On Inverse Distributions of e-Balanced Cayley Maps

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Abstract

At each vertex in a Cayley map the darts emanating from that vertex are labeled by a generating set of a group. This generating set is closed under inverses. Two classes of Cayley maps are balanced and antibalanced maps. For these cases the distributions of the inverses about the vertex are well understood. For a balanced Cayley map either all the generators are involutions or each generator is directly opposite across the vertex from its inverse. For an antibalanced Cayley map there is a line of reflection in the tangent plane of the vertex so that the inverse generator for each dart label is symmetric across that line. An e-balanced Cayley map is a recent generalization that has received much study, see for example [2, 6, 7, 13]. In this note we examine the symmetries of the inverse distributions of e-balanced maps in a manner analogous to those of balanced and antibalanced maps.

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1 Introduction and Definitions

Intuitively a map of a graph is a "drawing" of the graph onto an orientable closed surface so that edges intersect only at their vertices. In this paper we are concerned with a special type of map of a Cayley graph known as a Cayley map. In [11, 12] Širáň and Škoviera define and classify two types of regular Cayley maps: balanced and antibalanced. Balanced and antibalanced Cayley maps exhibit marked symmetry in the distribution of inverses at a vertex. A recent generalization of balanced and antibalanced maps is that of an e-balanced Cayley map [7]. Two recent papers on e-balanced Cayley maps are [2] and [6]. In this paper we investigate the symmetries inherent in the inverse distributions of e-balanced Cayley maps. Every Cayley map can be covered by a regular Cayley map [9]. Thus we may limit our investigation to regular e-balanced Cayley maps. A regular Cayley map is one that exhibits a high degree of symmetry (see below for the definition).

We follow the theoretical background and terminology of Nedela and Škoviera, set forth in [8]. A *graph* is a quadruple $K = (D, V; S, L)$, where the *dart set* $D = D(K)$ and the *vertex set* $V = V(K)$ are disjoint nonempty finite sets, $S : D \to V$ is a surjection that assigns to each dart its initial vertex, and L is a permutation on D of order 2 that determines the edges of the graph (dart δ and $L(\delta)$ together correspond to one and the same edge of the graph). Under this definition, there are three possible kinds of edges: links, loops, and semiedges. Links are incident with two distinct vertices ($SL(\delta) \neq S(\delta)$). Loops and semiedges are incident with a single vertex ($SL(\delta) = S(\delta)$); when $L(\delta) = \delta$, the dart δ corresponds to a semiedge, and when $L(\delta) \neq \delta$, the dart δ corresponds to a loop. In this paper we will only be concerned with simple graphs that have no loops, semiedges, or multiple edges between vertices.

An *oriented map* M is a 2-cell imbedding of a graph $(D, V; S, L)$ in an oriented surface; all maps in this paper will be oriented. Specifically, M is an ordered triple $(D; R, L)$, where D is the set of darts, L is the permutation of D with order 2 that determines the edges of the graph, and R is a permutation of D that specifies the cyclic ordering of darts at each initial vertex, i.e., $S(R(\delta)) = S(\delta)$, for all $\delta \in D$. We also call R the *dart rotation function* for map M. The cycles of R determine the vertices of the graph, and the cycles of RL constitute the region boundaries.

An *automorphism* of the map $M = (D; R, L)$ is a bijection $\Theta : D \to D$ that commutes with R and L, i.e., $\Theta R = R\Theta$ and $\Theta L = L\Theta$. By this definition, a map automorphism preserves a map's vertices, edges, and oriented boundary regions. The set of all map automorphisms of M forms the automorphism group of M, which is denoted by Aut M. Since an automorphism of M is uniquely determined by specifying the image of one dart, $|\text{Aut } M| \leq |D|$. When $|\text{Aut } M| = |D|$, the map M has the largest possible automorphism group and is called a *regular* map. Recently, certain classes of regular maps have become known [4, 7, 10, 12, 1] through the study of Cayley maps.

Let G be a finite group and let Δ be a generating set for G such that $1 \notin \Delta$ and Δ is closed under inverses. The *Cayley graph* $CG(G, \Delta)$ is a graph with the elements of G as its vertex set with an edge between vertices g_1 and g_2 if and only if $g_2^{-1}g_1 \in \Delta$. The dart set of the Cayley graph is $G \times \Delta$, the initial vertex function S is defined by $S(g, x) = g$, and the permutation L is defined by $L(g, x) = (gx, x^{-1})$, i.e., $CG(G, \Delta) = (G \times \Delta, G; S, L)$. Let $p : \Delta \to \Delta$ be a cyclic permutation. Then the *Cayley map* $M = CM(G, \Delta, p) = (G \times \Delta; R, L)$ is the 2-cell imbedding of the Cayley graph $CG(G, \Delta)$ such that $R(g, x) = (g, p(x))$, that is, the cyclic permutation of darts incident from each vertex is determined by p. Again, an *automorphism* of M is a bijection $\Theta : G \times \Delta \to G \times \Delta$ such that $\Theta R = R\Theta$ and $\Theta L = L\Theta$, and M is *regular* if $|\text{Aut } M| = |G||\Delta|$.

