On Some Catalan Identities of Shapiro

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Abstract

New identities involving the Catalan sequence ordinary generating function are developed, and a previously known one established from first principles using a hypergeometric approach.

Introduction

The Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, c_5, \ldots\} = \{1, 1, 2, 5, 14, 42, \ldots\}$, with general $(n+1)$th term

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

is well known to have ordinary generating function (o.g.f.)

$$G(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{1-\sqrt{1-4x}}{2x}. \tag{2}$$

Defining associated 'even' and 'odd' functions (so that $E(x)+O(x) = G(x)$)

$$\begin{aligned} E(x) &= \frac{1}{2}[G(x) + G(-x)] \\ &= \frac{1}{4x}[\sqrt{1+4x} - \sqrt{1-4x}] \\ &= \sum_{n=0}^{\infty} c_{2n} x^{2n}, \end{aligned}$$

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$$
\begin{aligned}
O(x) &= \frac{1}{2}[G(x) - G(-x)] \\
&= \frac{1}{4x}[2 - \sqrt{1+4x} - \sqrt{1-4x}] \\
&= \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1},
\end{aligned} \tag{3}
$$

the following identities, readily verified, are seen to hold:

Identity 1.1

$$1 + 2xE(x)O(x) = E(x).$$

Identity 1.2

$$x[E^2(x) + O^2(x)] = O(x).$$

Identity 1.3

$$[1 - 2xE(x)][1 - xO(x)]G^2(x) = 1.$$

Richard Stanley, in the informative and expanding "Catalan Addendum" [1][1] to a variety of problems involving, and combinatorial interpretations of, the Catalan numbers layed out in [2, Ch.6], cites Louis Shapiro as the originator of these (see the solution to Problem 6.19(ppp) of [1]; also Remark 5 later). In addition, Shapiro is said to have communicated a more elegant result still, which is equally trivial to check by hand:

Identity 2

$$E^2(x) = G(4x^2).$$

In this paper we first present new identities for $E(x)$ and $O(x)$, each of which arises naturally and easily from Identities 1.1,1.2 (by uncoupling them; an analogue to Identity 1.3 is also given). We then go on to show that, rather interestingly, Identity 2 can be established without *a priori* knowledge of the closed form for $G(x)$—the proof of this fact, constructed from first principles using a hypergeometric approach, constitutes the main contribution of the article. Accordingly, the author would like to express his thanks to Christian Krattenthaler for his (computer-assisted) advice on a line of attack, without which the particular route taken through successive hypergeometric transformations would almost certainly not have been found, such is the nature of the argument made.

[1]This is available electronically as a link from his web homepage, and is constantly being added to. We reference the address of the .pdf version of the document.

Lemma 1

$$4x^3O^4(x) - 8x^2O^3(x) + 5xO^2(x) - O(x) + x = 0.$$

Proof From Identity 1.1 $E(x) = [1 - 2xO(x)]^{-1}$, which when substituted into Identity 1.2 gives the result after some manipulation.□

Lemma 2

$$4x^2E^4(x) - E^2(x) + 1 = 0.$$

Proof From Identity 1.1 $O(x) = [E(x) - 1]/2xE(x)$, which when substituted into Identity 1.2 gives the result after some manipulation.□

Remark 1 Being a quadratic in $E^2(x)$ the equation can be solved, and so Lemma 2 checked, to give $E^2(x) = [1 \pm \sqrt{1 - 16x^2}]/8x^2$—with reference to the definition of $E(x)$ (3), it is the negative sign in front of the radical which is the correct one.

