An upper bound for the domination number of a graph in terms of order and girth

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Abstract

A vertex set D of a graph G is a dominating set if every vertex not in D is adjacent to some vertex in D. The domination number γ of a graph G is the minimum cardinality of a dominating set in G. In 1989, Brigham and Dutton [1] proved

$$\gamma \leq \left\lceil \frac{3n-g}{6} \right\rceil$$

for each graph G of order n, minimum degree $\delta \geq 2$, and girth $g \geq 5$. For this class of graphs, Volkmann [8] recently gave the better bound

$$\gamma \leq \left\lceil \frac{3n-g-6}{6} \right\rceil,$$

if G is neither a cycle nor one of two exceptional graphs.

If G is a graph of order n, minimum degree $\delta \geq 2$, girth $g \geq 5$, then we show in this paper that

$$\gamma \leq \left\lceil \frac{3n-g-9}{6} \right\rceil,$$

if G is neither a cycle nor one of 40 exceptional graphs of order between 8 and 21.

AMS Subject Classification: 05C69

Keywords: Domination number; Girth of a graph

1. Terminology

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N_G(v)$ of the vertex v consists of the vertices adjacent to v, and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N_G[S] = N(S) \cup S$. The vertex v is an *end-vertex* if $d_G(v) = 1$, and an *isolated vertex* if $d_G(v) = 0$, where $d(v) = d_G(v) = |N(v)|$ is the degree of $v \in V(G)$. An edge incident with an end-vertex is called a *pendant edge*. By $\delta = \delta(G)$ we denote the *minimum degree* of the graph G. The *distance* $d_G(x,y) = d(x,y)$ between two vertices x and y of a graph G is the length of a shortest path from x to y. The *girth* $g = g(G)$ of a graph G is the length of a shortest cycle of G. We write C_n for a cycle of length n and K_n for the complete graph of order n. A cycle with length n is also called an n-cycle. If G is a graph and k a positive integer, then let kG be the disjoint union of k copies of G.

A set $D \subseteq V(G)$ is a *dominating set* of G if $N_G[D] = V(G)$. The *domination number* $\gamma = \gamma(G)$ of G is the cardinality of any smallest dominating set.

The *corona* $H \circ K_1$ of the graph H is the graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added.

For detailed information on domination and related topics see the comprehensive monograph [3] by Haynes, Hedetniemi, and Slater.

In 1989, Brigham and Dutton [1] proved

$$\gamma(G) \leq \left\lceil \frac{3n(G) - g(G)}{6} \right\rceil$$

for each graph G of minimum degree $\delta \geq 2$ and girth $g(G) \geq 5$ (for a proof of this inequality cf. also [3], pp. 56-57). For this class of graphs, Volkmann [8] recently gave the better bound

$$\gamma(G) \leq \left\lceil \frac{3n(G) - g(G) - 6}{6} \right\rceil,$$

if G is not a cycle and not isomorphic to the graphs $2C_7$ or B_8 (see the figure below).

If G is a graph of minimum degree $\delta \geq 2$ and girth $g(G) \geq 5$, then we show in this paper that

$$\gamma(G) \leq \left\lceil \frac{3n(G) - g(G) - 9}{6} \right\rceil,$$

if G is not a cycle and not isomorphic to a member of the family $\mathcal{A} \cup \mathcal{B} \cup \mathcal{D} \cup \mathcal{L}$ of 40 exceptional graphs defined in the next section.

2. Preliminary results

The following well-known results play an important role in our investigations.

Proposition 2.1 (Ore [4] 1962). If G is a graph without isolated vertices, then

$$\gamma(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor.$$

Theorem 2.2 (Payan, Xuong [5] 1982, Fink, Jacobson, Kinch, Roberts [2] 1985). For a graph G with even order n and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.

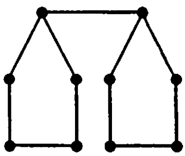
In 1998, Randerath and Volkmann [6] and independently, in 2000, Xu, Cockayne, Haynes, Hedetniemi, and Zhou [9] (cf. also [3], pp. 42-48) characterized the odd order graphs G for which $\gamma(G) = \lfloor n/2 \rfloor$. In the next theorem we only note the part of this characterization which we will use in Section 3.

Theorem 2.3 (Randerath, Volkmann [6] 1998 and Xu, Cockayne, Haynes, Hedetniemi, Zhou [9] 2000). Let G be a connected graph of odd order n with $\delta(G) \geq 2$. Then $\gamma(G) \leq (n-3)/2$, unless $G = C_5$, $G = C_7$, or G belongs to a family of 10 graphs of order at most 7 with girth less than or equal 4.

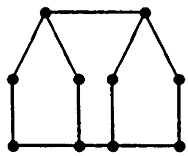
Theorem 2.4 (Volkmann [8] 2005) Let G be a graph of order n, minimum degree $\delta \geq 2$, and girth $g \geq 5$. If G is not a cycle and not isomorphic to $2C_7$ or to B_8 (see the figure), then

$$\gamma \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil.$$

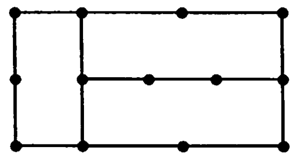
A proof of Theorems 2.2 can also be found in [7], pp. 223-224. In order to formulate our main result, we define a collection of graphs in the following figure.