In a Cayley graph $CG(G, \Delta)$, every dart incident from vertex $g \in G$ is of the form (g, x), $x \in \Delta$; we call x a dart label. In this paper we describe how the inverse of dart label x is distributed relative to x about the imbedded vertex of the Cayley map in the plane tangent at the vertex to the surface of imbedding. This distribution is called the *inverse distribution*. Several authors [2, 6] in the current century have touched on this topic, because the inverse distribution of a regular Cayley map may be used in determining the automorphism group of the map.

This paper is divided into the following sections.

1. Introduction and Definitions (current section)
2. Inverse Distributions for Regular Cayley Maps
3. e-Balanced Cayley Maps of Prime-Power Degree
4. e-Balanced Cayley Maps of General Degree

2 Inverse Distributions for Regular Cayley Maps

A Cayley map $M = CM(G, \Delta, p)$ that satisfies the condition $p(x^{-1}) = p(x)^{-1}$ for all $x \in \Delta$ is said to be *balanced* [1]. An *antibalanced* Cayley map satisfies the condition $p(x^{-1}) = (p^{-1}(x))^{-1}$ [12]. In [7] Martino, Schultz, and Škoviera introduce the generalization of an *e-balanced* Cayley map, one that satisfies the condition $p(x^{-1}) = (p^e(x))^{-1}$ for all $x \in \Delta$. Thus a balanced Cayley map is a 1-balanced Cayley map, and an antibalanced Cayley map is a (-1)-balanced Cayley map. For an e-balanced Cayley map

$$e^2 = 1 \bmod k$$

where $k = |\Delta|$ is the degree of M.

If k is a power of a prime, a well-known result of classical number theory [3] gives us the solutions to this equation.

Theorem 2.1. *Let q be a prime and $n \geq 1$ an integer; let $k = q^n$. The solutions of $e^2 = 1$ mod k are:*

1. *if q is odd, then $e = \pm 1$ mod k*
2. *if $k = 2$, then $e = 1$ mod 2*
3. *if $k = 4$, then $e = \pm 1$ mod 4*
4. *if $q = 2$ and $n \geq 3$, then $e = \pm 1$ mod k or $e = \frac{k}{2} \pm 1$ mod k.*

If k has more than one prime divisor, then the Chinese Remainder Theorem may be used to find the number of solutions for e.

Let $M = CM(G, \Delta, p)$ be a regular Cayley map. Label the elements of Δ by subscripts i such that $\Delta = \{x_1, x_2, \ldots, x_k\}$ and $p(x_i) = x_{i+1}$. The *inverse distribution* τ of M is a permutation of order 2 on the set $\{1, 2, \ldots, k\}$ such that $x_i^{-1} = x_{\tau(i)}$. It is shown in [7] that Aut M, the automorphism group of Cayley map M, has a right group action on $\langle p \rangle = \{p^0, p^1, p^2, \ldots, p^{k-1}\}$. This action induces a group homomorphism Ψ : Aut $M \to S_k$, where S_k is the symmetric group on k letters. An equivalent action of the dart group is also defined in [9]. The image of Ψ is generated by τ and the k-cycle

$\eta = (1, 2, \ldots, k)$. The e-balanced condition then becomes a statement about how τ conjugates η, namely

$$\tau\eta\tau = \eta^e.$$

Líšková, Mačaj, and Škoviera [6] have completely characterized inverse distributions of e-balanced Cayley maps, as follows.

Theorem 2.2. *Let $M = CM(G, \Delta, p)$ be an e-balanced Cayley map where $k = |\Delta|$, and let $\tau_e : S_k \to S_k$ be defined as $\tau_e(i) = ei \bmod k$. The distribution of inverses $\tau \in S_k$ associated with M has one of the following forms:*

1. *$\tau = \tau_e$, or*
2. *when k is even, $2^n \| k$, and $e = \pm 1 \bmod 2^n$, then $\tau = \eta^d \tau_e$, where $d = gcd(e - 1, k)/2$.*

In case 1, $\tau(i) = ei$ and Δ contains $gcd(k, e - 1)$ involutions; in case 2, $\tau(i) = ei + d$ and Δ has no involutions.

The above results were also independently discovered by one of the authors of this paper, see [13]. The number of involutions in an e-balanced Cayley map has been calculated by Martino, Schultz and Škoviera in [7].

3 e-Balanced Cayley Maps of Prime-Power Degree

Let $M = CM(G, \Delta, p)$ be a regular Cayley map. In this section we show, when $k = |\Delta|$ is the power of a prime number, the symmetries inherent in the inverse distribution τ when M is e-balanced. These distributions fall into a limited number of cases, depending on the values of e and k. Some are already well known: when $e = 1$ or -1. We review those cases, and consider the cases when e is not 1 or -1. We provide illustrations of the cases, adopting the convention that a dart that represents an involution in Δ (an *involutory* dart) is depicted as a double line and a letter from the beginning of the alphabet, and a dart that is not its own inverse (a *non-involutory* dart) is represented by a single-line arrow and a letter from the end of the alphabet. Darts of the same pattern are inverses of each other, or a *non-involutory dart pair.* If non-involutory pairs reflect across a line in the tangent plane of the vertex, that line is depicted by a dotted line. Further, in some of the figures we adopt the convention of numbering the darts counter-clockwise from 1 to k with k in the vertical position.