Remark 2 As a point of interest, we note that Identity 1.3 is a consequence of Lemma 2 and Identity 1.2, for consider Lemma 2 which is re-arranged to read

$$\begin{aligned}
1 &= E^2(x) - 4x^2E^4(x) \\
&= E^2(x)[1 - 4x^2E^2(x)] \\
&= E^2(x)[1 - 2xE(x)][1 + 2xE(x)] \\
&= \frac{O(x)}{x}[1 - xO(x)][1 - 2xE(x)][1 + 2xE(x)] && \text{(by Identity 1.2)} \\
&= [1 - xO(x)][1 - 2xE(x)]\left(\frac{O(x)}{x} + 2E(x)O(x)\right) \\
&= [1 - xO(x)][1 - 2xE(x)][E^2(x) + O^2(x) + 2E(x)O(x)] && \text{(ditto)} \\
&= [1 - xO(x)][1 - 2xE(x)][E(x) + O(x)]^2 \\
&= [1 - xO(x)][1 - 2xE(x)]G^2(x). && (4)
\end{aligned}$$

Remark 3 A further identity, analogous to Identity 1.3, can be derived in a similar fashion:

Lemma 3

$$[1 + 2xE(x)][1 - xO(x)]G^2(-x) = 1.$$

This is also immediate from Identity 1.3 itself, on replacing x with $-x$ and appealing to the parity of $E(x), O(x)$.

We now detail our proof of Identity 2.

Proof of Identity 2

By definition (3),

$$\begin{aligned} E^2(x) &= \sum_{n,n'=0}^{\infty} c_{2n}c_{2n'}x^{2(n+n')} \\ &= \sum_{s=0}^{\infty} S(s)x^s, \end{aligned} \tag{P1}$$

where

$$S(s) = \sum_{2(n+n')=s} c_{2n}c_{2n'}. \tag{P2}$$

We initially re-write $S(s)$ as

$$\begin{aligned} S(s) &= \sum_{n=0}^{[\frac{1}{2}s]} c_{2n}c_{s-2n} \\ &= c_s\, {}_4F_3\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, -\frac{1}{2}s, -\frac{1}{2}-\frac{1}{2}s \\ \frac{3}{2}, \frac{1}{4}-\frac{1}{2}s, \frac{3}{4}-\frac{1}{2}s \end{matrix} \,\middle|\, 1\right) \end{aligned} \tag{P3}$$

when converted to hypergeometric form (the sum upper limit $[\frac{1}{2}s]$ in the first line is the greatest integer not exceeding $\frac{1}{2}s$). The s even/odd cases are now dealt with separately. Whilst from (P2) clearly $S(s) = 0$ for s odd, we show this formally—for both completeness and interest—as part of the proof. Note that $(u)_m$ will denote the usual rising factorial function $u(u+1)(u+2)\cdots(u+m-1)$.

<u>Case A</u>: Suppose s (odd) $= 1, 3, 5, \ldots$ In 1927 Whipple published a transformation [3, (6.1), p.264] (for convenience, we actually use the version in Bailey [4, (4.6.1), p.32])

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} a, b, c, -m \\ \kappa-b, \kappa-c, \kappa+m \end{matrix} \,\middle|\, 1\right) = \\ &\frac{(\kappa)_m(\kappa-b-c)_m}{(\kappa-b)_m(\kappa-c)_m}\, {}_5F_4\left(\begin{matrix} \frac{1}{2}\kappa-\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}a, b, c, -m \\ \kappa-a, \frac{1}{2}\kappa, \frac{1}{2}+\frac{1}{2}\kappa, b+c-\kappa+1-m \end{matrix} \,\middle|\, 1\right) \end{aligned} \tag{P4}$$

valid for integer $m \geq 0$ (both sides are unity for $m = 0$). If we set $\kappa = 1-\frac{1}{2}s$, $a = -\frac{1}{2}s$, $b = \frac{1}{4}$, $c = \frac{3}{4}$ and $m = \frac{1}{2} + \frac{1}{2}s = 1, 2, 3, \ldots$, it contracts to