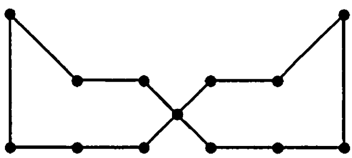
A_1



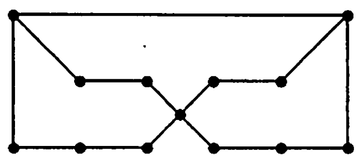
A_2



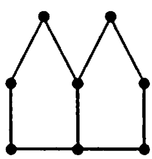
A_3



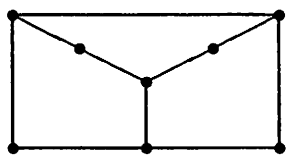
A_4



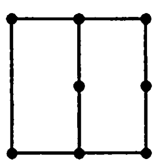
A_5



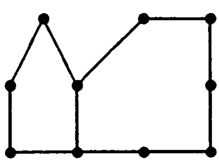
$B_1 = C_{5,5,8}$



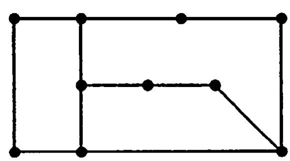
A_6



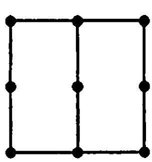
$B_2 = C_{5,6,7}$



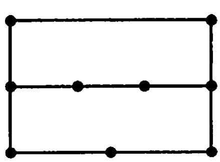
$B_3 = C_{5,7,10}$



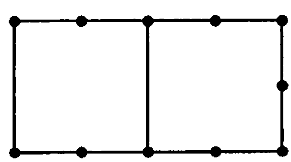
A_7



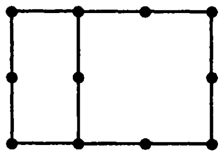
$B_4 = C_{6,6,8}$



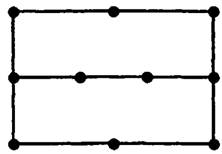
$B_5 = C_{6,7,7}$



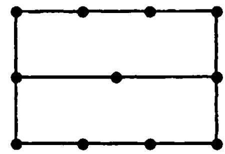
$B_6 = C_{6,7,11}$



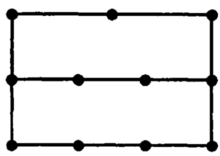
$B_7 = C_{6,8,10}$



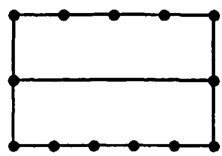
$B_8 = C_{7,7,8}$



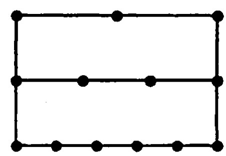
$B_9 = C_{7,7,10}$



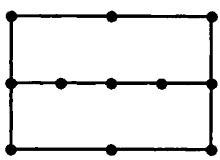
$B_{10} = C_{7,8,9}$



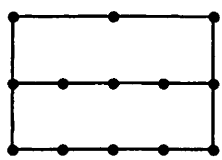
$B_{11} = C_{7,8,13}$



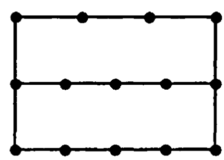
$B_{12} = C_{7,10,11}$



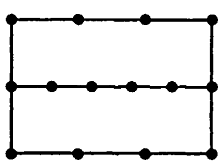
$B_{13} = C_{8,8,8}$



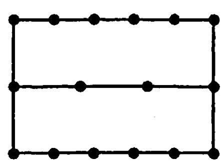
$B_{14} = C_{8,10,10}$



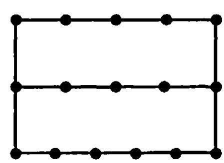
$B_{15} = C_{9,10,11}$



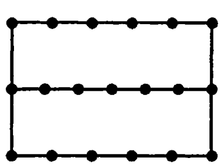
$B_{16} = C_{10,10,10}$



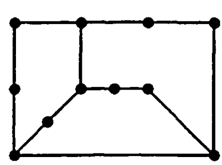
$B_{17} = C_{10,10,14}$



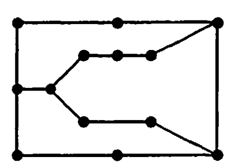
$B_{18} = C_{10,11,13}$



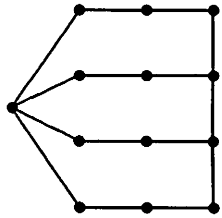
$B_{19} = C_{13,13,14}$



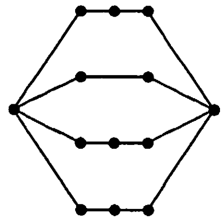
A_8



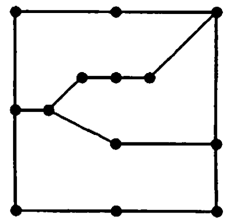
D_1



D_2



D_3



D_4

The denotation $C_{p,q,r}$ in this figure means that the corresponding graph has exactly three cycles of length p, q, and r. This notation is helpful to see that the graphs $B_1, B_2, \ldots, B_{19}$ are not isomorphic.

We define by $\mathcal{A} = \{A_1, A_2, \ldots, A_8\}$, $\mathcal{B} = \{B_1, B_2, \ldots, B_{19}\}$, and $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$. In addition, let $\mathcal{L}$ be the the family of graphs, consisting of the nine graphs $2C_5$, $2C_7$, $2C_{10}$, $3C_7$, $C_5 \cup C_7$, $C_6 \cup C_7$, $C_7 \cup C_8$, $C_7 \cup C_{10}$, and $C_7 \cup B_8$.

3. Main result

Theorem 3.1 Let G be a graph of order n, minimum degree $\delta \geq 2$, and girth $g \geq 5$. If G is not a cycle and not isomorphic to a member of the family $\mathcal{A} \cup \mathcal{B} \cup \mathcal{D} \cup \mathcal{L}$, then

$$\gamma \leq \left\lceil \frac{3n - g - 9}{6} \right\rceil \tag{1}$$

Proof. It is straightforward to verify that the graphs of the family $\mathcal{A} \cup \mathcal{B} \cup \mathcal{D} \cup \mathcal{L}$ do not satisfy inequality (1).