By Theorem 2.1, when k is the power of an odd prime, e is either 1 or -1, and so the associated Cayley map must be either 1-balanced or (-1)-balanced. The following results are already known from [11] and [12].

Theorem 3.1. *Let M be a regular Cayley map $CM(G,\Delta,p)$. Let $k = |\Delta|$ and e be an element of $\mathbb{Z}_k$ such that $e^2 = 1 \mod k$. If M is e-balanced and if k is a power of an odd prime q, then, when $e = 1$, all darts incident from a vertex are involutions; and when $e = -1$, there is exactly one involutory dart a, while each non-involutory dart pair reflect across a line through a.*

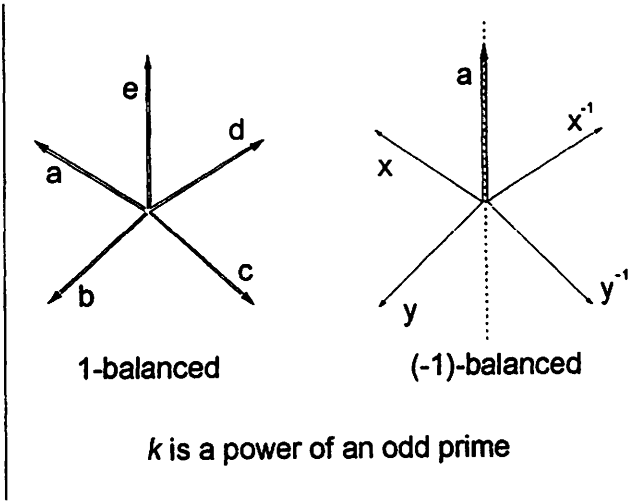


Figure 1:

By Theorem 2.2, if $e = 1$ (M is 1-balanced or "balanced"), the inverse distribution $\tau = (1,1)(2,2)\dots(k,k) = \iota$, the identity permutation of S_k, and Δ contains $\gcd(k,k) = k$ involutions; $p = (a_1, a_2, \dots, a_k)$. If $e = -1$ (M is (-1)-balanced or "antibalanced"), the inverse distribution $\tau = (1,-1)(2,-2)\dots(\frac{k-1}{2}, -\frac{k-1}{2})$ and Δ contains $\gcd(k,-2) = 1$ involution.

In this case, $p = (x_1, x_2, \ldots, x_l, a, x_l^{-1}, \ldots, x_2^{-1}, x_1^{-1})$ where $l = \frac{k-1}{2}$. Note that the line of reflection between darts and their inverses coincides with the single involution.

We now consider the case when k is a power of 2. For the first two cases, 2 and 4, e is either 1 or -1 and, as above, the results are known.

When $k = 2$, by Theorem 2.1, $e = 1$; hence, M is 1-balanced. But since 2 is even and divides itself, and $e = \pm 1 \bmod 2$, Theorem 2.2 provides two different possible inverse distributions: $\tau = \iota$, the identity permutation of S_k, Δ contains two involutions, and $p = (a, b)$; or $\tau = (1, 2)$, Δ contains no involutions, and $p = (x, x^{-1})$.

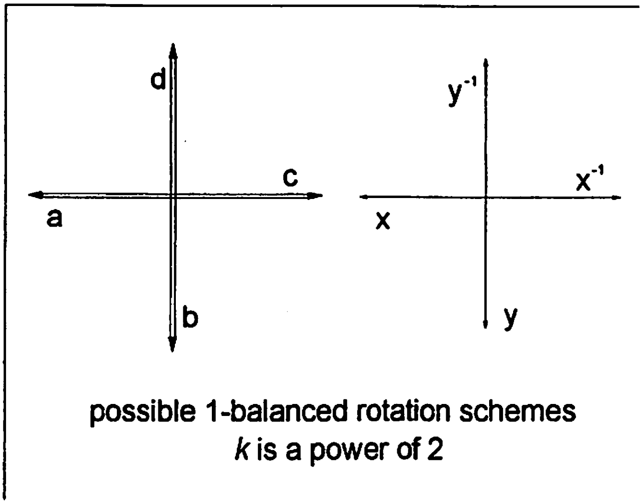


Figure 2:

When $k = 4$, e is either 1 or -1. Again, since 4 is even and divides itself, and $e = \pm 1 \bmod 4$, there are two different possible inverse distributions for each case. When $e = 1$, either $\tau = \iota$, Δ contains $\gcd(4,4) = 4$ involutions, and $p = (a, b, c, d)$; or $\tau = (1,3)(2,4)$, Δ contains no involutions, and $p = (x, y, x^{-1}, y^{-1})$. When $e = -1$, either $\tau = (1, -3)$, Δ contains $\gcd(4, -2) = 2$ involutions, and $p = (x, a, x^{-1}, b)$; or $\tau = (1,4)(2,3)$, Δ contains no involutions, and $p = (x, y, y^{-1}, x^{-1})$. Note in the former case that the line of reflection between darts and their inverses coincides with the pair of involutions, and in the latter the line coincides with no dart at all.