$$\begin{aligned}
&{}_4F_3\left(\begin{matrix} -\frac{1}{2}s, \frac{1}{4}, \frac{3}{4}, -\frac{1}{2}-\frac{1}{2}s \\ \frac{3}{4}-\frac{1}{2}s, \frac{1}{4}-\frac{1}{2}s, \frac{3}{2} \end{matrix} \,\middle|\, 1\right) \\
&= \frac{(1-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}(-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}}{(\frac{3}{4}-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}(\frac{1}{4}-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}}\, {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, -\frac{1}{2}-\frac{1}{2}s \\ \frac{1}{2}-\frac{1}{4}s, 1-\frac{1}{4}s \end{matrix} \,\middle|\, 1\right) \\
&= \frac{(1-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}(-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}}{(\frac{3}{4}-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}(\frac{1}{4}-\frac{1}{2}s)_{\frac{1}{2}+\frac{1}{2}s}} \frac{(\frac{1}{4}-\frac{1}{4}s)_{\frac{1}{2}+\frac{1}{2}s}(-\frac{1}{4}-\frac{1}{4}s)_{\frac{1}{2}+\frac{1}{2}s}}{(\frac{1}{2}-\frac{1}{4}s)_{\frac{1}{2}+\frac{1}{2}s}(-\frac{1}{2}-\frac{1}{4}s)_{\frac{1}{2}+\frac{1}{2}s}} \\
&= 0 \qquad \text{(P5)}
\end{aligned}$$

on first applying the well known Pfaff-Saalschütz identity (see, for example, [4, (2.2.1), p.9]) to evaluate the ${}_3F_2(1)$ series which appears, and then noting that

$$\begin{aligned}
0 &= \left(\frac{1}{4}-\frac{1}{4}s\right)_{\frac{1}{2}+\frac{1}{2}s}, \qquad s = 1, 5, 9, 13, \ldots, \\
0 &= \left(-\frac{1}{4}-\frac{1}{4}s\right)_{\frac{1}{2}+\frac{1}{2}s}, \qquad s = 3, 7, 11, 15, \ldots \qquad \text{(P6)}
\end{aligned}$$

Case B: Suppose s (even) $= 0, 2, 4, \ldots$, and set $a = \frac{1}{4}$, $b = -\frac{1}{2}s = 0, -1, -2, \ldots$, $c = \frac{3}{4}$, $d = -\frac{1}{2} - \frac{1}{2}s$ in the 1985 transformation of Gasper [5, (3.3), p.1065]

$$\begin{aligned}
&{}_4F_3\left(\begin{matrix} a, b, c, d \\ 1-a+b, 1-a+c, 1-a+d \end{matrix} \,\middle|\, 1\right) = \\
&\frac{\Gamma(1-d)\Gamma(a+b-d)\Gamma(a+c-d)\Gamma(1+b+c-d)}{\Gamma(a-d)\Gamma(1+b-d)\Gamma(1+c-d)\Gamma(a+b+c-d)} \\
&\qquad \times {}_9F_8\left(\begin{matrix} \alpha_1, \ldots, \alpha_9 \\ \beta_1, \ldots, \beta_8 \end{matrix} \,\middle|\, 1\right) \qquad \text{(P7)}
\end{aligned}$$

(where $\alpha_1 = b+c-d$, $\alpha_2 = 1+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}d$, $\alpha_3 = \frac{1}{2}-\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}d$, $\alpha_4 = 1-\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}d$, $\alpha_5 = a+b-d$, $\alpha_6 = a+c-d$, $\alpha_7 = a$, $\alpha_8 = b$, $\alpha_9 = c$, $\beta_1 = \frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}d$, $\beta_2 = \frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}d$, $\beta_3 = \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}c-\frac{1}{2}d$, $\beta_4 = 1-a+c$, $\beta_5 = 1-a+b$, $\beta_6 = 1-a+b+c-d$, $\beta_7 = 1+c-d$, $\beta_8 = 1+b-d$), which holds so long as at least one of a, b or c is a negative integer (or zero). In this instance it gives

$${}_4F_3\left(\begin{matrix} \frac{1}{4}, -\frac{1}{2}s, \frac{3}{4}, -\frac{1}{2}-\frac{1}{2}s \\ \frac{3}{4}-\frac{1}{2}s, \frac{3}{2}, \frac{1}{4}-\frac{1}{2}s \end{matrix} \,\middle|\, 1\right)$$

