1, \ldots, v_i$. The removal of any set of j edges ensures us that $d(x, y) = 2$ because G has property $B_{2,i,j}$. If we remove edges of the form xv_k for $k = i+1, i+2, \ldots, m$, we deleted at most j edges from $G - \{v_1, \ldots, v_i\}$. Our necessary x-y path of length 2 does not contain any of the vertices $v_1, \ldots, v_i, \ldots, v_m$ which gives us a contradiction. Thus, $\kappa(G) \geq i + j + 1$. □

If we repeat the above argument except we now delete all edges from x and y into $v_{i+1}, v_{i+2}, \ldots, v_m$, the lower bound on $\kappa(G)$ for a graph with property $B_{3,i,j}$ is proven.

Proposition 2. *If G is a graph with property $B_{3,i,j}$, then $\kappa(G) \geq i + \left\lfloor \frac{j}{2} \right\rfloor + 1$.*

We observe that both lower bounds are achievable. For $d = 2$, let G be the graph $K_{i+j+1} + (K_m \cup K_m)$ where m is large. After the deletion of any set of i vertices and j edges from G, every pair of vertices is adjacent to, in particular, at least one vertex from the K_{i+j+1}. Thus, G is a graph with property $B_{2,i,j}$, and $\kappa(G) = i + j + 1$. For $d = 3$, let G be the graph $K_{i+\lfloor \frac{j}{2} \rfloor+1} + (K_m \cup K_m)$ where m is large. Due to the structure of G, we only need to check that property $B_{3,i,j}$ is satisfied for pairs of vertices in distinct K_m's. For every pair of vertices x and y in distinct K_m's, at least one edge into $K_{i+\lfloor \frac{j}{2} \rfloor+1}$ remains after the deletion of any set of i vertices and j edges from G which allows us to find an x-y path of length at most 3 in the altered graph. Thus, G is a graph with property $B_{3,i,j}$, and $\kappa(G) = i + \left\lfloor \frac{j}{2} \right\rfloor + 1$.

For $d \geq 4$, we cannot say much about the lower bound on $\kappa(G)$ for those graphs G with property $B_{d,i,j}$ beyond what is obvious. If a graph G has $\kappa(G) \leq i$, and if we delete such a cut-set, it is clear that G cannot have property $B_{d,i,j}$. Thus, we have the following proposition.

Proposition 3. *If G is a graph with property $B_{d,i,j}$ where $d \geq 4$, then $\kappa(G) \geq i + 1$.*

If we consider the graph G in Figure 1, which is the blown-up path $P_d(m, i+1, m, m, \ldots, m)$ where m is large, the lower bound on $\kappa(G)$ when $d \geq 4$ is achievable.

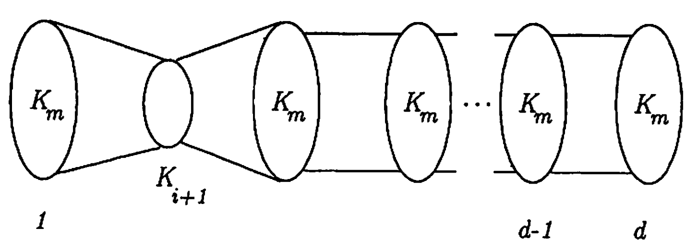

Figure 1: A graph with property $B_{d,i,j}$ and $\kappa(G) = i + 1$

Clearly, $\kappa(G) = i + 1$. If we delete any set of i vertices from G, at least one vertex from the K_{i+1} remains. Let the subgraph G_k correspond to the remains of the k^{th} complete graph in G after the vertex deletion. So G_2

corresponds to the remains of the K_{i+1}. To verify that G has property $B_{d,i,j}$, we only need to check that for any 2 vertices $x \in G_1$ and $y \in G_k$ for $k = 3, \ldots, d$, $d(x,y) \leq d$ after the deletion of any set of $j \geq 0$ edges from the altered graph. We first suppose that $x \in G_1$ and $y \in G_3$. Since G_1 and G_3 are still large, a neighbor of x and a neighbor of y are adjacent to the same vertex in G_2. Thus, $d(x,y) \leq 4$. Now, suppose that $x \in G_1$ and $y \in G_k$ where $4 \leq k \leq d$. Due to the size of G_1, there exists a G_1 neighbor of x that is adjacent to a vertex in G_2 after all edge deletions. We then easily find a path of length $k-2$ between this vertex in G_2 and y. Thus, we have an x-y path of length $k \leq d$ so G has property $B_{d,i,j}$ for $d \geq 4$.

