# The Decomposition of Graphs with Extended Cyclic Triples

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**Abstract**

In this paper, we study the decomposition of the graph $(\lambda D_v)^{+\alpha}$ into extended cyclic triples, for all $\lambda \geq \alpha$. By an extended cyclic triple, we mean a loop, a loop with symmetric arcs attached (known as a lollipop), or a directed 3-cycle (known as a cyclic triple).

## 1 Introduction

Let $G$ be a graph and $G^{+\alpha}$ be a graph which is obtained by attaching $\alpha$ loops to each vertex of $G$ (denoted $G^{+1}$ by $G^{+}$ for brevity). As usual, $\lambda D_v$ and $\lambda K_v$ denote the $\lambda$ fold complete digraph and graph on $v$ vertices, respectively, and an edge $\{a, b\}$ in the graph $K_v$ can be regarded as symmetric arcs $(a, b)$ and $(b, a)$ in the digraph $D_v$. A Mendelsohn triple system of order $v$, $MTS(v)$, is a pair $(V, B)$, where $V$ is a $v$-set and $B$ is a collection of cyclically ordered triples of distinct elements of $V$, such that every ordered pair of distinct elements of $V$ is contained in only one member of $B$. This concept was introduced by N.S. Mendelsohn [7], who proved that a $MTS(v)$ exists if and only if $v \not\equiv 2(mod\ 3)$, $v \neq 6$. F.E. Bennett [1] introduced the concept of a system, similar to an $MTS$, in which a triple may have repeated elements.

An extended Mendelsohn triple system of order $v$, $EMTS(v)$, is a pair $(V, B)$, where $V$ is a $v$-set and $B$ is a collection of cyclically ordered triples of elements of V, where each triple may have repeated elements and will

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be called a block, such that every ordered pair of elements of $V$, not necessarily distinct, is contained in exactly one block of $B$. It has been well established that an extended Mendelsohn triple system is co-extensive with the variety of quasigroup satisfying the identity $x(yx) = y$. (It is called a semi-symmetric quasigroup). The blocks of an $EMTS$ are of three types: $[x, y, z]$, $[x, x, y]$, $[x, x, x]$ which we call cyclic triple, lollipop and loop, respectively. By an extended cyclic triple, we mean a cyclic triple, a lollipop, or a loop. Observe that $[x, y, z]$ contains the ordered pairs $(x, y)$, $(y, z)$, $(z, x)$; $[x, x, y]$ contains the pairs $(x, x)$, $(x, y)$, $(y, x)$; and $[x, x, x]$ contains only the pair $(x, x)$. An $EMTS(v)$ which has $a$ loops will be denoted by $EMTS(v, a)$. If $(V, B)$ is an extended Mendelsohn triple system with parameters $v$ and $a$, we say $B$ is an $EMTS(v, a)$ and write $B \in EMTS(v, a)$. We say $EMTS(v, a)$ exists if there exists an extended Mendelsohn triple system with parameters $v$ and $a$.

F. E. Bennett [1] gave the necessary and sufficient conditions for the existence of an $EMTS(v, a)$ as follows.

**Theorem 1.1** *[1] There exists an EMTS(v, a), if and only if,* $0 \leq a \leq v$ *and*
*(i) if* $v \equiv 0 (mod\, 3)$*, then* $a \equiv 0 (mod\, 3)$*;*
*(ii) if* $v \not\equiv 0 (mod\, 3)$*, then* $a \equiv 1 (mod\, 3)$*;*
*(iii) if* $v = 6$*, then* $a \leq 3$.

In graph notation, an $EMTS(v, a)$ is equivalent to the decomposition of the digraph $D_v^+$ into cyclic triples, $v - a$ lollipops and $a$ loops. Now, we consider the following question.

Can the graph $(\lambda D_v)^{+\alpha}$ be decomposed into extended cyclic triples with all possible number of loops?

