Vertex magic total labeling of products of regular VMT graphs and regular supermagic graphs

Petr Kovář*
Department of Mathematics and Statistics
University of MN Duluth, Minnesota 55812, USA
Department of Mathematics and Descriptive Geometry
Technical University Ostrava, 708 33, Czech Republic
email: pkovar@d.umn.edu

January 30, 2004

Abstract

A vertex-magic total labeling of a graph $G(V, E)$ is defined as one-to-one mapping from $V \cup E$ to the set integers $\{1, 2, ..., |V| + |E|\}$ with the property that the sum of the label of a vertex and the labels of all edges incident to this vertex is the same constant for all vertices of the graph. A supermagic labeling of a graph $G(V, E)$ is defined as one-to-one mapping from E to the set integers $\{1, 2, ..., |E|\}$ with the property that the sum of the labels of all edges incident to a vertex is the same constant for all vertices of the graph.

In the paper we present a technique for constructing vertex magic total labelings of products of certain vertex magic total r-regular graphs G and certain $2s$-regular supermagic graphs H. H has to be decomposable into two s-regular factors and if r is even $|H|$ has to be odd.

1 Introduction and known results

A labeling of a graph $G(V, E)$ is a mapping from the set of vertices, edges or both vertices and edges to the set of labels. In most applications labels are positive (or nonnegative) integers, though in general real numbers could

*Partially supported by a UMD summer project [8].

be used. Various labelings are obtained based on requirements put on labelings.

The notion of magic squares can be naturally extended to graphs. We want the sum of labels related to an edge or a vertex be constant all over the graph. If the sum of labels of an edge and both end vertices does not depend on the edge, we call the labeling edge-magic type labeling. If the sum of labels of a vertex and all incident edges is constant, we call the labeling vertex-magic type labeling. This text is focused on vertex magic total labelings. The word *total* is used since labels are assigned to both edges and vertices.

Magic labelings were introduced by Sedláček in 1963 [13]. A *magic* labeling assigns distinct integers to edges with the property that the sum of the labels of all edges incident with the vertex v is the same for all v in a given graph. If the set of labels consists of consecutive integers, the labeling is called *supermagic*. This type of labeling is also called vertex-magic edge labeling by Y. Lin, M. Miller, R. Simanjuntak, and Slamin [12].

Let $G(V, E)$ be a graph with the vertex set V and the edge set E. Let v be a vertex in G. We denote the edges incident to the vertex v as $N_E(v) = \{e \in E(G) |\ e \text{ is incident to } v\}$.

Definition 1.1 *Let $G(V, E)$ be a graph with the vertex set V and the edge set E. We denote $v = |V|$ and $e = |E|$. A one-to-one mapping $\lambda : V \cup E \to \{1, 2, \ldots, v + e\}$ is called a* vertex magic total labeling (VMT) *of G if there exists a constant k such that for every vertex x of G*

$$\lambda(x) + \sum_{y \in N(x)} \lambda(xy) = h. \tag{1}$$

The constant h is the magic constant *for λ. We call a graph to be a* VMT graph *if it has a VMT labeling.*

An example of VMT labeling is in figure 1.1.

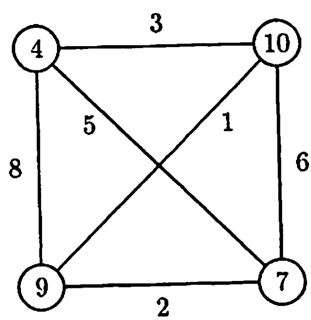

Figure 1.1: *Vertex magic total labeling of K_4 with $h = 20$*

Definition 1.2 *Let $G(V, E)$ be a graph with the vertex set V and the edge set E. We denote $e = |E|$. A one-to-one mapping $\lambda : E \to \{1, 2, \ldots, e\}$ is called a* supermagic (SPM) labeling of G *if there exists a constant k such that for every vertex x of G*

$$\sum_{y \in N(x)} \lambda(xy) = k. \tag{2}$$

The constant k is the magic constant *for λ. We call a graph to be an* SPM graph *if it has an SPM labeling.*

An example of an SPM labeling is in figure 1.2.