2.1 Minimum degree results

Here, we examine lower bounds on $\delta(G)$ to imply that G has property $B_{d,i,j}$. We obtain different bounds depending upon the value of d. In each case the lower bound for $\delta(G)$ is sharp. We first consider the case when $d = 2$. Faudree, Jacobson, Ordman, Schelp, and Tuza in [8] showed that for $d \geq 2$, $\delta(G) \geq \lfloor \frac{n+m}{2} \rfloor$ implies G has property $P_{d,m}$. By letting $m = i+j+1$ and $d = 2$, we have that $\delta(G) \geq \left\lfloor \frac{n+i+j+1}{2} \right\rfloor$ implies that G has property $P_{2,i+j+1}$. By definition, property $P_{2,i+j+1}$ implies property $B_{2,i,j}$. Therefore, $\delta(G) \geq \left\lfloor \frac{n+i+j+1}{2} \right\rfloor$ implies that G has property $B_{2,i,j}$. Also in [8], Faudree, Jacobson, Ordman, Schelp, and Tuza used the graph $K_{i+j} + \left(K_{\lfloor \frac{n-i-j}{2} \rfloor} \cup K_{\lceil \frac{n-i-j}{2} \rceil}\right)$ to demonstrate that the minimum degree bound for property $P_{2,i+j+1}$ is sharp when $n-i-j$ is even. It is routine to check that $\delta\left(K_{i+j} + \left(K_{\lfloor \frac{n-i-j}{2} \rfloor} \cup K_{\lceil \frac{n-i-j}{2} \rceil}\right)\right) = \lfloor \frac{n+i+j}{2} \rfloor - 1$. For $n-i-j$ odd and $j > 0$, we offer the following graph to demonstrate that bound on $\delta(G)$ is sharp. Let G be the graph composed of two $K_{\frac{n+i+j-1}{2}}$'s that share $i+j-1$ vertices. We include a complete matching between the vertices of the complete graphs that are not shared. One can easily verify that $\delta(G) = \frac{n+i+j-1}{2}$. Let x be any vertex in G that is not contained in both complete graphs, and let y be the unique vertex in G that is adjacent to x along a matching edge. If we remove any set of i vertices from $N(x) \cap N(y)$, the matching edge xy, and, in the case that $j > 1$, the remaining $j-1$ edges in $N(x) \cap N(y)$ incident to x, then $d(x,y) \geq 3$.

Since property $B_{d,i,j}$ is a weaker property than property $P_{d,i+j+1}$ when $d \geq 3$, we expect our lower bound on $\delta(G)$ to decrease as well. Indeed, for $d = 3$ and 4, we are able to decrease our minimum degree requirement by $\frac{j}{4}$ each time although the proof when $d = 3$ becomes more complicated than the proofs of the other cases.

Theorem 4. *If* $\delta(G) \geq \frac{n+i+\frac{j}{2}}{2}$, *then for* n *sufficiently large,* G *has property* $B_{3,i,j}$.

Proof: We assume the result is not true and argue to a contradiction. So there exists a pair $x,y \in V(G)$ such that after the removal of some set of i vertices and j edges in G, $d(x,y) \geq 4$. We remove such a set of i vertices and call the altered graph G'. We note that G' contains $n-i$ vertices and $\delta(G') \geq \frac{n-i+\frac{j}{2}}{2}$. In G', let $I = N(x) \cap N(y)$. Since deg x, deg $y \geq \frac{n-i+\frac{j}{2}}{2}$, we have that $|I| \geq \frac{j}{2}$. Let $|I| = \frac{j}{2}+k$ where $k \geq 0$. Also, define $S_x = N(x) - I$ and $S_y = N(y) - I$.