From now on, a decomposition of $G$ is a decomposition of $G$ into extended cyclic triples. In the process of each construction, we need the packing for the digraph $\lambda D_v$. A packing of the graph $\lambda D_v$ with cyclic triples (or a packing of $\lambda D_v$ for brevity) is a set $M$ of cyclic triples formed from the arcs of $\lambda D_v$ with the multiplicity of each arc in $M$ less then or equal to $\lambda$. The leave of the packing of $\lambda D_v$ are the arcs of $\lambda D_v \backslash M$. When the cardinality of the set $M$ is a maximum, the packing is called the maximum packing of $\lambda D_v$ and the corresponding leave set denoted by $L(\lambda D_v)$. For $v \not\equiv 2$(mod 3) and $v \neq 6$, there is an $MTS(v)$. $L(\lambda D_v) = \emptyset$ by taking each triple of the MTS($v$) $\lambda$ times. When $v = 6$, $L(D_6)$ is a 1-factor of $K_6$. The decompositions $\mathcal{B}$ and $\mathcal{B}^*$ [11] of $2D_6$ and $3D_6$ are obtained by taking

$$\{[0,1,2],[0,2,1],[\infty,0,2],[\infty,0,3]\}$$

and

$$\{[0,1,2],[0,2,4],[0,3,1],[\infty,0,2],[\infty,0,4],[\infty,0,4]\}$$

as starter blocks on $Z_5 \cup \{\infty\}$, respectively. Taking the union of the decompositions of $2D_6$ and $3D_6$, we obtain $L(\lambda D_6) = \emptyset$ for all $\lambda \geq 2$. When $v \equiv 2(mod\ 3)$, $L(D_v)$ contains only one edge by the existence of $EMTS(v,1)$. Therefore, the set $L(\lambda D_v)$ is

(i) $\emptyset$ if $v \not\equiv 2(mod\ 3)$ and $(v,\lambda) \neq (6,1)$, or $v \equiv 2(mod\ 3)$ and $\lambda \equiv 0(mod\ 3)$;

(ii) 1-factor of $K_6$ if $(v,\lambda) = (6,1)$;

(iii) one edge if $v \equiv 2(mod\ 3)$ and $\lambda \equiv 1(mod\ 3)$;

(iv) two edges if $v \equiv 2(mod\ 3)$ and $\lambda \equiv 2(mod\ 3)$.

Recently, some papers investigated the structure of generalized triple systems. M. E. Raines and C. A. Rodger [8, 9, 10] considered this problem for extended triple systems and V. E. Castellana and M. E. Raines [3] for extended Mendelsohn triple systems. In sections 2, we will give the decomposition of the graph $(\lambda D_v)^{+\alpha}$ for $\alpha = 1$, 2 and 3. Lastly, we will give the general results.

# 2 Decomposition of $(\lambda D_v)^{+\alpha}$, for $\alpha = 1, 2, 3$

It can be determined by the means of a computer that the necessary condition for the existence of the decomposition of $(\lambda D_v)^+$ with $a$ loops, $0 \leq a \leq v$, is

(i) if $a \equiv 0(mod\ 3)$ then $v \equiv 0(mod\ 3)$ or $v \equiv 2(mod\ 3)$ and $\lambda \equiv 2(mod\ 3)$;

(ii) if $a \equiv 1(mod\ 3)$ then $v \equiv 1(mod\ 3)$ or $v \equiv 2(mod\ 3)$ and $\lambda \equiv 1(mod\ 3)$;

(iii) if $a \equiv 2(mod\ 3)$ then $v \equiv 2(mod\ 3)$ and $\lambda \equiv 0(mod\ 3)$.

**Theorem 2.1** *If $v \equiv 0(mod\ 3)$ or $v \equiv 2(mod\ 3)$ and $\lambda \equiv 2(mod\ 3)$ then the digraph $(\lambda D_v)^+$ can be decomposed into cyclic triples, lollipops and $a$ loops, for all $a \equiv 0\ (mod\ 3)$ and $0 \leq a \leq v$.*

**Proof.** The digraph $(\lambda D_v)^+$ can be regarded as the union of the subgraphs $D_v^+$ and $(\lambda-1)D_v$ on the same underlying vertices set $X$. And we will denote it by $(\lambda D_v)^+ = D_v^+ \cup (\lambda-1)D_v$.

