Factorizations of Complete Graphs into $[r, s, 2, 2]$-Caterpillars of Diameter 5

MICHAEL KUBESA

Technical University Ostrava

October 2003

ABSTRACT. A *caterpillar* R is a tree with the property that after deleting of all vertices of degree 1 we obtain a path P or a single vertex. The path P is called the *spine* of caterpillar R. If the spine has length 3 and R contains vertices of degrees $r, s, 2, 2$, where $r, s > 2$, then we say that R is a $[r, s, 2, 2]$-*caterpillar* of diameter 5. We completely characterize $[r, s, 2, 2]$-caterpillars of diameter 5 on $4k + 2$ vertices that factorize K_{4k+2}.

1. INTRODUCTION

Let G be a simple graph with at most n vertices. A graph H with n vertices has a *decomposition* into subgraphs $G_0, G_1, G_2, \ldots, G_s$ if each edge of H belongs to exactly one G_i. When all subgraphs $G_i, 0 \leq i \leq s$, are isomorphic to a graph G, we say that H has a G-*decomposition.* If G has exactly n vertices and none of them is isolated, then G is called a *factor* and the decomposition is called a G-*factorization* of H.

Graph factorizations have been extensively studied for many years. Special attention has been paid to isomorphic factorizations. Among graphs whose G-factorizations have been sought, the most popular ones are the obvious candidates—complete graphs and complete bipartite graphs (see, e.g., [2]). In this paper we concentrate on isomorphic factorizations of complete graphs into spanning trees and in particular into spanning caterpillars of diameter 5.

A simple arithmetic condition shows that only complete graphs with an even number of vertices can be factorized into spanning trees. Moreover, every spanning tree, which factorizes K_{2n}, satisfies the *maximum degree*

1991 *Mathematics Subject Classification.* 05C70, 05C78.

Key words and phrases. Decompositions and factorizations of a complete graphs, spanning trees, blended ρ-labeling, caterpillars.

condition, which means that for each vertex v in such a tree on $2n$ vertices it holds that $\deg(v) \leq n$.

It is a part of graph theory folklore that each graph K_{2n} can be factorized into hamiltonian paths P_{2n}. On the other hand, it is easy to observe that each K_{2n} can be also factorized into double stars; that is, two stars $K_{1,n-1}$ joined by an edge with the endvertices in the centers of both stars. The first attempt to fill the gap between these two extremal cases was P. Eldergill's thesis [1], where he dealt with symmetric trees. Some classes of non-symmetric trees were examined by Fronček [3,4], Fronček and the author [6], and by the author [7]. In [5] Fronček proves that some classes of caterpillars of diameter 4 and 5 do not factorize complete graphs of order $2n$. Results in this paper together with Fronček's results [5] give a complete characterization of caterpillars of diameter 5 and order $4k+2$ with exactly two vertices of degree 2 that factorize K_{4k+2}.

The labeling used in constructions in this paper exists only for graphs with $4k+2$ vertices. Therefore, we examine just a special class of caterpillars of diameter 5, namely the caterpillars of order $4k+2$ with exactly two vertices of degree 2. The reason why we do not present here a more general class is that the caterpillars with one or none vertex of degree 2 require many different and usually very long constructions. The results for the remaining classes are already in preparation.

2. Definitions and notation

A *labeling* of G with at most $2n+1$ vertices is an injection $\lambda : V(G) \to S$, where S is often a subset of the set $\{0, 1, ..., 2n\}$—however, in this paper we have $S = \{0_0, 1_0, \dots, (n-1)_0, 0_1, 1_1, \dots, (n-1)_1\}$. The labels of vertices u, v, denoted $\lambda(u) = i, \lambda(v) = j$, respectively, where $i, j \in S$, induce uniquely the *length* $\ell(e)$ of the edge $e = (u, v)$ with endvertices u, v. All labelings used here are generalizations of labelings introduced by A. Rosa [8,9].

The following definition was introduced in [4].