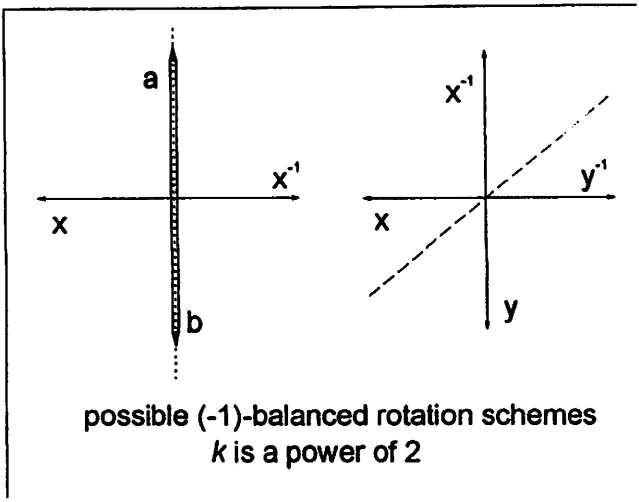


Figure 3:

When $k = 2^n$ with $n \geq 3$, then e is 1, -1, $\frac{k}{2} - 1$, or $\frac{k}{2} + 1$; M need not necessarily be 1-balanced or (-1)-balanced.

Theorem 3.2. *Let M be a regular Cayley map $CM(\Gamma, \Delta, p)$. Let $k = |\Delta|$ and e be an element of $\mathbb{Z}_k$ such that $e^2 = 1 \bmod k$. If M is e-balanced and if k is a power of 2, then:*

1. *when $e = 1$, either Δ consists entirely of involutions, or else each dart and its inverse lie symmetrically about the vertex;*

2. *when $e = -1$, either Δ contains no involutions, but all the darts lie symmetrically about a line in the tangent plane of the vertex from their inverses; or else two involutions lie opposite each other and all the other darts lie symmetrically about a line containing the involutions from their inverses;*

3. *when $e = \frac{k}{2} + 1$, there is one e-balanced form for p, in which the darts alternate between involutions and inverse pairs across the vertex;*

4. *when $e = \frac{k}{2} - 1$, there is exactly one e-balanced form for p: for each inverse pair of darts, the labels of the darts are either both even or both*

odd, and there are two lines through the vertex, perpendicular to each other, such that the even inverse pairs of darts lie symmetrically about one line, and the odd inverse pairs lie symmetrically about the other.

Proof. Let $k = 2^n$ for some positive integer n.

1) When $e = 1$, by Theorem 2.2, either $\tau(i) = 1 * i = i$ or $\tau(i) = 1 * i + \gcd(e-1,k)/2 = i + \frac{k}{2}$. In the former case $\tau = \iota$, Δ consists entirely of involutions, and $p = (a_1, a_2, \ldots, a_k)$. In the latter case, $\tau = (1, \frac{k}{2}+1)(2, \frac{k}{2}+2)\ldots(i, \frac{k}{2}+i)\ldots(\frac{k}{2}, k)$, Δ contains no involutions, and

$$p = (x_1, x_2, \ldots, x_l, x_1^{-1}, x_2^{-1}, \ldots, x_l^{-1})$$

where $l = \frac{k}{2}$, i.e., the inverse of each dart is the dart symmetrically across the vertex.

2) When $e = -1$, by Theorem 2.2, either $\tau(i) = -1 * i = -i$ or $\tau(i) = -1 * i + \gcd(e-1,k)/2 = 1 - i$. In the former case, $\tau = (1,-1)(2,-2)\ldots(\frac{k}{2}-1, \frac{k}{2}+1)$, Δ contains 2 involutions, and $p = (a, y_1, y_2, \ldots, y_{l-1}, b, y_{l-1}^{-1}, \ldots, y_2^{-1}, y_1^{-1})$, where $l = \frac{k}{2}$; it is clear from τ that the two involutions (k) and $(\frac{k}{2})$ lie opposite each other and all the other darts reflect their inverses about a line through the involutions. In the latter case, $\tau = (1,k)(2,k-1)(3,k-2)\ldots(\frac{k}{2}, \frac{k}{2}+1)$, Δ clearly contains no involutions, and

$$p = (x_1, x_2, \ldots, x_l, x_l^{-1}, \ldots, x_2^{-1}, x_1^{-1})$$

where $l = \frac{k}{2}$. Here, all the darts reflect their inverses across the line in the tangent plane of the vertex between 1 and k at one point and between $\frac{k}{2}$ and $\frac{k}{2}+1$ at the other.