Now, we first suppose $xy \in E(G')$. Since deg x, deg $y \geq \frac{n-i+\frac{j}{2}}{2}$, we have that

$$|S_x|, |S_y| \geq \frac{n-i+\frac{j}{2}}{2} - 1 - \left(\frac{j}{2}+k\right) = \frac{n-i-\frac{j}{2}}{2} - k - 1.$$

Let $|S_x| = \frac{n-i-\frac{j}{2}}{2} - k - 1 + \alpha$ and let $|S_y| = \frac{n-i-\frac{j}{2}}{2} - k - 1 + \beta$ for $\alpha, \beta \geq 0$. Counting the vertices in G', we see that

$$\begin{aligned} |V(G')| \geq & \left(\frac{n-i-\frac{j}{2}}{2} - k - 1 + \alpha\right) + \left(\frac{n-i-\frac{j}{2}}{2} - k - 1 + \beta\right) \\ & + \left(\frac{j}{2}+k\right) + 2 \\ = & \, n - i + \alpha + \beta - k. \end{aligned}$$

If $\alpha+\beta-k > 0$, we have a contradiction. Thus, we have $k-\alpha-\beta$ vertices remaining in G' that are not adjacent to x nor y where $0 \leq \alpha \leq k$ and $0 \leq \beta \leq k-\alpha$.

Now, we consider an arbitrary vertex $v \in S_x$ and its possible adjacencies in G. To avoid creating any additional short x-y paths, v can be adjacent to: all vertices in S_x; the vertex x; the i vertices that were removed; and the $k-\alpha-\beta$ vertices not adjacent to either x or y. This gives a total of

$$\left(\frac{n-i-\frac{j}{2}}{2} - k - 1 + \alpha - 1\right) + 1 + i + (k-\alpha-\beta) = \frac{n+i-\frac{j}{2}}{2} - \beta - 1.$$

Since $\delta(G') \geq \frac{n-i+\frac{j}{2}}{2}$, v must be adjacent to at least $\frac{j}{2}+\beta+1$ additional vertices in G'. Since $v \in S_x$, v is not adjacent to y; therefore, the additional adjacencies of v must come from $S_y \cup I$. Applying a similar argument, an arbitrary $u \in S_y$ must be adjacent to at least $\frac{j}{2} + \alpha + 1$ additional vertices in $S_x \cup I$.

Next, we wish to show that we have more edge disjoint x-y paths of length at most 3 than we can destroy due to the size of S_x and S_y and the number of additional adjacencies required for each vertex in these sets. We choose a maximum matching M between S_x and S_y. If $|M| + |I| \geq j$, we have at least $j+1$ edge disjoint x-y paths of length at most 3. Thus, we assume that $|M| + |I| < j$. Note, however, that $|M| + |I| \geq \frac{j}{2}$ since $|I| \geq \frac{j}{2}$. Now, we proceed to construct a matching of size at least $\frac{j}{2}$ between $S_x - V(M)$ and $(S_y \cap V(M)) \cup I$. First, each $v \in S_x - V(M)$ must send their required additional edges to vertices in $(S_y \cap V(M)) \cup I$; otherwise, we obtain a larger matching. We choose a $v_\ell \in S_x - V(M)$ and distinguish the edge $v_\ell u_\ell$ where u_ℓ is any neighbor of v_ℓ in $(S_y \cap V(M)) \cup I$. Next, we select a $v_m \in S_x - V(M)$ which is distinct from v_ℓ. Since v_m must send at least $\frac{j}{2} + \beta + 1$ edges into $(S_y \cap V(M)) \cup I$ and since $|M| + |I| \geq \frac{j}{2}$, there exists a $u_m \in (S_y \cap V(M)) \cup I$ such that u_m is adjacent to v_m and u_m has not yet been distinguished with some $u_\ell \in S_x - V(M)$. Since n is large, $|S_x - V(M)|$ is large, and so we can iterate this procedure at least $\frac{j}{2}$ times. Once this is done, the set of distinguished edges gives us a matching of size at least $\frac{j}{2}$ between $S_x - V(M)$ and $(S_y \cap V(M)) \cup I$. Repeating the same procedure, we also obtain a matching of size at least $\frac{j}{2}$ between $S_y - V(M)$ and $(S_x \cap V(M)) \cup I$. Recall that $xy \in E(G)$. Thus, we have at least $j+1$ edge disjoint x-y paths of length at most 3 which gives us a contradiction.