Let T be a tree with $2n = 4k+2$ vertices, $V(T) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$, and $|V_0| = |V_1| = 2k+1$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, 2_i, \dots, (2k)_i\}, i = 0, 1$. The *pure length* of an edge (x_i, y_i) with $x_i, y_i \in V_i, i \in \{0, 1\}$ is defined as follows: If $\lambda(x_i) = a_i$ and $\lambda(y_i) = b_i$, then $\ell_{ii}(x_i, y_i) = \min\{|a-b|, 2k+1-|a-b|\}$ for $i = 0, 1$. The *mixed length* of an edge (x_0, y_1) with $\lambda(x_0) = a_0$ and $\lambda(y_1) = b_1$, is defined as $\ell_{01}(x_0, y_1) = b - a \mod (2k+1)$ for $x_0 \in V_0, y_1 \in V_1$. We say that T has a *blended ρ-labeling* or just *blended labeling* if

(1) $\{\ell_{ii}(x_i, y_i) | (x_i, y_i) \in E(T)\} = \{1, 2, \dots, k\}$ for $i = 0, 1$,

(2) $\{\ell_{01}(x_0, y_1) | (x_0, y_1) \in E(T)\} = \{0, 1, 2, \dots, 2k\}$.

To simplify our notation, we often unify vertices with their respective labels. We will say "a vertex a_i" rather than "a vertex x with $\lambda(x) = a_i$". Similarly, we will say "an edge (a_i, b_j)" rather than "an edge xy, where $\lambda(x) = a_i$ and $\lambda(y) = b_j$".

Notice that the lengths of pure and mixed edges are computed differently. Suppose we have the complete graph K_{14} with the vertex labels $0_0, 1_0, \ldots, 6_0, 0_1, 1_1, \ldots 6_1$. Then both the edges $(1_0, 3_0)$ and $(1_0, 6_0)$ have the pure length 2. On the other hand, the edge $(1_0, 3_1)$ has the mixed length 2 while the edge $(1_1, 3_0)$ has the mixed length 5. Similarly, the edge $(1_0, 6_1)$ has the mixed length 5 while the edge $(6_0, 1_1)$ has the mixed length 2.

It was proved in [4] that a tree T of order $4k+2$ with a blended labeling allows a T-factorization of K_{4k+2}.

We want to characterize some classes of trees on $4k+2$ vertices of diameter 5, which allow a blended ρ-labeling. Since the factorization into hamiltonian paths P_{4k+2} is well-known, we start our work with the caterpillars. From now on we will only consider caterpillars with $4k+2$ vertices.

A tree R such that after deleting all leaves we obtain a path P or a trivial graph is called a *caterpillar*. The path P is called the *spine* of the caterpillar R.

It is clear that the caterpillars on $4k+2$ vertices of diameter 2 are the stars $K_{1,4k+1}$, which clearly do not satisfy the maximum degree condition. The caterpillars of order $4k+2$ with diameter 3 are the double stars mentioned above. Therefore, the first interesting case is the class of caterpillars of diameter 4. The results obtained in [5] and [7] give the complete characterization of the caterpillars of order $4k+2$ with diameter 4, which factorize the complete graphs K_{4k+2}. Hence, we continue with the caterpillars on $4k+2$ vertices of diameter 5. Recall that if R is a *caterpillar of diameter 5* then the spine of R has four vertices.

Let the spine of a caterpillar R of diameter 5 have vertices A, a, b, B and edges Aa, ab, bB. Then we see that the endvertices of the spine of R of diameter 5 are denoted by A, B and the internal vertices are denoted by a, b. If $\deg(A) = d_1, \deg(a) = d_2, \deg(b) = d_3, \deg(B) = d_4$, then such a caterpillar will be called a (d_1, d_2, d_3, d_4)*-caterpillar*. If we specify just the degrees of the vertices, say as $r_1 \geq r_2 \geq r_3 \geq r_4$, without specifying their location in the spine, then we will denote R as an $[r_1, r_2, r_3, r_4]$*-caterpillar*.

It is obvious that no $[4k-2, 2, 2, 2]$-caterpillar can factorize K_{4k+2} for any $k \geq 2$, because it does not satisfy the maximum degree condition. Note that $[4k-2, 2, 2, 2]$-caterpillar of order $4k+2$ for $k = 1$ is the hamiltonian path P_6.

Notice that we deal only with trees with $4k+2$ vertices, since trees with $4k$ vertices do not allow a blended labeling (see [6]). We conclude this section with the main result of this paper that will be proved in Section 3.