3) In the final cases the order k must be greater than or equal to 8; then $l = \frac{k}{2}$ is greater than or equal to 4. When $e = l + 1$, then by Theorem 2.2, $\tau(i) = ei = li + i = \frac{k}{2}i + i$. Hence $\tau = (1, l+1)(2, 2l+2)(3, 3l+3)\ldots(k, kl+k) = (1, l+1)(3, 3l+3)\ldots(k-1, 1-l)$ (deleting duplicate transpositions); τ is separable into even-labeled transpositions and odd-labeled transpositions. All the even-labeled darts are involutions, while the odd-labeled darts are symmetric across the vertex from their odd-labeled inverses. Thus there are l involutions in Δ, and

$$p = (x_1, a_1, x_2, a_2, \ldots, x_{\frac{l}{2}}, a_{\frac{l}{2}}, \ldots, x_1^{-1}, \ldots, x_{\frac{l}{2}}^{-1}, a_l).$$

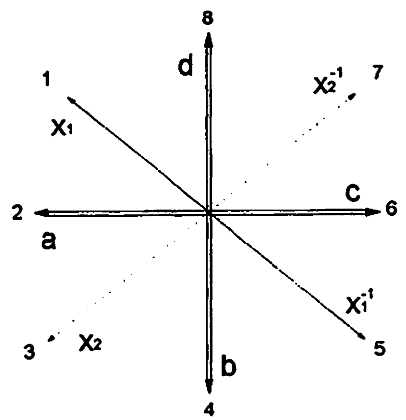
Note that the even-labeled darts, as all involutions, are of the exact form as one of the possible 1-balanced for l darts rotations, while the odd-labeled darts, symmetric across the vertex, are of the other form.

4) When $e = l - 1$, then by Theorem 2.2, $\tau(i) = ei = li - i = \frac{k}{2}i - i$. Hence $\tau = (1, l-1)(2, 2l-2)(3, 3l-3)\dots(\frac{k}{2}, \frac{k}{2}l - \frac{k}{2})\dots(k, kl-k) = (1, l-1)(3, l-3)\dots(k-1, l+1)(2, -2)(4, -4)\dots(\frac{k}{2}, \frac{k}{2})\dots(k, k)$ (deleting duplicate transpositions). Again, τ is separable into even-labeled transpositions and odd-labeled transpositions. Note that there are exactly two involutions among the even-labeled darts, $\frac{k}{2}$ and k, and the other even-labeled darts reflect their inverses across a line that coincides with $\frac{k}{2}$ and k. At the same time, the odd-labeled darts reflect their inverse across a line between $\frac{k}{4}$ and $\frac{3k}{4}$, both of which are even labels. Thus Δ contains 2 involutions and

$$p = (x_1, y^{-1}_{\frac{l}{4}-1}, \dots, x_{\frac{l}{4}}, a, \dots, x_{\frac{l}{2}}, y_{\frac{l}{4}}, \dots, x^{-1}_{\frac{l}{4}+1}, b, \dots, x_1^{-1}, y^{-1}_{\frac{l}{4}})$$

alternates between the entries in the two possible (-1)-balanced for l darts rotations, with the second rotation cycled by $\frac{k}{8} - 1$. Thus for each inverse pair of darts, the labels of the darts are either both even or both odd, and there are two lines through the vertex, perpendicular to each other, such that the even inverse pairs of darts lie symmetrically about one line, and the odd inverse pairs lie symmetrically about the other.

□



5-balanced for 8 darts

Figure 4:

Example 3.3. *Let $k = 8$ and $e = 5$; then p alternates between the two 1-balanced rotations $(x_1, x_2, x_1^{-1}, x_2^{-1})$ and (a, b, c, d), where a, b, c, and d are all involutions. That is,*

$$p = (x_1, a, x_2, b, x_1^{-1}, c, x_2^{-1}, d).$$

On the other hand if $k = 8$ and $e = 3$, then p alternates between the two (-1)-balanced rotations $(x_1, x_2, x_2^{-1}, x_1^{-1})$ and (y_1, a, y_1^{-1}, b), where a and b are involutions; the second rotation is not cycled since $\frac{k}{8} - 1 = 0$. That is,

$$p = (x_1, y_1, x_1^{-1}, a, x_2, y_1^{-1}, x_2^{-1}, b).$$

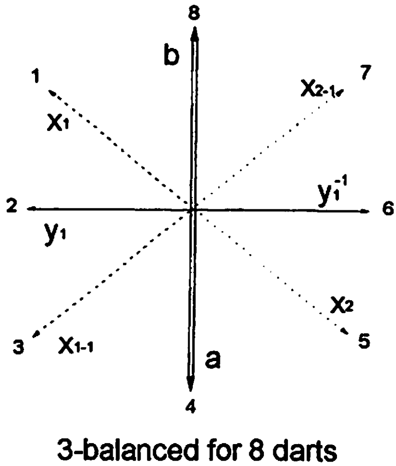


Figure 5:

For $k = 32$ and $e = 15$, the second rotation is cycled by $\frac{k}{8} - 1 = 3$.

$$p = (x_1, y_3^{-1}, x_2, y_2^{-1}, x_3, y_1^{-1}, x_4, a, x_5, y_1, x_6, y_2, x_7, y_3, x_8, y_4, \\ x_8^{-1}, y_5, x_7^{-1}, y_6, x_6^{-1}, y_7, x_5^{-1}, b, x_4^{-1}, y_7^{-1}, x_3^{-1}, y_6^{-1}, x_2^{-1}, y_5^{-1}, x_1^{-1}, y_4^{-1}).$$

4 e-Balanced Cayley Maps of General Degree

We now consider the case that $k = |\Delta|$ is not a power of a prime. Let $k = k_1 k_2$ where k_1 and k_2 are relatively prime. The distribution of the inverses of an e-balanced map of degree k is a "composite" of the distributions for maps of degree k_1 and k_2.