Observe that in general, a g-cycle can be dominated by $\lceil g/3 \rceil$ vertices. Assume that G is not a cycle, and remove a g-cycle C_g from G to a form a graph H. Since $g \geq 5$ and $\delta \geq 2$, the graph H has minimum degree at least $\delta - 1 \geq 1$.

Case 1. Assume that one of the components F of the graph H is of odd order such that $\delta(F) \geq 2$. According to Theorem 2.3, we conclude that $\gamma(F) \leq (n(F) - 3)/2$ or $F = C_5$ or $F = C_7$. In the first case, Proposition 2.1 implies

$$\gamma \leq \left\lfloor \frac{n(H) - 3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil = \left\lfloor \frac{n - g - 3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 9}{6} \right\rceil,$$

and inequality (1) is proved. Therefore it remains to consider the cases that $F = C_5$ or $F = C_7$ and thus $5 \le g \le 7$.

Subcase 1.1. Assume that $H = F$. If $F = C_5$, then it follows that $C_g = C_5$. If G is disconnected or there exists exactly one edge between F and C_g, then we obtain the exceptional graphs $2C_5$ or A_1. If there are at least two edges between F and C_g, then, since $g = 5$, we arrive at the exceptional graph A_2 or we obtain the desired inequality

$$\gamma \le 3 = \left\lceil \frac{16}{6} \right\rceil = \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Assume now that $F = C_7$. If there is no edge between F and C_g, then we arrive at the exceptional graphs $2C_7$, $C_5 \cup C_7$, or $C_6 \cup C_7$. However, in the case that G is connected, it is a simple matter to show that inequality (1) is valid.

Subcase 1.2. Assume that the graph H has a further component F_1 of even order such that $\delta(F_1) \ge 2$. According to Theorem 2.2, we deduce that $\gamma(F_1) \le \lfloor (n(F_1) - 2)/2 \rfloor$. Since $\gamma(F) \le \lfloor (n(F) - 1)/2 \rfloor$, Proposition 2.1 leads to the desired bound

$$\gamma \le \left\lfloor \frac{n(H)-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \le \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Subcase 1.3. Assume that the graph H has two further components F_1 and F_2 of odd order. In view of Proposition 2.1, we observe that $\gamma(F_i) \le \lfloor (n(F_i) - 1)/2 \rfloor$ for $i = 1, 2$, and we arrive at (1) as in Subcase 1.2.

Subcase 1.4. Assume that H has a further component F_1 of even order such that $\delta(F_1) = 1$. If F_1 is not a corona graph, then, by Theorem 2.2, $\gamma(F_1) \le \lfloor (n(F_1)-2)/2 \rfloor$. Analogously to Subcase 1.2, we obtain the desired result.

It remains the case that F_1 is a corona graph. Firstly, assume that $n(F_1) \ge 4$ and let u be an end-vertex of F_1. Because of $\delta \ge 2$, the vertex u is adjacent with a vertex $x \in V(C_g)$. If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. Since $F_1 - u$ is connected and of odd order at least three, Proposition 2.1 implies $\gamma(F_1 - u) \le \lfloor (n(F_1 - u) - 1)/2 \rfloor$. This easily leads to (1).

Secondly, assume that $V(F_1) = \{u, v\}$. Because of $\delta \ge 2$, the vertices u and v have neighbors x and y in C_g. Since $g \ge 5$, we conclude that $d_G(x, y) \ge 2$ and $g \le 6$.

If $g = 5$, then $\{x, y\}$ is a dominating set of $V(C_g) \cup \{u, v\}$. In the case that $F = C_5$, Proposition 2.1 yields

$$\gamma \le 2 + \left\lfloor \frac{n-g-7}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil = \left\lfloor \frac{n-g-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \le \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

In the case that $F = C_7$, Proposition 2.1 also leads to the desired result

$$\gamma \leq 3 + \left\lfloor \frac{n-g-9}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil = \left\lfloor \frac{n-g-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

If $g = 6$, then $d_{C_g}(x,y) = 3$ and $\{x,y\}$ is again a dominating set of $V(C_g) \cup \{u,v\}$. Furthermore, we deduce that $F = C_7$, and (1) follows as in the last case.

Subcase 1.5. Assume that H has exactly one further component F_1 of odd order.

Subcase 1.5.1. Assume that $\delta(F_1) \geq 2$. If $F_1 \neq C_5, C_7$, then we obtain (1) as in Case 1. If $C_g = C_5 = F = F_1$, $C_g = C_5 = F$ and $F_1 = C_7$, $C_g = C_5 = F_1$ and $F = C_7$, $C_g = C_5$ and $F = F_1 = C_7$, or $C_g = C_6$ and $F = F_1 = C_7$, then it is easy to see that (1) is valid. Now let $F_1 = F = C_g = C_7$. In this case we have the exceptional graph $3C_7$ or there exists an edge between two of these cycles. But in the last case, we observe that $\gamma \leq 8 = \lceil (3n-g-9)/6 \rceil$.

Subcase 1.5.2. Assume that $\delta(F_1) = 1$. If u is an end-vertex of F_1, then u is adjacent with a vertex $x \in V(C_g)$.

If $F_1 - u$ is not a corona graph, then we conclude from Theorem 2.2 the inequality $\gamma(F_1 - u) \leq \lfloor (n(F_1 - u) - 2)/2 \rfloor$. Applying Proposition 2.1, we obtain

$$\gamma \leq \left\lfloor \frac{n(H)-4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-12}{6} \right\rceil$$

and (1) is proved.