$$= \frac{\Gamma(\frac{3}{4})\Gamma(\frac{9}{4})\Gamma^2(\frac{3}{2}+\frac{1}{2}s)}{\Gamma^2(\frac{3}{2})\Gamma(\frac{3}{4}+\frac{1}{2}s)\Gamma(\frac{9}{4}+\frac{1}{2}s)}\ {}_7F_6\left(\begin{matrix}\frac{5}{4},\frac{13}{8},1,\frac{3}{2}+\frac{1}{2}s,\frac{1}{4},-\frac{1}{2}s,\frac{3}{4}\\ \frac{5}{8},\frac{5}{4},\frac{3}{4}-\frac{1}{2}s,2,\frac{9}{4}+\frac{1}{2}s,\frac{3}{2}\end{matrix}\ \middle|\ 1\right)$$

$$= \frac{(\frac{3}{2})^2_{\frac{1}{2}s}}{(\frac{3}{4})_{\frac{1}{2}s}(\frac{9}{4})_{\frac{1}{2}s}}\ {}_7F_6\left(\begin{matrix}\frac{5}{4},\frac{13}{8},1,\frac{3}{2}+\frac{1}{2}s,\frac{1}{4},-\frac{1}{2}s,\frac{3}{4}\\ \frac{5}{8},\frac{5}{4},\frac{3}{4}-\frac{1}{2}s,2,\frac{9}{4}+\frac{1}{2}s,\frac{3}{2}\end{matrix}\ \middle|\ 1\right) \tag{P8}$$

(recall the relation $(u)_t = \Gamma(u+t)/\Gamma(u)$). Now, Dougall's formula (see, for example, [6, (2.3.4.4), p.56]) states that, for integer $m \geq 0$,

$${}_7F_6\left(\begin{matrix}a,1+\frac{1}{2}a,b,c,d,1+2a-b-c-d+m,-m\\ \frac{1}{2}a,1+a-b,1+a-c,1+a-d,b+c+d-a-m,1+a+m\end{matrix}\ \middle|\ 1\right)$$
$$= \frac{(1+a)_m(1+a-b-c)_m(1+a-b-d)_m(1+a-c-d)_m}{(1+a-b)_m(1+a-c)_m(1+a-d)_m(1+a-b-c-d)_m}, \tag{P9}$$

whereby, choosing $a=\frac{5}{4}$, $b=\frac{3}{2}+\frac{1}{2}s$, $c=\frac{1}{4}$, $d=\frac{3}{4}$ and $m=\frac{1}{2}s$, the ${}_7F_6(1)$ series is evaluated in (P8) which then becomes

$${}_4F_3\left(\begin{matrix}\frac{1}{4},-\frac{1}{2}s,\frac{3}{4},-\frac{1}{2}-\frac{1}{2}s\\ \frac{3}{4}-\frac{1}{2}s,\frac{3}{2},\frac{1}{4}-\frac{1}{2}s\end{matrix}\ \middle|\ 1\right) = f(s), \tag{P10}$$

where

$$f(s) = \frac{(\frac{3}{2})_{\frac{1}{2}s}(\frac{5}{4})_{\frac{1}{2}s}(\frac{1}{2}-\frac{1}{2}s)_{\frac{1}{2}s}(-\frac{1}{2}s)_{\frac{1}{2}s}}{(\frac{3}{4})_{\frac{1}{2}s}(2)_{\frac{1}{2}s}(\frac{3}{4}-\frac{1}{2}s)_{\frac{1}{2}s}(-\frac{1}{4}-\frac{1}{2}s)_{\frac{1}{2}s}} \tag{P11}$$

after a little simplification. In turn,

$$f(s) = \frac{(\frac{3}{2})_{\frac{1}{2}s}(\frac{1}{2}-\frac{1}{2}s)_{\frac{1}{2}s}(1)_{\frac{1}{2}s}}{(\frac{3}{4})_{\frac{1}{2}s}(2)_{\frac{1}{2}s}(\frac{3}{4}-\frac{1}{2}s)_{\frac{1}{2}s}} \tag{P12}$$

noting that (i) $(-\frac{1}{2}s)_{\frac{1}{2}s} = (-1)^{\frac{1}{2}s}(1)_{\frac{1}{2}s}$ and, via the result