If $xy \notin E(G')$, then $|S_x|$ and $|S_y|$ each increase by one. Similar to the previous case, we take a maximum matching M between S_x and S_y. We now have $|M| + |I| \leq j$, or else we are finished. Once again, each $v \in S_x - V(M)$ and each $u \in S_y - V(M)$ must send out at least $\frac{j}{2} + 1$ additional edges. So as not to contradict the maximality of M, it must be the case that $|M| + |I| \geq \frac{j}{2} + 1$. Thus, we obtain matchings of size at least $\frac{j}{2} + 1$ between $S_x - V(M)$ and $(S_y \cap V(M)) \cup I$ and between $S_y - V(M)$ and $(S_x \cap V(M)) \cup I$ which gives us a contradiction similar to the previous case. Hence, G has property $B_{3,i,j}$. □

The bound in Theorem 4 is sharp as evidenced by the graph in Figure 2. The value of m in Figure 2 is $\dfrac{n - i - 2\lfloor \frac{j}{2} \rfloor + \lceil \frac{j}{2} \rceil}{2}$. To avoid clutter, we use a line in Figure 2 to indicate the inclusion of all possible edges between designated sets of vertices, and we mention that a set of i vertices is adjacent to every vertex of G. We leave it to the reader to verify that $\delta(G) =$

$\left\lceil \frac{n+i+\frac{j}{2}-1}{2} \right\rceil - 1$, regardless of the parity of $n+i+\frac{j}{2}$. If we remove the i vertices that are adjacent to every vertex; the $\left\lfloor \frac{j}{2} \right\rfloor$ edges from y to the $K_{\lfloor \frac{j}{2} \rfloor}$; and the $\left\lceil \frac{j}{2} \right\rceil$ edges from x to the $K_{\lceil \frac{j}{2} \rceil}$, then $d(x, y) = 4$.

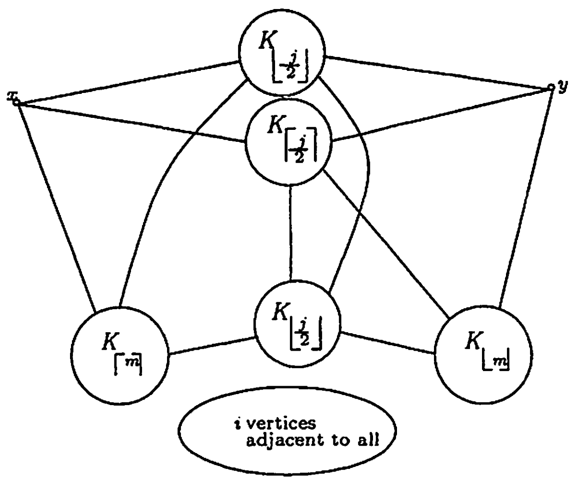

Figure 2: A graph G with $\delta(G) = \left\lceil \frac{n+i+\frac{j}{2}-1}{2} \right\rceil - 1$ but not property $B_{3,i,j}$

We now turn our attention to the lower bound on $\delta(G)$ when $d = 4$.