By the Chinese Remainder Theorem, there is a ring isomorphism $\alpha : \mathbb{Z}_k \to \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ defined by $\alpha(r) = (r_1, r_2)$ where $r_1 = r \mod k_1$ and $r_2 = r \mod k_2$. Since $\gcd(k_1, k_2) = 1$, there exists integers a and b such that $ak_1 + bk_2 = 1$. The inverse α^{-1} is defined by $\alpha^{-1}(r_1, r_2) = r_1 bk_2 + r_2 ak_1 \mod k$.

From the inverse distribution $\tau : S_k \to S_k$, we define $\tau_1(r_1) = \tau(r) \mod k_1$ and $\tau_2(r_2) = \tau(r) \mod k_2$. Theorem 2.2 shows that τ_1 and τ_2 are well-defined. The functions τ_1 and τ_2 may be defined as permutations on the sets $\{1, 2, \ldots, k_1\}$ and $\{1, 2, \ldots, k_2\}$ respectively. We will denote these sets Δ_1 and Δ_2. The distributions τ_1 and τ_2 induce e_1- and e_2-balanced rotations on Δ_1 and Δ_2.

We adopt the notation: i stands for 1 or 2. For convenience , we denote the e-balanced for k darts rotation as the "k-rotation" and the e_i-balanced for k_i darts rotation as the "k_i-rotation." Label the darts in the k_i-rotation 1 through k_i. In the k_i-rotation, the inverse of the dart labeled k_i is located at label w_i. If dart k_i is an involution, then $w_i = k_i$. By the form of the involution τ_i associated with the k_i-rotation, the inverse of dart r_i is located at $w_i + r_i e_i$ in the k_i-rotation, by Theorem 2.2.

Proposition 4.1. *With the above notation, if $\tau(k) = w$, then*

$$\tau(r) = (w_1 + r_1 e_1) bk_2 + (w_2 + r_2 e_2) ak_1.$$

Proof.

$$\alpha(\tau(r)) = (\tau_1(r_1), \tau_2(r_2)).$$

□

Remark 4.2. *That is, to find the inverse of $(r_1, r_2) \in \Delta_1 \times \Delta_2 \cong \Delta$ first use the symmetry of the k_1-rotation on $\Delta_1 \times \{r_2\}$ to find $(\tau_1(r_1), r_2)$, then use the symmetry of the k_2-rotation on $\{\tau_1(r_1)\} \times \Delta_2$ to find $(\tau_1(r_1), \tau_2(r_2))$. Or one may first use the e_2-symmetry on $\{r_1\} \times \Delta_2$ and then the e_1-symmetry on $\Delta_1 \times \{\tau_2(r_2)\}$.*

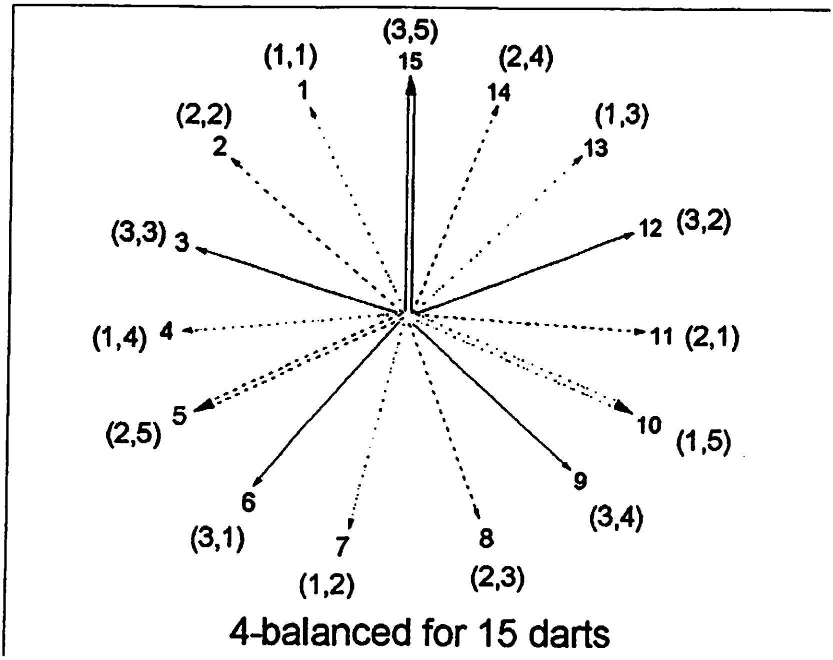


Figure 6:

Figure 6 shows a 4-balanced for 15 darts rotation. From Theorem 2.2, we know that, considered as a permutation,

$$\tau = (1,4)(2,8)(3,12)(6,9)(7,13)(11,14)$$

(since k is not even, $\tau(i) = 4i$). The inner letters depict this inverse distribution: if we consider the dart pointing straight "up" to lie at position 15, and advance in a counter-clockwise direction, dart x_1 lies at position 1 while dart x_1^{-1} lies at position 4, dart x_2 lies at position 2 while dart x_2^{-1} lies at position 8, etc. The involutory darts, a, b, and c, lie at positions 5, 10 and 15, respectively.