When $v \equiv 0$ $(mod\ 3)$, $v \neq 6$, the first part $D_v^+$ can be decomposed into cyclic triples, lollipops and $3k$ loops according to the existence of $EMTS(v, 3k)$ and the second part $(\lambda-1)D_v$ can be decomposed into cyclic triples by $L((\lambda-1)D_v) = \emptyset$. When $v = 6$, $T_1 = \mathcal{B} \cup \{[x,x,x] \mid x \in X\}$ forms a decomposition of $2D_6$ with 6 loops. Now, $T_2$ is obtained from $T_1$ by removing the blocks $\{[0,1,2], [0,2,1], [0,0,0], [1,1,1], [2,2,2]\}$ and replacing them with $\{[0,0,1], [1,1,2], [2,2,0]\}$. Then $T_2$ is a decomposition of $2D_6$ with 3 loops. Let $X = \{x_1,x_2,x_3,x_4,x_5,x_6\}$, taking two of the same maximum packing of $D_6$ with two leave sets $\{\{x_1,x_2\},\{x_3,x_4\},\{x_5,x_6\}\}$ and replacing those leave sets and loops with $\{[x_1,x_1,x_2],[x_2,x_2,x_1],[x_3,x_3,x_4], [x_4,x_4,x_3], [x_5,x_5,x_6], [x_6,x_6,x_5]\}$, we obtain a decomposition of $(2D_6)^+$ without a loop. $T_3 = \mathcal{B}^* \cup \{[x,x,x] \mid x \in X\}$ forms a decomposition of $3D_6$ with 6 loops. From $(3D_6)^+ = D_6^+ \cup 2D_6$, the decomposition of $(3D_6)^+$ with 3 loops and without a loop can be obtained from the existence of $EMTS(6,3)$ and $EMTS(6,0)$, respectively. As for $\lambda \geq 4$, it follows by $(\lambda D_6)^+ = (2D_6)^+ \cup (\lambda-2)D_6$.

When $\lambda \equiv 2$ $(mod\ 3)$ and $v \equiv 2$ $(mod\ 3)$, the first part $D_v^+$ can be decomposed into cyclic triples, lollipops and $3k+1$ loops, $0 \leq 3k+1 \leq v$, (say one loop at $x$) by the existence of $EMTS(v, 3k+1)$ and the second part $(\lambda-1)D_v$ can be decomposed into cyclic triples and one-edge (say $\{x,y\}$) due to the results of packing. The digraph $(\lambda D_v)^+$ can be decomposed into cyclic triples, lollipops and $3k$ loops by combining the loop $x$ and the edge $\{x,y\}$ to a lollipop $[x,x,y]$. ∎

**Theorem 2.2** *If $v \equiv 1(mod\ 3)$ or $v \equiv 2(mod\ 3)$ and $\lambda \equiv 1(mod\ 3)$ then the digraph $(\lambda D_v)^+$ can be decomposed into cyclic triples, lollipops and a loops, for all $a \equiv 1$ (mod 3) and $0 \leq a \leq v$.*

**Proof.** Consider $(\lambda D_v)^+ = D_v^+ \cup (\lambda-1)D_v$. When $v \equiv 1(mod\ 3)$ or $v \equiv 2(mod\ 3)$ and $\lambda \equiv 1(mod\ 3)$, the first part $D_v^+$ can be decomposed into cyclic triples, lollipops and $3k+1$ loops, $0 \leq 3k+1 \leq v$, according to the existence of $EMTS(v, 3k+1)$ and the second part $(\lambda-1)D_v$ can be decomposed into cyclic triples by $L((\lambda-1)D_v) = \emptyset$. Therefore, the digraph $(\lambda D_v)^+$ can be decomposed into cyclic triples, lollipops and $3k+1$ loops. ∎

**Theorem 2.3** *If $v \equiv 2(mod\ 3)$ and $\lambda \equiv 0(mod\ 3)$ then the digraph $(\lambda D_v)^+$ can be decomposed into cyclic triples, lollipops and a loops, for all $a \equiv 2$ (mod 3) and $0 \leq a \leq v$.*

**Proof.** Consider $(\lambda D_v)^+ = D_v^+ \cup (\lambda - 1)D_v$. When $v \equiv 2(mod\,3)$ and $\lambda \equiv 0(mod\,3)$, the first part $D_v^+$ can be decomposed into cyclic triples, lollipops and $3k+1$ loops, $0 \le 3k+1 \le v$ and $k \ge 1$, with two loops at $x$ and $y$ according to the existence of $EMTS(v, 3k+1)$ and the second part $(\lambda-1)D_v$ can be decomposed into cyclic triples and two edges $\{\{x,y\},\{y,z\}\}$ due to the results of packing. Replacing the edges $\{\{x,y\},\{y,z\}\}$, the loops $x$ and $y$ with lollipops $[x,x,y]$ and $[y,y,z]$, the digraph $D_v^+$ can be decomposed into cyclic triples, lollipops and $a$ loops, where $a \equiv 2$ (mod 3) and $2 \le a \le v-3$. From $L(\lambda D_v) = \emptyset$, the digraph $(\lambda D_v)^+$ can be decomposed into cyclic triples and $v$ loops. ∎