Theorem 2.1. *An $[r, s, 2, 2]$-caterpillar R with diameter 5 and $4k+2$ vertices factorizes K_{4k+2} if and only if $3 \leq r, s \leq 2k+1$ and R is not isomorphic to $(2k+1, 2, 2, 2k-1)$- or $(2k+1, 2, 2k-1, 2)$-caterpillars.*

3. $[r, s, 2, 2]$-CATERPILLARS FOR $3 \leq r, s \leq 2k+1$

If $r, s > 2$ then we see that a $[r, s, 2, 2]$-caterpillar R of order $4k+2$ has exactly two vertices $u, v \in \{A, a, b, B\}$ such that $\deg_R(u) = r > 2$ and $\deg_R(v) = s > 2$ for $k \geq 2$.

Notice that P. Eldergill in [1] complete characterized trees on 10 vertices that factorize K_{10}. Therefore we can deal only with $[r, s, 2, 2]$-caterpillars on $4k+2$ vertices for $k \geq 3$.

We express $\deg_R(u)$ and $\deg_R(v)$, respectively, as $x+2$ and $y+2$. If R satisfies the maximum degree condition then it holds that $x+y = 4k-4$ for $x \leq 2k-1, y \leq 2k-1$. Thus the solutions of previous equation are either

(1) $$x = 2k-1, y = 2k-3 \text{ or}$$

(2) $$x = 2k-2, y = 2k-2.$$

Therefore, we can further consider only $[2k+1, 2k-1, 2, 2]$- and $[2k, 2k, 2, 2]$-caterpillars. To prove Theorem 2.1, we will use the following result.

Theorem 3.1. (Fronček [5]) *Let R be a caterpillar of order $2n$ and diameter 5 that is isomorphic either to $(n, 2, 2, n-2)$- or $(n, 2, n-2, 2)$-caterpillar. Then R does not factorize K_{2n}.*

In our proofs we will use the following lemma.

Lemma 3.2. *Let T be a tree on $4k+2$ vertices, which allows a blended ρ-labeling. Then $\sum_{i \in V_0} \deg(i) = \sum_{j \in V_1} \deg(j) = 4k+1$.*

Proof. If a tree T of order $4k+2$ has a blended labeling then it has k pure 00-edges, k pure 11-edges and $2k+1$ mixed 01-edges. Each pure 00-edge contributes to the sum $\sum_{i \in V_0} \deg(i)$ by 2 and every mixed 01-edge by 1. Therefore $\sum_{i \in V_0} \deg(i) = 2k + 2k + 1 = 4k+1$. For the sum $\sum_{j \in V_1} \deg(j)$ the proof is essentially similar. □

Lemma 3.2 will be called the *degree condition for a blended labeling.*

Recall that every tree T with a blended labeling has vertices labeled so that $V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $V(T) = V_0 \cup V_1, V_0 \cap V_1 = \emptyset$. Therefore in all following constructions we assume that the vertices are already labeled and then join them by edges, keeping in mind that we need to construct the $[r, s, 2, 2]$-caterpillar while obtaining exactly one edge of each mixed length from 0 to $2k$ and exactly one edge of every pure length from 1 to k in each partite set.

Lemma 3.3. *All* $[2k+1, 2k-1, 2, 2]$*-caterpillars with the exception of* $(2k+1, 2, 2, 2k-1)$*- and* $(2k+1, 2, 2k-1, 2)$*-caterpillars factorize* K_{4k+2} *for* $k \geq 3$.

Proof. By constructions.

Case 1. Let R be a $(2k+1, 2k-1, 2, 2)$-caterpillar. Furthermore, let $V(R) = V_0 \cup V_1$, where $V_0 = \{0_0, 1_0, ..., (2k)_0\}, V_1 = \{0_1, 1_1, ..., (2k)_1\}$ and $A = 0_0, a = 0_1, b = (2k-2)_1, B = (2k)_1$.

The diametrical path P' of R is $(k+1)_0, 0_0, 0_1, (2k-2)_1, (2k)_1, (2k-1)_1$. Thus P' contains 00-edge of length k, 01-edge of length 0 and 11-edges of lengths $3, 2, 1$.