Finally, assume that $F_1 - u$ is a corona graph. Let $n(F_1) = 2p+1$ with $p \geq 1$. If $C_g = C_5$ and $F = C_5$ or $F = C_7$, then we deduce inequality (1) as follows:

$$\gamma \leq p + 2 + \left\lfloor \frac{n(F)}{2} \right\rfloor = \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

This is also valid if $C_g = C_6$ and thus $F = C_7$. So it remains the case that $C_g = C_7$ and $F = C_7$. If there is an edge between C_g and F, then it is straightforward to verify that

$$\gamma \leq p + 5 = \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Consequently, we consider now the case that $F = C_7$ is a component of G.

Subcase 1.5.2.1. Assume that u is adjacent to a vertex of $F_1 - u$ which is no end-vertex of $F_1 - u$. This implies that $n(F_1 - u) \geq 4$. Since $F_1 - u$ is a connected corona graph, there exist two further end-vertices v and w of F_1 such that $d_G(u,v) = 2$ and $d_G(u,w) = 3$. If $x,y,z \in V(C_g)$ are the neighbors of u,v,w, respectively, then it is straightforward to verify that

$\{x, y, z\}$ is a dominating set of $V(C_g) \cup \{u, v, w\}$. Since $F_1 - \{u, v, w\}$ is connected, Proposition 2.1 implies $\gamma(F_1 - \{u, v, w\}) \leq (n(F_1 - \{u, v, w\})/2$. Altogether, we obtain

$$\gamma \leq 3 + \left\lfloor \frac{n(F_1) - 3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil = \left\lfloor \frac{n(H) - 4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 12}{6} \right\rceil.$$

Subcase 1.5.2.2. Assume that u is adjacent to an end-vertex of $F_1 - u$. If $n(F_1) = 3$, then we arrive at the exceptional graph $C_7 \cup B_8$. If $n(F_1) \geq 5$, then there exists a further end-vertex v of F_1 with $d_G(u, v) = 4$. If y is adjacent with $y \in V(C_g)$, then we can choose a minimum dominating set D_g of C_g such that $x, y \in D_g$. Since $F_1 - \{u, v\}$ is connected and of odd order, Proposition 2.1 implies $\gamma(F_1 - \{u, v\}) \leq (n(F_1) - 3)/2$. This leads to (1) as in Subcase 1.5.2.1.

Case 2. Assume that one of the components F of H is of even order such that $\delta(F) \geq 2$, and H has no component of odd order and minimum degree at least two.

Subcase 2.1. Assume that $H = F$. The hypothesis $g \geq 5$ implies $n(F) \geq 6$. Since $\delta(F) \geq 2$, Theorem 2.2 yields $\gamma(F) \leq \lfloor (n(F) - 2)/2 \rfloor$, and we obtain

$$\gamma \leq \left\lfloor \frac{n(F) - 2}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 6}{6} \right\rceil. \quad (2)$$

If $g = 3s$ or $g = 3s + 2$, then n and s are of the same parity, and we observe that

$$\left\lceil \frac{3n - g - 6}{6} \right\rceil = \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

This identity and inequality (2) lead to the desired bound (1) in these two cases. In the remaining case that $g = 3s + 1 \geq 7$, we investigate three cases.

Subcase 2.1.1. Assume that F is a cycle C_p of length $p \geq g \geq 7$.

Subcase 2.1.1.1. Assume that $p = 3t$ with $t \geq 3$. Since F is of even order, we observe that $t \geq 4$ is even and this leads to

$$\gamma \leq \left\lceil \frac{g}{3} \right\rceil + \left\lceil \frac{p}{3} \right\rceil = s + 1 + t \leq \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

Subcase 2.1.1.2. Assume that $p = 3t + 1$ with $t \geq 2$. Since F is of even order, we observe that $t \geq 3$ is odd. In the case that $t \geq 5$, we arrive at (1) as follows:

$$\gamma \leq \left\lceil \frac{g}{3} \right\rceil + \left\lceil \frac{p}{3} \right\rceil = s + 1 + t + 1 \leq \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

There remain the cases $p = 10$ and $g = 7$ or $g = 10$. If C_g and $F = C_{10}$ are two components of G, then we have the exceptional graphs $C_7 \cup C_{10}$ or

$2C_{10}$. However, if there is an edge between C_g and $F = C_{10}$, then (1) is immediate.

Subcase 2.1.1.3. Assume that $p = 3t+2 \geq 8$. In the case that $p \geq 14$, we arrive at (1) as in the last case. There remains the cases $p = 8$ and $g = 7$. If $C_g = C_7$ and $F = C_8$ are components of G, then we have the exceptional graph $C_7 \cup C_8$. If there is an edge between $C_g = C_7$ and $F = C_8$, then (1) is immediate.

Subcase 2.1.2. Assume that $F = B_8$. Since B_8 has a cycle of length 7, it follows that $g = 7$. If $C_g = C_7$ and $F = B_8$ are two components of G, then we have the exceptional graph $C_7 \cup B_8$. If there is an edge between $C_g = C_7$ and $F = B_8$, then it is easy to see that (1) is valid.

Subcase 2.1.3. Assume that F is neither a cycle nor isomorphic to B_8. Because of $\delta(F) \geq 2$, there is a cycle C_p of length $p \geq g \geq 7$ in F with the property that p is the girth of F. Because of $p \geq g = 3s + 1$ and $s \geq 2$, Theorem 2.4 leads to

$$\begin{aligned}
\gamma &\leq s+1+\left\lceil \frac{3n(F)-p-6}{6} \right\rceil \\
&\leq s+1+\left\lceil \frac{3n-9s-3-p-6}{6} \right\rceil \\
&\leq s+1+\left\lceil \frac{3n-9s-3-3s-1-6}{6} \right\rceil \\
&= \left\lceil \frac{3n-6s-4}{6} \right\rceil \\
&= \left\lceil \frac{3n-3s-1-3s-3}{6} \right\rceil \\
&\leq \left\lceil \frac{3n-g-9}{6} \right\rceil .
\end{aligned}$$

Subcase 2.2. Assume that H has a further component F_1 of odd order. Because of $\delta(F) \geq 2$, Theorem 2.2 and Proposition 2.1 yield immediately

$$\gamma \leq \left\lfloor \frac{n(H)-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-9}{6} \right\rceil .$$

Subcase 2.3. Assume that H has a further component F_1 of even order.