In the same way, the necessary condition for the existence of the decomposition of $(\lambda D_v)^{+2}$ with $a$ loops, $0 \le a \le 2v$, is

**(i)** if $a \equiv 0(mod\,3)$ then $v \equiv 0(mod\,3)$ or $v \equiv 2(mod\,3)$ and $\lambda \equiv 1(mod\,3)$;

**(ii)** if $a \equiv 1(mod\,3)$ then $v \equiv 2(mod\,3)$ and $\lambda \equiv 0(mod\,3)$;

**(iii)** if $a \equiv 2(mod\,3)$ then $v \equiv 1(mod\,3)$ or $v \equiv 2(mod\,3)$ and $\lambda \equiv 2(mod\,3)$.

As for the decomposition of $(\lambda D_v)^{+2}$ for those data, we consider $(\lambda D_v)^{+2} = D_v^+ \cup ((\lambda - 1)D_v)^+$. Combining the decomposition of first part by Theorem 1.1 and the decomposition of second part by Theorems 2.1, 2.2 and 2.3, the digraph $(\lambda D_v)^{+2}$ can be decomposed into cyclic triples, lollipops and $a$ loops, for all $a$ in each case above. So, we have the following three Theorems.

**Theorem 2.4** *If $v \equiv 0(mod\,3)$ or $v \equiv 2(mod\,3)$ and $\lambda \equiv 1(mod\,3)$ then the digraph $(\lambda D_v)^{+2}$ can be decomposed into cyclic triples, lollipops and $a$ loops, for all $a \equiv 0$ (mod 3) and $0 \le a \le 2v$.*

**Theorem 2.5** *If $v \equiv 2(mod\,3)$ and $\lambda \equiv 0(mod\,3)$ then the digraph $(\lambda D_v)^{+2}$ can be decomposed into cyclic triples, lollipops and $a$ loops, for all $a \equiv 1$ (mod 3) and $0 \le a \le 2v$.*

**Theorem 2.6** *If $v \equiv 1(mod\,3)$ or $v \equiv 2(mod\,3)$ and $\lambda \equiv 2(mod\,3)$ then the digraph $(\lambda D_v)^{+2}$ can be decomposed into cyclic triples, lollipops and $a$ loops, for all $a \equiv 2$ (mod 3) and $0 \le a \le 2v$.*

Similarly, the necessary condition for the existence of the decomposition of $(\lambda D_v)^{+3}$ with $a$ loops, $0 \le a \le 3v$, is

(i) if $a \equiv 0(mod\,3)$ then $v \not\equiv 2(mod\,3)$ or $v \equiv 2(mod\,3)$ and $\lambda \equiv 0(mod\,3)$;

(ii) if $a \equiv 1(mod\,3)$ then $v \equiv 2(mod\,3)$ and $\lambda \equiv 2(mod\,3)$;

(iii) if $a \equiv 2(mod\,3)$ then $v \equiv 2(mod\,3)$ and $\lambda \equiv 1(mod\,3)$.

From the constructions of $EMTS$ in [4], the following theorem is obtained.

**Lemma 2.7** *[4] An $EMTS(v, a)$ can be embedded in an $EMTS$ $(2v+i, a)$, for $i = 0, 3$.*

**Lemma 2.8** *For $v \not\equiv 0$ (mod 3), the digraph $D_v^+$ can be decomposed into cyclic triples, lollipops and one loop, which contain the patterns of blocks $T$, where $T = \{[x, x, x],\ [x, y, z],\ [x, z, y]\}$ for some vertices $x$, $y$ and $z$.*