Further we join each vertex from the sets $\{(k+2)_0, (k+3)_0, ..., (2k)_0\}$, $\{1_1, 2_1, ..., k_1\}$ by an edge to the vertex $A = 0_0$ and each vertex from the sets $\{1_0, 2_0, ..., k_0\}$, $\{(k+1)_1, (k+2)_1, ..., (2k-3)_1\}$ to the vertex $a = 0_1$. We see that R also contains 00-edges of lengths $k-1, k-2, ..., 1$, 01-edges of lengths $1, 2, ..., k$ and $2k, 2k-1, ..., k+1$, and 11-edges of lengths $k, k-1, ..., 4$.

Case 2. Let R be a $(2, 2k+1, 2, 2k-1)$-caterpillar and let $A = (k+1)_1, a = 0_0, b = k_1, B = 0_1$.

Again first we construct the diametrical path P' of R so that $P' = (k+1)_0, (k+1)_1, 0_0, k_1, 0_1, (2k)_1$. Then P' contains 01-edges of lengths $0, k+1, k$ and 11-edges of lengths $k, 1$.

Further we join each vertex from the sets $\{k_0, (k+2)_0, (k+3)_0, ..., (2k)_0\}$, $\{1_1, 2_1, ..., (k-1)_1\}$ by an edge to the vertex $a = 0_0$ and each vertex from the sets $\{1_0, 2_0, ..., (k-1)_0\}$, $\{(k+2)_1, (k+3)_1, ..., (2k-1)_1\}$ to the vertex $B = 0_1$. Hence, R also contains 00-edges of lengths k and $k-1, k-2, ..., 1$, 01-edges of lengths $1, 2, ..., k-1$ and $2k, 2k-1, ..., k+2$, and 11-edges of lengths $k-1, k-2, ..., 2$.

Case 3. Let R be a $(2, 2k+1, 2k-1, 2)$-caterpillar and let $A = 2_1, a = 0_0, b = 0_1, B = (2k)_1$.

First we construct the diametrical path P' of R so that $P' = (k+2)_1, 2_1, 0_0, 0_1, (2k)_1, (k+1)_1$. Then P' contains 01-edges of lengths $0, 2$ and 11-edges of lengths k and $k-1, 1$.

Further we join each vertex from the sets $\{(k+1)_0, (k+2)_0, ..., (2k)_0\}$, $\{1_1, 3_1, 4_1, ..., k_1\}$ by an edge to the vertex $a = 0_0$ and each vertex from the sets $\{1_0, 2_0, ..., k_0\}$, $\{(k+3)_1, (k+4)_1, ..., (2k-1)_1\}$ to the vertex $b = 0_1$. Hence, R also contains 00-edges of lengths $k, k-1, k-2, ..., 1$, 01-edges of lengths $1, 3, 4, ..., k$ and $2k, 2k-1, ..., k+1$, and 11-edges of lengths $k-2, k-3, ..., 2$.

Case 4. Let R be a $(2k-1, 2k+1, 2, 2)$-caterpillar and let $A = 0_1, a = 0_0, b = 1_1, B = (2k)_1$.

We construct the diametrical path P' of R so that $P' = (2k-1)_1, (2k)_1, 1_1, 0_0, 0_1, (k+1)_1$. Then P' contains 01-edges of lengths $0, 1$ and 11-edges of lengths $1, 2$ and k.

Further we join each vertex from the sets $\{(k+1)_0, (k+2)_0, \ldots, (2k)_0\}$, $\{2_1, 3_1, \ldots, k_1\}$ by an edge to the vertex $a = 0_0$ and each vertex from the sets $\{1_0, 2_0, \ldots, k_0\}$, $\{(k+2)_1, (k+3)_1, \ldots, (2k-2)_1\}$ to the vertex $A = 0_1$. Hence, R also contains 00-edges of lengths $k, k-1, k-2, \ldots, 1$, 01-edges of lengths $2, 3, \ldots, k$ and $2k, 2k-1, \ldots, k+1$, and 11-edges of lengths $k-1, k-2, \ldots, 3$.

It is easy to check that the caterpillar R in each previous case has a blended labeling. Therefore R factorizes K_{4k+2}. □

Lemma 3.4. *Every* $[2k, 2k, 2, 2]$*-caterpillar factorizes* K_{4k+2} *for* $k \geq 3$.

Proof. By constructions.

Case 1. Since $(2k, 2, 2, 2k)$- and $(2, 2k, 2k, 2)$-caterpillars are symmetric, it follows from Eldergill's results [1] that they factorize K_{4k+2}.