Theorem 5. *If $\delta(G) \geq \frac{n+i-1}{2}$, then for n sufficiently large, G has property $B_{4,i,j}$.*

Proof: As in the proof of the previous theorem, we assume the result is not true. So there exists a pair $x,y \in V(G)$ such that after the removal of some set of i vertices and j edges in G, $d(x,y) \geq 5$. Remove such a set of vertices and edges and call the altered graph G'. Note that $|V(G')| = n-i$. Suppose that α edges incident to x and β edges incident to y have been removed. Due to the cardinalities of $N_{G'}(x)$ and $N_{G'}(y)$ (i.e., $\frac{n-i-1}{2}$ is much greater than j because n is large compared to i and j.), there exists a vertex $x' \in N_{G'}(x)$ and a vertex $y' \in N_{G'}(y)$ such that $\deg_{G'}(x'), \deg_{G'}(y') \geq \frac{n-i-1}{2}$. If the neighborhoods $N_{G'}(x')$ and $N_{G'}(y')$ are not disjoint or contain x' or y', then we obtain an x-y path of length at most 4. Thus, $N_{G'}(x')$, $N_{G'}(y')$, and $\{x', y'\}$ are disjoint. Counting the vertices of G', we have $|V(G')| \geq 2(\frac{n-i-1}{2}) + 2 = n - i + 1$ which gives a contradiction. Thus, an

x-y path of length 4 exists in G', and therefore, G has property $B_{4,i,j}$. □

From our minimum degree results thus far, one might be inclined to conjecture that we may continue to decrease our minimum degree bound by a factor of $\frac{j}{4}$ for each incrimental increase in d. However, this is not true by considering the following graph. Let the graph G be $\left(K_{\lfloor \frac{n-i}{2} \rfloor} \cup K_{\lceil \frac{n-i}{2} \rceil}\right) + K_i$. Then, $\delta(G) = \left\lfloor \frac{n+i}{2} \right\rfloor - 1$, and if we remove the K_i and any j edges, the resulting graph is disconnected. Thus, any hope of reducing the minimum degree condition needed to imply $B_{d,i,j}$ for $d \geq 5$ must also incorporate a connectivity condition for $\kappa(G) \geq i + 1$. Also, the graph $\left(K_{\lfloor \frac{n-i}{2} \rfloor} \cup K_{\lceil \frac{n-i}{2} \rceil}\right) + K_i$ demonstrates that our minimum degree bound for $d = 4$ is sharp.

2.2 Connectivity results

Now, we turn our attention to lower bounds on $\kappa(G)$ to imply $B_{d,i,j}$ for specific values of d. For $d \geq 2$, Faudree, Jacobson, Ordman, Schelp, and Tuza [8] proved that for a graph of order n, $\kappa(G) \geq \frac{n-m}{d} + m$ implies that G has property $P_{d,m}$. In particular, letting $d = 2$ and $m = i + j + 1$, we have that $\kappa(G) \geq \frac{n+i+j+1}{2}$ implies that G has property $P_{2,i+j+1}$, and therefore, implies that G has property $B_{2,i,j}$. They also demonstrate the bound is sharp when $n + i + j$ is odd by considering the graph $K_{i+j-1} + C_4\left(\frac{n-i-j-1}{2}, \frac{n-i-j-1}{2}, 1, 1\right)$ which we denote by G. We leave it to the reader to verify that $\kappa(G) = \frac{n+i+j-1}{2}$ and that G fails to have property $B_{2,i,j}$. As in the $\delta(G)$ discussion in the previous section, the lower bound on $\kappa(G)$ to imply $B_{d,i,j}$ is smaller than that which imples $P_{d,i+j+1}$ for $d \geq 3$. The result for $d = 3$ follows.

Theorem 6. *If $\kappa(G) \geq \frac{n+2i+\frac{3}{2}j}{3}$, then for n sufficiently large, G has property $B_{3,i,j}$.*

Proof: Let G be a graph with $\kappa(G) \geq \frac{n+2i+\frac{3j}{2}}{3}$. We assume the result is not true and argue to obtain a contradiction. So there exists a set of i vertices and j edges in G such that upon their removal, the diameter of the altered graph is at least 4. We remove such a set of i vertices from G and call the altered graph G'. Then

$$\kappa(G') \geq \frac{n+2i+\frac{3j}{2}}{3} - i = \frac{n-i+\frac{3j}{2}}{3}.$$