$$(x)_n = (-1)^n(-x-n+1)_n, \tag{P13}$$

that (ii) $(\frac{5}{4})_{\frac{1}{2}s} = (-1)^{\frac{1}{2}s}(-\frac{1}{4}-\frac{1}{2}s)_{\frac{1}{2}s}$. Moreover, from (P13) again,

$$\left(\frac{1}{2}-\frac{1}{2}s\right)_{\frac{1}{2}s} = (-1)^{\frac{1}{2}s}\left(\frac{1}{2}\right)_{\frac{1}{2}s},$$
$$\left(\frac{3}{4}-\frac{1}{2}s\right)_{\frac{1}{2}s} = (-1)^{\frac{1}{2}s}\left(\frac{1}{4}\right)_{\frac{1}{2}s}, \tag{P14}$$

whence

$$f(s) = \frac{(\frac{1}{2})_{\frac{1}{2}s}(1)_{\frac{1}{2}s}(\frac{3}{2})_{\frac{1}{2}s}}{(\frac{1}{4})_{\frac{1}{2}s}(\frac{3}{4})_{\frac{1}{2}s}(2)_{\frac{1}{2}s}}. \tag{P15}$$

To reduce $f(s)$ further still to a desired form, we deploy the identity

$$\left(\frac{1}{2}x\right)_n \left(\frac{1}{2}x+\frac{1}{2}\right)_n = 2^{-2n}(x)_{2n} \tag{P16}$$

twice with $x = \frac{1}{2}, 1$, $n = \frac{1}{2}s$, yielding

$$\begin{aligned} f(s) &= \frac{(1)_s(\frac{3}{2})_{\frac{1}{2}s}}{(\frac{1}{2})_s(2)_{\frac{1}{2}s}} \\ &= \frac{(\frac{1}{2}s+2)_{\frac{1}{2}s}}{(\frac{1}{2}s+\frac{1}{2})_{\frac{1}{2}s}} \end{aligned} \tag{P17}$$

after re-writing, and Case B is finished; we have, finally,

$$S(s) = c_s \frac{(\frac{1}{2}s+2)_{\frac{1}{2}s}}{(\frac{1}{2}s+\frac{1}{2})_{\frac{1}{2}s}}, \qquad s = 0, 2, 4, \ldots, \tag{P18}$$

combining (P3),(P10),(P17).

We are now in a position to complete the proof without difficulty. Since, as demonstrated by Case A, $S(s) = 0$ for s odd, (P1) is revised to read

$$E^2(x) = \sum_{s=0}^{\infty} S(2s)x^{2s}. \tag{P19}$$

Thus, in order to establish Identity 2 it is required merely to show that

$$\sum_{s=0}^{\infty} S(2s)x^{2s} = G(4x^2) = \sum_{s=0}^{\infty} c_s(4x^2)^s \tag{P20}$$

using (2)—in other words,

$$S(2s) = 4^s c_s = \frac{4^s}{s+1}\binom{2s}{s}, \qquad s \geq 0. \tag{P21}$$

From (P18),

$$\begin{aligned} S(2s) &= c_{2s}\frac{(s+2)_s}{(s+\frac{1}{2})_s} \\ &= \frac{1}{2s+1}\frac{(s+2)_s}{(s+\frac{1}{2})_s}\binom{4s}{2s}, \qquad s = 0, 1, 2, \ldots, \end{aligned} \tag{P22}$$

and (P21) duly follows (we leave this last step as a straightforward algebraic exercise for the reader).□

We end with a couple of remarks.

Remark 4 Equation (P2) gives that $S(2s) = \sum_{n=0}^{s} c_{2n}c_{2(s-n)}$. By (P21), therefore, an equivalent statement of Identity 2 is the binomial coefficient identity

$$\sum_{n=0}^{s} c_{2n}c_{2(s-n)} = 4^s c_s, \qquad s \geq 0, \tag{5}$$

a proof of which, via generating functions, is posed as Problem 6.C10(a) in [1] (see also its solution, where (5), as here, is commented upon in the context of Identity 2). Note that a similar type of result is set as 2(c) (p.44) in the Exercises section of [7], to which combinatorial and non-combinatorial proofs have been found (see p.52). Presently the question of a bijective proof of (5) remains an open one [1, Problem 6.C10(b)], as does that of Identity 1.3.