Subcase 2.3.1. Assume that F_1 is not a corona graph. Analogously to Subcase 2.2, we even obtain

$$\gamma \leq \left\lfloor \frac{n(H)-4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-12}{6} \right\rceil .$$

Subcase 2.3.2. Assume that F_1 is a corona graph such that $n(F_1) \geq 4$. If u is an end-vertex of F_1, then u is adjacent with a vertex $x \in V(C_g)$.

If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. Since $F_1 - u$ is connected and of odd order at least three, Proposition 2.1 implies

$$\gamma(F_1 - u) \leq \left\lfloor \frac{n(F_1 - u) - 1}{2} \right\rfloor = \left\lfloor \frac{n(F_1) - 2}{2} \right\rfloor.$$

This leads to (1) as in Subcase 2.3.1.

Subcase 2.3.3. Assume that F_1 is a corona graph such that $n(F_1) = 2$. Let $V(F_1) = \{u, v\}$. Because of $\delta \geq 2$, the vertices u and v have neighbors x and y in C_g. Since $g \geq 5$, we conclude that $d_{C_g}(x, y) \geq 2$ and $g \leq 6$. If $g = 5$, then $\{x, y\}$ is a dominating set of $V(C_g) \cup \{u, v\}$, and we conclude that

$$\gamma \leq 2 + \left\lfloor \frac{n-9}{2} \right\rfloor \leq \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

If $g = 6$, then $d_{C_g}(x, y) = 3$ and $\{x, y\}$ is again a dominating set of $V(C_g) \cup \{u, v\}$. Hence we obtain

$$\gamma \leq 2 + \left\lfloor \frac{n-10}{2} \right\rfloor \leq \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

Case 3. Assume that all components of H have minimum degree one, and let F be such a component of odd order.

Subcase 3.1. Assume that $F = H$. Let u be an end-vertex of F. If $F - u$ is not a corona graph, then let $x \in V(C_g)$ be a neighbor of u. If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. Since $F - u$ is of even order and not a corona graph, it follows from Theorem 2.2 that $\gamma(F - u) \leq \lfloor (n(F - u) - 2)/2 \rfloor$. This leads to

$$\gamma \leq \left\lfloor \frac{n(H) - 3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

In the case that $F - u$ is a corona graph, we distinguish four cases.

Subcase 3.1.1. Assume that $n(F - u) = 2$. Let $V(F - u) = \{v, w\}$ such that v is adjacent with u. Since u and w are adjacent with vertices in C_g, we deduce that $g \leq 8$.

If $g = 5$, then we arrive at the three exceptional graphs B_1, A_6, and B_2.

If $g = 6$, then we arrive at the two exceptional graphs B_4 and B_5.

If $g = 7$, then we arrive at the exceptional graph B_8, and if $g = 8$, then we arrive at the exceptional graph B_{13}.

Subcase 3.1.2. Assume that $n(F - u) \geq 4$ and u is adjacent with a vertex in $F - u$ which is no end-vertex of $F - u$. If w is an end-vertex of $F - u$ with $d_F(u, w) \geq 3$, then $F - w$ is not a corona graph. Applying

Theorem 2.2., we observe that $\gamma(F-w) \le \lfloor (n(F-w)-2)/2 \rfloor$. Now let $y \in V(C_g)$ be a neighbor of w. If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $y \in D_g$, then D_g dominates the vertex w. Altogether, we obtain

$$\gamma \le \left\lfloor \frac{n(H)-3}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \le \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Subcase 3.1.3. Assume that $n(F-u) \ge 6$ and u is adjacent with an end-vertex of $F-u$.

If there exists an end-vertex $w \ne u$ of F such that $d_F(u,w) = 5$, then $F-w$ is not a corona graph. Analogously to Subcase 3.1.2 we arrive at (1).

Otherwise, it follows that $n(F-u) = 6$ and thus $g \le 9$. In the case $7 \le g \le 9$, it is a straightforward to verify that

$$\gamma \le 5 = \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

If $g = 6$, then we arrive at the exceptional graph A_3 or the bound (1). In the remaining case $g = 5$, we observe that $\gamma \le 4$ and (1) is also valid.

Subcase 3.1.4. Assume that $n(F-u) = 4$ and u is adjacent with an end-vertex of $F-u$. This implies that F is a path $uu_1u_2u_3w$ of length 4 and that $g \le 12$. Let x and y be the neighbors of u and w on the cycle C_g, respectively.

Subcase 3.1.4.1. Assume that $g = 3t+1$ with $t \ge 2$. Since G is not a cycle and not isomorphic to $2C_7$ or to B_8, it follows from Theorem 2.4 that

$$\gamma \le \left\lceil \frac{3n-g-6}{6} \right\rceil = t+2 = \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Subcase 3.1.4.2. Assume that $g = 3t+2$ with $t \ge 1$. If $g = 11$, then we observe that $d_{C_g}(x,y) = 5$. Because of $\gamma \le 5$, inequality (1) holds. If $g = 8$ and $2 \le d_{C_g}(x,y) \le 3$, then $\gamma \le 4$ and (1) is valid. Since $d_{C_g}(x,y) \le 1$ is not possible, it remains the case that $d_{C_g}(x,y) = 4$ and there is no further edge in G. In this case we arrive at the exceptional graph B_{14}. If $g = 5$ and $d_{C_g}(x,y) = 0$ or $d_{C_g}(x,y) = 2$, then it is a simple matter to show that the desired inequality holds. If $g = 5$ and $d_{C_g}(x,y) = 1$, then we obtain the exceptional graphs B_3 and A_7.