**Proof:** The digraph $D_4^+$ can be decomposed into $T \cup \{[2,2,1],\ [3,3,2],\ [4,4,2]\}$, the digraph $D_5^+$ can be decomposed into $T \cup \{[2,2,3],\ [3,3,5],\ [4,4,2],\ [5,5,4],\ [1,2,5], [1,\ 5,2]\}$ and the digraph $D_7^+$ can be decomposed into $T \cup \{[1,6,7],\ [1,7,6],\ [4,6,2],\ [4,2,6],\ [4,5,5],\ [3,2,2],\ [2,7,7],\ [3,7,5],\ [6,3,3],\ [4,4,7],\ [5,6,6],\ [1,2,5],\ [3,5,7],\ [1,5,2]\}$, where $T = \{[1,1,1], [1,3,4], [1,4,3]\}$. Applying Lemma 2.7 with these small cases, we can construct the new systems of the other order containing $T$. ∎

**Theorem 2.9** *If $v \not\equiv 2(mod\,3)$ or $v \equiv 2(mod\,3)$ and $\lambda \equiv 0(mod\,3)$ then the digraph $(\lambda D_v)^{+3}$ can be decomposed into cyclic triples, lollipops and $a$ loops, for all $a \equiv 0$ (mod 3) and $0 \le a \le 3v$.*

**Proof.** Consider $(\lambda D_v)^{+3} = G_1 \cup G_2$, where $G_1 = D_v^+$ and $G_2 = ((\lambda - 1)D_v)^{+2}$.

**Case 1:** When $v \equiv 0$(mod 3), by Theorem 1.1 and Theorem 2.4, the digraph $(\lambda D_v)^{+3}$ can be decomposed into cyclic triples, lollipops and $3k$ loops.

**Case 2:** When $v \equiv 1$(mod 3) or $v \equiv 2(mod\,3)$ and $\lambda \equiv 0(mod\,3)$, the digraph $G_1$ can be decomposed into cyclic triples, lollipops and $3i+1$ loops, $0 \le 3i+1 \le v$, by Theorem 1.1 and the digraph $G_2$ can be decomposed into cyclic triples, lollipops and $3j+2$ loops, $0 \le 3j+1 \le 2v$, by Theorem 2.6. Combining the two structures, the digraph $(\lambda D_v)^{+3}$ can be decomposed into cyclic triples, lollipops and $3k$ loops with $k \ge 1$. As for $k = 0$, by Lemma 2.8 and Theorem 2.6, the digraph $G_1$ can be decomposed into cyclic triples, lollipops and one loop containing the patterns $T_1 = \{[x,x,x], [x,y,z], [x,z,y]\}$ and the digraph $G_2$ can be decomposed into cyclic triples, lollipops and 2 loops, say $T_2 = \{[y,y,y],[z,z,z]\}$. Replacing $T_1 \cup T_2$ with $\{[x,x,y],[y,y,z],[z,z,x]\}$, the digraph $(\lambda D_v)^{+3}$ can be decomposed into cyclic triples and lollipops. ∎

In the same way as in case 1 of Theorem 2.9, we can obtain the following two theorems.

**Theorem 2.10** *If $v \equiv 2 (mod\ 3)$ and $\lambda \equiv 2 (mod\ 3)$ then the digraph $(\lambda D_v)^{+3}$ can be decomposed into cyclic triples, lollipops and a loops, for all $a \equiv 1\ (mod\ 3)$ and $0 \leq a \leq 3v$.*

**Theorem 2.11** *If $v \equiv 2 (mod\ 3)$ and $\lambda \equiv 1 (mod\ 3)$ then the digraph $(\lambda D_v)^{+3}$ can be decomposed into cyclic triples, lollipops and a loops, for all $a \equiv 2\ (mod\ 3)$ and $0 \leq a \leq 3v$.*

# 3 Conclusions.

Using the above theorems, we obtained the following results:

**Theorem 3.1** *When $v > 2$ and $\lambda \geq \alpha$, the digraph $(\lambda D_v)^{+\alpha}$ can be decomposed into extended cyclic triples.*

**Proof.** When $\alpha \geq 4$, we consider

$$(\lambda D_v)^{+\alpha} = ((\alpha - i)D_v)^{+(\alpha-i)} \cup ((\lambda - (\alpha - i))D_v)^{+i},$$

where $i = 1,2,3$ depend on $\alpha \equiv 1,2,0 (\text{mod } 3)$, respectively. The first part can be decomposed into the unions of $(3D_v)^{+3}$ and the second part can be obtained by the results of section 2. So, the digraph $(\lambda D_v)^{+\alpha}$ can be decomposed into extended cyclic triples. ∎


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