Since, by assumption, G does not have property $B_{3,i,j}$, there exists a pair of vertices x and y in G' such that the removal of some set of j edges from G' leaves the distance between x and y to be at least 4. By Menger's Theorem, there exists at least $\frac{n-i+\frac{3j}{2}}{3}$ vertex disjoint x-y paths. No more than j of these paths may be of length at most 3; otherwise, we contradict our assumption. We select a path system of $p = \kappa(G') \geq \frac{n-i+\frac{3j}{2}}{3}$ vertex disjoint x-y paths such that the paths are of minimal length and the number of paths of length at most 3 is maximal. (So no chords exist along the paths.) Let p_k be the number of vertex disjoint x-y paths of length k in our path system for $k = 1, 2, 3$, and let p_4 be the number of x-y paths in our path system of length at least 4. As we previously mentioned, $p_1 + p_2 + p_3 \leq j$, and $p_4 = p - p_1 - p_2 - p_3$. Also, we suppose that the p_4 x-y paths of length at least 4 contain a total of $3p_4 + a$ internal vertices where $a \geq 0$.

Counting the vertices on the p paths in our path system in addition to the vertices x and y, we have that

$$\begin{aligned}|V(G')| &\geq p_2 + 2p_3 + 3p_4 + a + 2 = p_2 + 2p_3 + 3(p - p_1 - p_2 - p_3) + a + 2 \\ &\geq \left(n - i + \frac{3}{2}j\right) - 3p_1 - 2p_2 - p_3 + a + 2.\end{aligned}$$

If $n - i + \frac{3}{2}j - 3p_1 - 2p_2 - p_3 + a + 2 > n - i$, we have a contradiction. Thus, G' contains at most $3p_1 + 2p_2 + p_3 - \frac{3}{2}j - a - 2$ vertices outside of our path system.

Now, we utilize the connectivity of G' and the conditions imposed on the selection of our path system to obtain additional short x-y paths which will give us the contradiction we desire.

We define A to be the set of vertices in G' that are adjacent to y on the paths of length at least 4 in our path system. Note that $|A| = p_4 \geq \frac{n-i+\frac{3}{2}j}{3} > \frac{j}{2}$ because n is sufficiently large. Next, we observe that if we remove the following vertices from G': all vertices on the $p_4 = p - p_1 - p_2 - p_3$ vertex disjoint x-y paths of length at least 4 that are not adjacent to x or y; all vertices on the length 3 x-y paths that are adjacent to y; the additional vertices in G' not contained in our path system; and the vertex y, the total number of vertices removed from G' is at most

$$\begin{aligned}(p - p_1 - p_2 - p_3 + a) + p_3 + 3p_1 + 2p_2 + p_3 - \frac{3}{2}j - a - 2 + 1 \\ = p + 2p_1 + p_2 + p_3 - \frac{3}{2}j - 1.\end{aligned}$$

Since $p_1 + p_2 + p_3 \leq j$ and $p_1 \leq 1$, it follows that $2p_1 + p_2 + p_3 - \frac{3}{2}j - 1 < 0$. Thus, $p + 2p_1 + p_2 + p_3 - \frac{3}{2}j - 1 < \frac{n - i + \frac{3}{2}j}{3}$ so the remaining graph must still be connected. It follows that the vertices in A must be adjacent to at least $\frac{3}{2}j + 1 - 2p_1 - p_2 - p_3 > \frac{j}{2}$ distinct vertices from G' that were not deleted above; otherwise, we contradict the connectivity of G'.

We define X to be the set of vertices adjacent to x that are contained on paths of length 2 or 3 in our path system. Those vertices mentioned above that are adjacent to the vertices in A must come from X so as not to contradict the minimality of the paths chosen in our path system. Thus, $|X| > \frac{j}{2}$.

We now construct a matching between A and X. Let $v_k \in A$. The vertex v_k must be adjacent to more than $\frac{j}{2}$ vertices in X. From the neighbors of v_k in X, we distinguish the edge $v_k u_k$ for some $u_k \in X$. Because $|X| > \frac{j}{2}$, we have that for any other $v_m \in A$, we can select a $u_m \in X$ such that u_m is adjacent to v_m and u_m has yet to be distinguished to any other $v_k \in A$. Thus, we distinguish the edge $v_m u_m$ and continue until we run out of vertices in X. The cardinalities of A and X ensure that the set of distinguished edges form a matching of size greater than $\frac{j}{2}$ between A and X.