Subcase 3.1.4.3. Assume that $g = 3t$ with $t \ge 2$. If $g = 12$, then we observe that $d_{C_g}(x,y) = 6$. Because of $\gamma \le 5$, inequality (1) is immediate. If $g = 9$, then $3 \le d_{C_g}(x,y) \le 4$. If $d_{C_g}(x,y) = 3$, then $\gamma \le 4$ and (1) is also valid. In the case that $d_{C_g}(x,y) = 4$, there is no further edge, and we arrive at the exceptional graph B_{15}. If $g = 6$ and $d_{C_g}(x,y) = 0$ or $d_{C_g}(x,y) = 3$, we easily obtain (1). If $g = 6$ and $d_{C_g}(x,y) = 1$ or $d_{C_g}(x,y) = 2$, then we arrive at the exceptional graphs B_6, A_8, or B_7, respectively.

Subcase 3.2. Assume that there exists a further component F_1 of even order such that $\delta(F_1) = 1$. If F_1 is not a corona graph, then Theorem 2.2 and Proposition 2.1 imply inequality (1) analogously to above.

Subcase 3.2.1. Let F_1 be a corona graph with $V(F_1) = \{u, v\}$. It follows that $g \leq 6$. If x and y are the neighbors of u and v on C_g, respectively, then $\{x, y\}$ is a dominating set of $V(C_g) \cup \{u, v\}$. If $g = 5$, then we conclude together with Proposition 2.1 that

$$\gamma \leq 2 + \left\lfloor \frac{n-8}{2} \right\rfloor \leq \left\lceil \frac{3n-g-9}{6} \right\rceil,$$

and if $g = 6$, then we deduce that

$$\gamma \leq 2 + \left\lfloor \frac{n-9}{2} \right\rfloor \leq \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

Subcase 3.2.2. Let F_1 be a corona graph with $n(F_1) \geq 4$. If u is an end-vertex of F_1, then u is adjacent with a vertex $x \in V(C_g)$. If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $x \in D_g$, then D_g dominates the vertex u. Since $F_1 - u$ is connected and of odd order at least three, Proposition 2.1 implies

$$\gamma(F_1 - u) \leq \left\lfloor \frac{n(F_1 - u) - 1}{2} \right\rfloor = \left\lfloor \frac{n(F_1) - 2}{2} \right\rfloor.$$

This leads to (1) as above.

Subcase 3.3. Assume that there exists a further component F_1 of odd order such that $\delta(F_1) = 1$. In the case that there is a third component of odd order, the desired result is immediate. Hence we next discuss the case that H consists of the two components F and F_1. Let u and v be end-vertices of F and F_1, respectively. If one of $F - u$ and $F_1 - v$ is not a corona graph, then we arrive at (1) analogously to above. Thus it remains the case that $F - u$ and $F_1 - v$ are both corona graphs. Proposition 2.1 leads to

$$\gamma \leq \left\lfloor \frac{n(F) - 1}{2} \right\rfloor + \left\lfloor \frac{n(F_1) - 1}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-6}{6} \right\rceil. \quad (3)$$

If $g = 3s$ or $g = 3s + 2$, then n and s are of the same parity, and we observe that

$$\left\lceil \frac{3n-g-6}{6} \right\rceil = \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

This identity and inequality (3) lead to the desired bound (1) in these two cases. In the remaining case that $g = 3s + 1 \geq 7$, we investigate two cases.

Subcase 3.3.1. Assume that $F-u$ or F_1-v, say $F-u$, has at least four vertices. If u and w are two end-vertices of F, then let x and y their neighbors on C_g. Because of $g=3s+1$, there is a exists a minimum dominating set D_g of C_g with $x,y\in D_g$. Since $F-\{u,w\}$ is connected and of odd order at least three, Proposition 2.1 implies $\gamma(F-\{u,w\})\leq\lfloor(n(F)-3)/2\rfloor$, and we conclude that

$$\gamma\leq\left\lfloor\frac{n(F)-3}{2}\right\rfloor+\left\lfloor\frac{n(F_1)-1}{2}\right\rfloor+\left\lceil\frac{g}{3}\right\rceil\leq\left\lceil\frac{3n-g-12}{6}\right\rceil.$$

Subcase 3.3.2. Assume that $n(F)=n(F_1)=3$ and $g=3s+1$. It follows that $5\leq g\leq 8$ and thus $g=7$. Let $C_g=x_1x_2x_3x_4x_5x_6x_7x_1$, $F=u_1u_2u_3$, and $F_1=v_1v_2v_3$. Assume, without loss of generality, that x_1u_1 and x_4u_3 are edges of G. If $v_1x_1\in E(G)$, then we have the two possibilities that $v_3x_5\in E(G)$ or $v_3x_4\in E(G)$. These two graphs are isomorphic two D_2 or D_3, respectively. If $v_1x_2\in E(G)$, then $v_3x_5\in E(G)$ or $v_3x_6\in E(G)$. In the case that $v_3x_5\in E(G)$, we arrive at the exceptional graph D_4. In the other case that $v_3x_6\in E(G)$, we see that $\{u_1,v_1,x_4,x_6\}$ is a dominating set of G and (1) is valid. If $v_1x_3\in E(G)$, then $v_3x_6\in E(G)$ or $v_3x_7\in E(G)$. If $v_3x_6\in E(G)$, then $\{u_3,v_1,x_1,x_6\}$ is a dominating set of G and (1) is valid. In the case that $v_3x_7\in E(G)$, we again arrive at D_4. If $v_1x_4\in E(G)$ and $v_3x_7\in E(G)$, then G is also isomorphic to D_2. By symmetry, the remaining cases are immediate.