The outer pairs of numbers label the darts from $\mathbb{Z}_3 \times \mathbb{Z}_5$, using 3 or 5 instead of 0. To demonstrate the method of finding the inverse distribution according to Proposition 4.1, we first note that $\alpha(4) = (1, -1)$, $\tau_1(i) = 1 * i = i$, and $\tau_2(i) = -1 * i = -i$. Consider label $(r_1, r_2) = (1, 1)$; $(\tau_1(r_1), r_2) =$

$(1 * 1, 1) = (1, 1)$; then $(\tau_1(r_1), \tau_2(r_2)) = (1, -1 * 1) = (1, -1) = (1, 4)$. Likewise for $(r_1, r_2) = (1, 2)$: $(\tau_1(r_1), \tau_2(r_2)) = (1 * 1, -1 * 2) = (1, 3)$.

Observe that, because of the nature of α^{-1}, advancing from $\Delta_1 \times \{r_2\}$ to $\Delta_1 \times \{r_2 + 1\}$ rotates the copy of Δ_1 by $a(\frac{2\pi}{k_2})$ radians. Similarly, advancing from $\{r_1\} \times \Delta_2$ to $\{r_1 + 1\} \times \Delta_2$ rotates the copy of Δ_2 by $b(\frac{2\pi}{k_1})$ radians.

We are now ready to summarize the inverse distributions of e-balanced Cayley maps.

Theorem 4.3. *Let $M = CM(G, \Delta, p)$ be an e-balanced Cayley map of degree k. There are seven possible cases.*

1. *If $e = 1$, then either all the generators are involutions or inverse pairs are symmetrical about the vertex.*

2. *If $e = -1$, then inverse pairs are symmetrical about one line in the tangent plane that either coincides with two involutions or else coincides with no dart.*

3. *If $e = \frac{k}{2} + 1$, then the dart labels alternate between involutions and inverse pairs across the vertex.*

4. *If $e = \frac{k}{2} - 1$, then the inverse pairs lie symmetrically about one of two perpendicular lines. The lines that the pairs are associated with alternate as one rotates about the vertex.*

5. *If $k = k_1k_2$ with $gcd(k_1, k_2) = 1$, $\alpha(e) = (-1, 1)$, and for integers a, b, $ak_1 + bk_2 = 1$, then inverse pairs lie symmetrically about one of either k_2, if k_2 is odd, or $\frac{k_2}{2}$, if k_2 is even, lines. As one advances counterclockwise from dart to dart, the lines of reflection rotate counterclockwise by $a(\frac{2\pi}{k_2})$ radians.*

6. *If $k = k_1k_2$ with $gcd(k_1, k_2) = 1$, $\alpha(e) = (\frac{k}{2} + 1, -1)$, and for integers a, b, $ak_1 + bk_2 = 1$, then inverse pairs lie symmetrically about one of $\frac{k_1}{2}$ lines. As one advances counterclockwise from dart to dart, the lines of reflection rotate counterclockwise by $a(\frac{\pi}{2}) + b(\frac{2\pi}{k_1})$ radians.*

7. *If $k = k_1k_2$ with $gcd(k_1, k_2) = 1$, $\alpha(e) = (\frac{k}{2} - 1, 1)$, and for integers a, b, $ak_1 + bk_2 = 1$, then inverse pairs lie symmetrically about one of $2k_2$ lines. As one advances counterclockwise from dart to dart, the lines of reflection rotate counterclockwise by $a(\frac{2\pi}{k_2}) + b(\frac{\pi}{2})$ radians.*

Remark 4.4. *These seven cases exhaust all possibilities. If $k = k_1k_2$ with $gcd(k_1, k_2) = 1$, then $\alpha^{-1}(\frac{k}{2}+1, 1) = \frac{k}{2}+1$ and $\alpha^{-1}(\frac{k}{2}-1, -1) = \frac{k}{2}-1$. If $k = k_1k_2k_3$ with $gcd(k_1, k_2, k_3) = 1$, then $\alpha^{-1}(\frac{k}{2}+1, 1, -1) = \beta^{-1}(\frac{k}{2}+1, -1)$ and $\alpha^{-1}(\frac{k}{2}-1, -1, 1) = \beta^{-1}(\frac{k}{2}-1, 1)$, for ring isomorphisms $\alpha : \mathbb{Z}_k \to \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \mathbb{Z}_{k_3}$ and $\beta : \mathbb{Z}_k \to \mathbb{Z}_{k_1k_2} \times \mathbb{Z}_{k_3}$.*

Proof of Theorem 4.3. As we have already stated we may assume that M is regular.