Similarly, we define B to be the set of vertices that are adjacent to x on our paths of length at least 4 in our path system. Since n is sufficiently large, $|B| = p_4 > \frac{j}{2}$. Let Y denote the set of vertices adjacent to y that are contained on paths of length 2 or 3 in our path system. By removing vertices from G' similar to what was previously done, the vertices in B are adjacent to more than $\frac{j}{2}$ distinct vertices from Y. Employing a similar argument to the one used for the sets A and X, we obtain a matching of size greater than $\frac{j}{2}$ between B and Y. Thus, we have more than j edge disjoint x-y paths of length at most 3 which gives us a contradiction. Hence, G has property $B_{3,i,j}$. □

If we consider the graph G appearing in Figure 3, we show that the bound on $\kappa(G)$ for $d = 3$ is sharp. As before, we use a line in Figure 3 to indicate the inclusion of all possible edges between designated sets of vertices, and we mention that a set of i vertices is adjacent to every vertex of G. Also, the value of m in the $K_{\lfloor m \rfloor}$'s is $\frac{n - i - 3\lfloor \frac{j-1}{2} \rfloor - 2}{3}$, and q equals either $\lceil m \rceil$ or $\lceil m \rceil + 1$ so that the sum of the vertices in G is n.

One cut-set in G of minimum cardinality contains all vertices incident to the vertex x. Thus, $\kappa(G) = \left\lfloor \frac{n - i - 3\lfloor \frac{j-1}{2} \rfloor - 2}{3} \right\rfloor + i + 1 + 2\left\lfloor \frac{j-1}{2} \right\rfloor$.

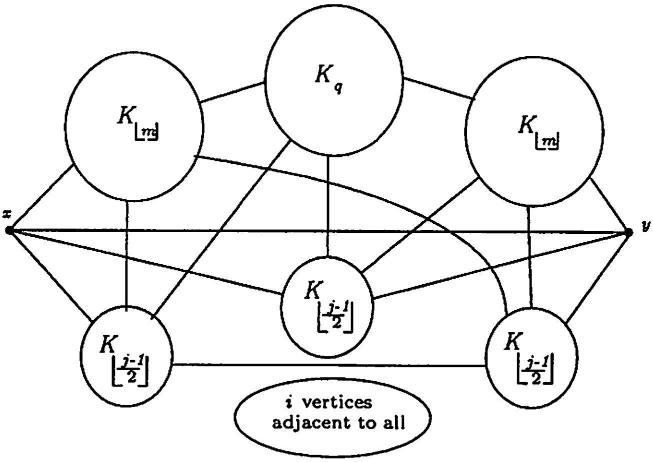

Figure 3: A graph G with $\kappa(G) < \frac{n+2i+\frac{3}{2}j}{3}$ but not property $B_{3,i,j}$

This sum equals $\left\lceil \frac{n+2i+\frac{3}{2}j-1}{3} \right\rceil - 1$ except when $n+2i \equiv 2 \pmod 3$ and j is odd. In the later case, $\kappa(G) = \left\lfloor \frac{n+2i+\frac{3}{2}j}{3} \right\rfloor$.

If we delete the K_i adjacent to all vertices in G, the edge xy, the edges between y and the right-most $K_{\lfloor \frac{j-i}{2} \rfloor}$ in Figure 3, and the edges between x and the middle $K_{\lfloor \frac{j-i}{2} \rfloor}$ in Figure 3, $d(x,y) = 4$. Thus, property $B_{3,i,j}$ is not satisfied.

For $d = 4$, we conjecture that $\kappa(G) \geq \frac{n+3i+\frac{3}{2}j}{4} + 1$ implies property $B_{4,i,j}$. Employing a proof technique similar to the one used for $d = 3$ was an accounting nightmare so we feel a different technique should be utilized to prove the $d = 4$ case.

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