Case 4. Assume that all components of H are of even order and minimum degree one. Let F be such a component.

Subcase 4.1. Assume that $F=H$. Let u be an end-vertex of F, and let $x\in V(C_g)$ be a neighbor of u.

Subcase 4.1.1. Assume that $V(F)=\{u,v\}$. It follows that $g\leq 6$. If $y\in V(C_g)$ is a neighbor of v, then $\{x,y\}$ is a dominating set of G and the desired inequality is immediate.

Subcase 4.1.2. Assume that $n(F-u)\geq 3$ and $g=3s+2$ or $g=3s$. If we choose, without loss of generality, a minimum dominating set D_g of C_g such that $x\in D_g$, then D_g dominates the vertex x. Because of $\gamma(F-u)\leq\lfloor(n(F-u)-1)/2\rfloor$, we deduce in these cases as above that

$$\gamma\leq\left\lfloor\frac{n(H)-2}{2}\right\rfloor+\left\lceil\frac{g}{3}\right\rceil\leq\left\lceil\frac{3n-g-6}{6}\right\rceil=\left\lceil\frac{3n-g-9}{6}\right\rceil.$$

Subcase 4.1.3. Assume that $n(F-u)\geq 3$ and $g=3s+1\geq 7$.

Subcase 4.1.3.1. Assume that F has a further end-vertex v. Let $y\in V(C_g)$ be a neighbor of v. If $F-\{u,v\}$ is not a corona graph, then Theorem 2.2 implies $\gamma(F-\{u,v\})\leq\lfloor(n(F)-4)/2\rfloor$. Since $g=3s+1$, it is not difficult to show that there exists a minimum dominating set D_g of C_g with $x, y \in D_g$. As D_g dominates u and v, we obtain

$$\gamma \leq \left\lfloor \frac{n(H)-4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-12}{6} \right\rceil.$$

Therefore we assume in the following that $F - \{u, v\}$ is a corona graph.

Subcase 4.1.3.1.1. Let $n(F) = 4$. We conclude that $g \leq 10$ and thus $g = 7$ or $g = 10$. If $g = 7$, then we arrive at the exceptional graphs B_9 and B_{10}, and if $g = 10$, then we arrive at B_{16}.

Subcase 4.1.3.1.2. Let $n(F) = 6$. Then $F - \{u, v\} = w_1w_2w_3w_4$ is a path of length three.

If $uw_1 \in E(G)$ and $vw_4 \in E(G)$, then it follows that $g \leq 14$ and thus $g = 7$, $g = 10$, or $g = 13$.

Firstly, let $g = 7$. If $x = y$, then it is straightforward to verify that we arrive at the two exceptional graphs A_4 or A_5. If $d_{C_g}(x, y) = 1$, then it is a simple matter to show that we arrive at the exceptional graphs B_{11}, D_1, or D_4. If $d_{C_g}(x, y) = 2$, then we observe that $\gamma \leq 4$ and so (1) is valid. In the remaining case that $d_{C_g}(x, y) = 3$, we obtain the exceptional graph B_{12}.

Secondly, let $g = 10$. It follows that $3 \leq d_{C_g}(x, y) \leq 5$. In the case that $d_{C_g}(x, y) = 5$, we observe that $\gamma \leq 5$ and (1) is valid. However, if $d_{C_g}(x, y) = 3$ or $d_{C_g}(x, y) = 4$, then we arrive at the exceptional graphs B_{17} or B_{18}, respectively.

Thirdly, let $g = 13$. It follows that $d_{C_g}(x, y) = 6$, and this leads to the exceptional graph B_{19}.

If $uw_1 \in E(G)$ and $vw_1 \in E(G)$, then it follows that $g \leq 8$ and thus $g = 7$. Since w_4 has a neighbor on C_g, there exists a dominating set D of G with $w_1, w_4 \in D$ and $|D| = 4 \leq \lceil (3n - g - 9)/6 \rceil$.

If $uw_3 \in E(G)$ and $vw_4 \in E(G)$, then it follows that $g \leq 10$ and thus $g = 7$ or $g = 10$. If $a \in V(C_g)$ is a neighbor of w_1, then there exists a minimum dominating set D_g of C_g such that $a, y \in D_g$. Hence $D_g \cup \{w_3\}$ is a dominating set of G and we conclude that

$$\gamma \leq 1 + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-9}{6} \right\rceil.$$

If $uw_3 \in E(G)$ and $vw_1 \in E(G)$, then it follows that $g = 7$. Since v has a neighbor on C_g, there exists a dominating set D of G with $v, w_3 \in D$ and $|D| = 4 \leq \lceil (3n - g - 9)/6 \rceil$.

If $uw_3 \in E(G)$ and $vw_3 \in E(G)$, then it follows that $g = 7$. If $a \in V(C_g)$ is a neighbor of w_1, then there exists a minimum dominating set D_g of C_g such that $a \in D_g$. Hence $D_g \cup \{w_3\}$ is a dominating set of G and this yields the desired inequality.

If $uw_2 \in E(G)$ and $vw_3 \in E(G)$, then it follows that $g = 7$. If $a \in V(C_g)$ is a neighbor of w_1, then there exists a minimum dominating set D_g of C_g such that $a, x \in D_g$. Hence $D_g \cup \{w_3\}$ is a dominating set of G and (1) is valid. Since we have investigated all possible cases, Subcase 4.1.3.1.2 is proved.

Subcase 4.1.3.1.3. Let $n(F) \geq 8$ and $g = 3s + 1 \geq 7$.