Cases (1) through (4) follow exactly as in the proofs of Theorems 3.1 and 3.2.

Case (5): The symmetry of k_2-rotation is either the identity or reflection across the vertex (or equivalently rotation Π by π radians. If R_l is reflection across a line l, then $\Pi R_l = R_m$, reflection across the line perpendicular to l. Thus the effect of Π is to rotate all the lines of reflection. Thus without loss of generality we may assume the symmetry of the k_2-rotation is the identity. Therefore, we have a line of reflection for each $\Delta_1 \times \{x\}$, $x \in \Delta_2$. If k_1 is even, the lines are repeated twice.

The subset $\Delta_1 \times \{x+1\}$ is rotated by $a(\frac{2\pi}{k_2})$ from $\Delta_1 \times \{x\}$.

Case (6) is a similar argument with the roles of k_1 and k_2 reversed, but the symmetry of the k_1-rotation proceeds $\Pi, \Pi^2, \Pi^3, \ldots, \Pi^{k_1}$, where Π is rotation by π radians. This introduces an extra $a(\frac{\pi}{2})$ in the rotation of the lines of reflection.

Case (7) is also a similar argument to case (5), but the symmetry of the k_2-rotation proceeds $\Pi R_l, \Pi^2 R_l, \Pi^3 R_l, \ldots, \Pi^{k_1} R_l$, which introduces an extra $b(\frac{\pi}{2})$ in the rotation of the lines of reflection. Since $bk_2 = 1 \mod k_1$, we have $\gcd(b, k_1) = 1$. This implies that b is odd as is k_2 so there will be $2k_2$ lines of reflection.

□

As an example of case 5, let $k = 15 = 5*3$, $e = 4$, $\alpha(e) = (-1, 1) \in \mathbb{Z}_5 \times \mathbb{Z}_3$, $a = 2$ and $b = -3$, and see Figure 7. Note that the line of symmetry for each (-1)-balanced for 5 darts rotation itself rotates counter-clockwise by $2(\frac{2\pi}{3})$ radians, or $(\frac{\pi}{3})$ radians clockwise.Note that the 15-rotation consists of three 5-rotations. A k_1-rotation is produced whenever there is an involution in the k_2-rotation by Proposition 4.1 and Remark 4.2.

As a final example if $k = 12 = 3 * 4$, $e = 5$, $\alpha(e) = (-1, 1) \in \mathbb{Z}_3 \times \mathbb{Z}_4$, $a = -1$, $b = 1$, then note first that $e = \frac{k}{2} - 1$, but we may also use item (5)

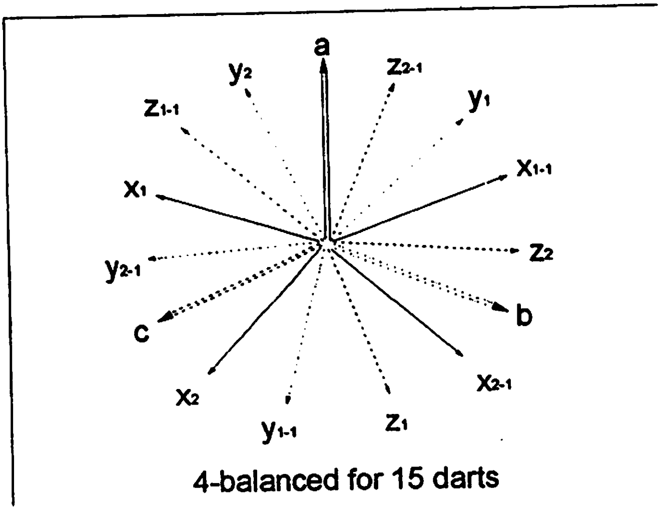


Figure 7:

of Theorem 4.3. The lines of symmetry rotates by $\frac{\pi}{2}$ radians clockwise (or $-\frac{\pi}{2}$ radians counter-clockwise) in Figure 9, but the 4-rotation is not repeated over and over again as in the previous example, since the inverse distribution $\tau_2 = (1,3)(2,4)$ does not consists of involutions.

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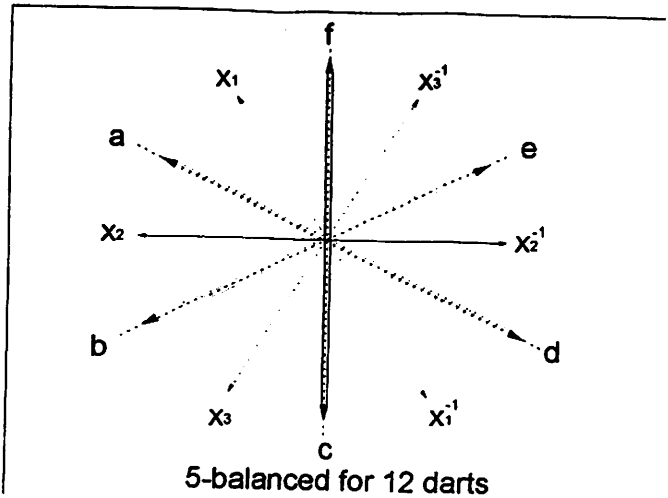

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