Case 2. Let R be a $(2k, 2, 2k, 2)$-caterpillar and let $A = 0_0, a = k_0, b = 0_1, B = (k+1)_1$.

We construct the diametrical path P' of R so that $P' = (2k)_0, 0_0, k_0, 0_1, (k+1)_1, (k+1)_0$. Then P' contains 00-edges of lengths $1, k$, 01-edges of lengths $k+1$ and 0, and 11-edge of length k.

Further we join each vertex from the sets $\{(k+2)_0, (k+3)_0, \ldots, (2k-1)_0\}$, $\{1_1, 2_1, \ldots, k_1\}$ by an edge to the vertex $A = 0_0$ and each vertex from the sets $\{1_0, 2_0, \ldots, (k-1)_0\}$, $\{(k+2)_1, (k+3)_1, \ldots, (2k)_1\}$ to the vertex $b = 0_1$. Hence, R also contains 00-edges of lengths $k-1, k-2, \ldots, 2$, 01-edges of lengths $1, 2, \ldots, k$ and $2k, 2k-1, \ldots, k+2$, and 11-edges of lengths $k-1, k-2, \ldots, 1$.

Case 3. Let R be a $(2k, 2k, 2, 2)$-caterpillar and let $A = 0_0, a = (k+1)_1, b = (k+1)_0, B = 0_1$.

We construct the diametrical path P' of R so that $P' = k_0, 0_0, (k+1)_1, (k+1)_0, 0_1, k_1$. Then P' contains 00-edge of length k, 01-edges of lengths $k+1, 0, k$ and 11-edge of length k.

Then we join each vertex from the sets $\{1_0, 2_0, \ldots, (k-1)_0\}$, $\{1_1, 2_1, \ldots, (k-1)_1\}$ by an edge to the vertex $A = 0_0$ and each vertex from the sets $\{(k+2)_0, (k+3)_0, \ldots, (2k)_0\}$, $\{(k+2)_1, (k+3)_1, \ldots, (2k)_1\}$ to the vertex $a = (k+1)_1$. Hence, R also contains 00-edges of lengths $1, 2, \ldots, k-1$, 01-edges of lengths $1, 2, \ldots, k-1$ and $2k, 2k-1, \ldots, k+2$, and 11-edges of lengths $1, 2, \ldots, k-1$.

We see that the caterpillar R in the cases 2 and 3 has a blended labeling. Therefore R factorizes K_{4k+2}. □

From above it follows that every $[r, s, 2, 2]$-caterpillar for $3 \leq r, s \leq 2k+1$, with the exception of $(2k+1, 2, 2, 2k-1)$- and $(2k+1, 2, 2k-1, 2)$-caterpillars, factorizes K_{4k+2}. This result together with Theorem 3.1 and the maximum degree condition gives the proof of Theorem 2.1.

References

1. P. Eldergill, *Decompositions of the complete graph with an even number of vertices* (1997), M.Sc. thesis, McMaster University, Hamilton.
2. Yukio Shibata and Yasuo Seki, *The isomorphic factorization of complete bipartite graphs into trees*, Ars Combinatoria **33** (1992), 3–25.
3. D. Frončeк, *Cyclic decompositions of complete graphs into spanning trees*, Discussiones Mathematicae Graph Theory **24** (2004), 345–353.
4. D. Frončeк, *Bi-cyclic decompositions of complete graphs into spanning trees*, submitted for publication.
5. D. Frončeк, *Note on factorization of complete graphs into caterpillars with small diameters*, preprint.
6. D. Frončeк, M. Kubesa, *Factorizations of complete graphs into spanning trees*, Congressus Numerantium **154** (2002), 125–134.
7. M. Kubesa, *Spanning tree factorizations of complete graphs*, Journal of Combinatorial Mathematics and Combinatorial Computing, accepted for publication.
8. A. Rosa, *Cyclic decompositions of complete graphs* (1965), Ph.D. thesis, Slovak Academy of Science, Bratislava.
9. A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Intl. Symp. Rome 1966), Gordon and Breach, Dunod, Paris, 1967, pp. 349–355.

Department of Applied Mathematics, Technical University Ostrava, 17.listopadu, Ostrava-Poruba 708 33, Czech Republic

E-mail address: michael.kubesa@vsb.cz