Firstly, assume that u or v, say u, is adjacent with a vertex w_1 of $F - \{u, v\}$ which is not an end-vertex of $F - \{u, v\}$. Furthermore, let w be the end-vertex of $F - \{u, v\}$ such that $d_G(u, w) = 2$, and let $z \neq u, v, w$ be an end-vertex of F with a neighbor $a \in V(C_g)$. Now we see that $F - \{v, z\}$ is not a corona graph, and Theorem 2.2 implies $\gamma(F - \{v, z\}) \leq \lfloor (n(F) - 4)/2 \rfloor$. Since $g = 3s + 1$, there exists a minimum dominating set D_g of C_g with $a, y \in D_g$. As D_g dominates v and z, we obtain

$$\gamma \leq \left\lfloor \frac{n(H) - 4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 12}{6} \right\rceil.$$

Secondly, assume that u and v are only adjacent with end-vertices of $F - \{u, v\}$.

If u and v are adjacent with a common end-vertex w of $F - \{u, v\}$, then let $w_1, w_2 \neq w$ be two further end-vertices of $F - \{u, v\}$ with the neighbors $z_1, z_2 \in V(C_g)$, respectively. Now $F - \{w_1, w_2\}$ is not a corona graph, and there is a minimum dominating set D_g of C_g with $z_1, z_2 \in D_g$. Analogously to the last case, we obtain inequality (1).

Next assume that u and v are adjacent with different end-vertices w_1 and w_2 of $F - \{u, v\}$, respectively. Since $n(F) \geq 8$, there exists an end-vertex w in F such that $d_G(u, w) = 4$ or $d_G(v, w) = 4$, say $d_G(u, w) = 4$. In this situation it is a simple matter to verify that $F - \{u, w\}$ is not a corona graph. If $z \in V(C_g)$ is a neighbor of w, then let D_g be a minimum dominating set of C_g with $x, z \in D_g$. Now the desired result follows as above.

Subcase 4.1.3.2. Assume that F has only the end-vertex u.

Subcase 4.1.3.2.1. Assume that $\delta(F - u) \geq 2$. Since $F - u$ is of odd order, we deduce from Theorem 2.3 that $\gamma(F - u) \leq (n(F - u) - 3)/2$, $F - u = C_5$, or $F - u = C_7$. In the first case we obtain

$$\gamma \leq \left\lfloor \frac{n(H) - 4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n - g - 12}{6} \right\rceil.$$

Since $F - u = C_5$ is impossible, it remains the case that $F - u = C_7$. This implies $g = 7$, $\gamma \leq 5$, and inequality (1) is immediate.

Subcase 4.1.3.2.2. Assume that $F - u$ has also an end-vertex v. It follows that v is the unique end-vertex of $F - u$. Therefore $F - \{u, v\}$ is of even order but not a corona graph and Theorem 2.2 leads to

$$\gamma(F - \{u,v\}) \le \frac{n(F) - 4}{2}.$$

Since $g = 3s + 1$, we can choose a minimum dominating set D^* of the path $C_g - x$ such that $D^* \cup \{u\}$ is a dominating set of $V(C_g) \cup \{u,v\}$ with $|D^* \cup \{u\}| = \lceil g/3 \rceil$. Consequently, we arrive at

$$\gamma \le \left\lfloor \frac{n(H) - 4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \le \left\lceil \frac{3n - g - 12}{6} \right\rceil.$$

Subcase 4.2. Assume that H consists of at least two even components F and F_1 with $\delta(F) = \delta(F_1) = 1$. If both of these two components are no corona graphs, then Theorem 2.2 and Proposition 2.1 immediately yield the desired result. If F is no corona graph and F_1 is a corona graph, then we obtain inequality (1) analogously to Subcase 2.3. Finally, we assume that all components of H are corona graphs.

Subcase 4.2.1. Assume that one component, say F, has order two. Let $V(F) = \{u,v\}$. It follows that $g \le 6$. Let $x, y \in V(C_g)$ be the neighbors of u and v, respectively. Because of $g \le 6$, we observe that $\{x,y\}$ is a dominating set of $V(C_g) \cup \{u,v\}$. If $g = 5$, then n is odd and we conclude together with Proposition 2.1 that

$$\gamma \le 2 + \left\lfloor \frac{n-7}{2} \right\rfloor \le \left\lceil \frac{3n - g - 9}{6} \right\rceil,$$

and if $g = 6$, then n is even and we deduce that

$$\gamma \le 2 + \left\lfloor \frac{n-8}{2} \right\rfloor \le \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

Subcase 4.2.2. Assume that all components of H have at least four vertices. Since F has at least two end-vertices u and v with $d_F(u,v) = 3$, it follows that $g \le 10$. Let $x, y \in V(C_g)$ be the neighbors of u and v, respectively. If we choose, without loss of generality, a minimum dominating set D_g of C_g with $x \in D_g$, then D_g dominates the vertex u. Because of $\gamma(F - u) \le \lfloor (n(F-u) - 1)/2 \rfloor$, we deduce together with Proposition 2.1 in the cases $g = 3s + 2$ or $g = 3s$ that

$$\gamma \le \left\lfloor \frac{n(H) - 2}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \le \left\lceil \frac{3n - g - 6}{6} \right\rceil = \left\lceil \frac{3n - g - 9}{6} \right\rceil.$$

In the remaining cases that $g = 7$ or $g = 10$, we assume that w is an end-vertex of F_1 with the neighbor $z \in V(C_g)$. Now we can choose a minimum dominating set D_g of C_g with $x, z \in D_g$. Since D_g dominates the vertices u and w, and because of $\gamma(F-u) \leq \lfloor (n(F-u)-1)/2 \rfloor$ and $\gamma(F_1-w) \leq \lfloor (n(F_1-w)-1)/2 \rfloor$, Proposition 2.1 finally yields the desired result

$$\gamma \leq \left\lfloor \frac{n(H)-4}{2} \right\rfloor + \left\lceil \frac{g}{3} \right\rceil \leq \left\lceil \frac{3n-g-12}{6} \right\rceil . \ \square$$

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