Remark 5 At the time of writing (November 2004), it is the August 2004 version of [1] to which reference is made here. In it, Identities 1.1,1.2 are cited as a private communication in the form of a preprint from Shapiro (published thereafter as [8]) entitled "Catalan Trigonometry"[2] in which a simple bijective proof of Identity 1.2 is given (a parallel argument is said to establish Identity 1.1). Other than in [1], to the author's knowledge Identity 2 has to date not appeared formally in the literature, although it can be recovered from a line in [8] using, for example, Identities 1.1,1.2 and Lemma 2 (see the Appendix for details). Other Catalan, and Catalan related, identities may be seen in [9].

Appendix

Here we recover Identity 2 from Shapiro [8, p.135] who writes that some algebraic manipulation results in the relation

$$g(x) = 2E(x) + \frac{O(x)}{x} - 2G(4x^2), \tag{A1}$$

where $g(x) = [E^2(x) - O^2(x)]/E(x)$ is a generating function defined (on p.132) in the context of a Riordan group. Re-arranging,

$$2G(4x^2) \quad = \quad 2E(x) + \frac{O(x)}{x} - g(x)$$

[2] So called because, according to Shapiro [8, p.130], these "[generating function] identities have a family resemblance [to] the double angle formulas for sine and cosine (or cosh and sinh)".

$$
\begin{aligned}
&= 2E(x) + \frac{O(x)}{x} - \frac{[E^2(x) - O^2(x)]}{E(x)} \\
&= E(x) + \frac{O(x)}{x} + \frac{O^2(x)}{E(x)} \\
&= E(x) + E^2(x) + O^2(x) + \frac{O^2(x)}{E(x)} && \text{(by Identity 1.2)} \\
&= 2E^2(x), && \text{(A2)}
\end{aligned}
$$

thus establishing Identity 2, if

$$E(x) + O^2(x) + \frac{O^2(x)}{E(x)} = E^2(x), \tag{A3}$$

that is to say, if

$$E^2(x) + E(x)O^2(x) + O^2(x) = E^3(x). \tag{A4}$$

To show (A4), consider the l.h.s.

$$
\begin{aligned}
&E^2(x) + E(x)O^2(x) + O^2(x) \\
&\quad = E^2(x) + [E(x) + 1]O^2(x) \\
&\quad = E^2(x) + [E(x) + 1]\left(\frac{E(x) - 1}{2xE(x)}\right)^2 && \text{(by Identity 1.1)} \\
&\quad = \frac{4x^2E^4(x) + [E(x) + 1][E(x) - 1]^2}{4x^2E^2(x)} \\
&\quad = \frac{4x^2E^4(x) + E^3(x) - E^2(x) - E(x) + 1}{4x^2E^2(x)} \\
&\quad = \frac{E^3(x) - E(x)}{4x^2E^2(x)} && \text{(by Lemma 2)} \\
&\quad = \frac{E^2(x) - 1}{4x^2E(x)} \\
&\quad = E^3(x), && \text{(A5)}
\end{aligned}
$$

as required, using Lemma 2 once more.

Acknowledgement

The author re-states his gratitude to Christian Krattenthaler for suggesting the proof of Identity 2 based on the package "HYP" developed by him.[3] We

[3]See **http://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html#HYP**.

have refrained from using a fully automated verification of the result (via an implementation of the now familiar and widely used WZ method of Wilf and Zeilberger (see, *e.g.*, [10] for information)) in favour of the approach taken here. Whilst supported by computation the latter is, structurally, in line with the manner in which the problem would have been tackled before the advent of symbolic software designed to deal with proofs of binomial coefficient identities, and in this respect is illuminating in its own way.

References

[1] http://www-math.mit.edu/~rstan/ec/catadd.pdf [this is the web address of the "Catalan Addendum" to work in [2]]

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