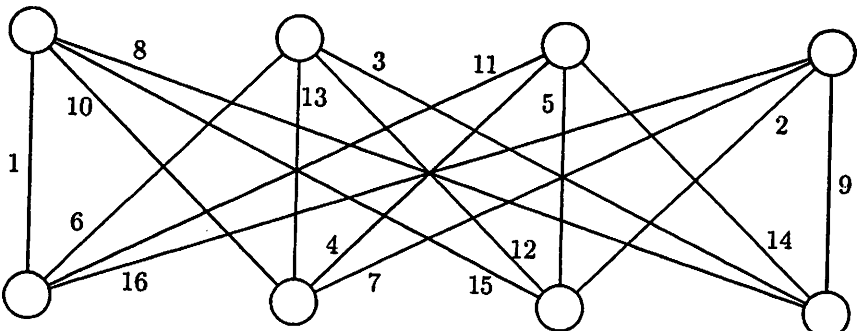

Figure 1.2: *Super magic labeling of $K_{4,4}$ with $k = 34$*

There was a large number of articles published on magic-type graph labelings. Probably the best source of information is *Dynamic Survey of Graph Labeling* by Joseph Gallian [7]. The table 1.1 appeared in the fall 2003 edition of the *Dynamic Survey*.

From the table it is apparent that the existence of a VMT labeling is known for most basic families of graphs as complete graphs, complete bipartite graphs, cycles, etc. Notice there are only two general results concerning VMT labeling of copies of a VMT graph (see [15] and [9]).

Graph	*Labeling*	*Notes*
C_n	VMT	[3]
P_n	VMT	$n > 2$ [3]
$K_{m,m}$	VMT	$m > 1$ [3][11]
$K_{m,m} - e$	VMT	$m > 2$ [3]
$K_{m,n}$	VMT not VMT	iff $\|m - n\| \leq 1$ [3] if $n > m + 1$
K_n	VMT	for n odd [3] for $n \equiv 2 \pmod 4$, $n > 2$ [11]
Petersen $P(n, k)$	VMT	for all n and k [2]
prisms $C_n \times P_2$	VMT	for all n
W_n	VMT	iff $n \leq 11$ [3]
F_n	VMT	iff $n \leq 10$ [3]
friendship graphs	VMT	iff # of triangles ≤ 3 [3]
$G + H$	VMT	$\|V(G)\| = \|V(H)\|$ and $G \cup H$ is VMT [14]
unions of stars	VMT	[14]
Tree with n internal vertices and more than $2n$ leaves	not VMT	[14] [14]
nG	VMT	n odd, G regular of even degree, VMT [15]
nG	VMT	G is regular of odd degree, VMT, but not K_1 [15]
$C_n \square C_{2m}$	VMT	[4]
$K_5 \square C_{2n+1}$	VMT	[5]
$G \square C_{2n}$	VMT	G $(2r + 1)$-regular, decomposable into $r + 1$ and r regular [9]

Table 1.1: *Summary of results on vertex magic total labelings.*

2 Copies of VMT and SPM graphs

The construction of VMT labelings of products of certain r-regular VMT graphs and certain s-regular SPM graphs shown below in this paper is based on a VMT labeling of copies of an r-regular graph VMT and on an

SPM labeling of copies of a $2s$-regular graph, which can be factored into two s-regular factors. The first result is due to Wallis [15]. We give another proof here in this paper, which yields a different magic constant. Graph nG denotes n disjoint copies of the graphs G.

Theorem 2.1 (W. D. Wallis 2002) *Suppose G is a regular graph of degree Δ, which has a magic total labeling. Provided G is not K_1,*

1. *if Δ is even then nG has a vertex magic total labeling whenever n is an odd positive integer;*

2. *if Δ is odd then nG has a vertex magic total labeling for every positive integer n.*

Proof 1. Given in [15]. □

Note 2.2 Wallis showed that the labeling

$$\begin{aligned} \lambda'(v_{i,p}) &= \lambda(v_p) + (e+v)a_{\eta(v_p),i} \text{ for } p = 1,2,\ldots,v \\ \lambda'(e_{i,q}) &= \lambda(e_q) + (e+v)a_{\eta(e_q),i} \text{ for } q = 1,2,\ldots,e \end{aligned} \tag{3}$$

is a VMT labeling of n copies of G with the magic constant $h + \frac{1}{2}(e+v)(r+1)(n-1)$. We give another proof here, which yields a different magic constant.

Proof 2. Let G be a VMT r-regular graph from Theorem 2.1. Take a VMT labeling λ of G with the magic constant h and a Kotzig array $A = (a_{i,j})$ of size $(r+1) \times n$. Such an array always exists under the conditions on Δ and n of Theorem 2.1. Take a total coloring of G $\eta : (V(G) \cup E(G)) \to \{1,2,\ldots,r+1\}$. We denote the vertices of G by $v_1, v_2, \ldots, v_v$ and edges by $e_1, e_2, \ldots, e_e$. We will construct n copies G_i of graph G for $i = 1,2,\ldots,n$. In G_i we denote the copies of vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,v}$ and edges $e_{i,1}, e_{i,2}, \ldots, e_{i,e}$.

Consider the following labeling

$$\begin{aligned} \lambda'(v_{i,p}) &= n(\lambda(v_p) - 1) + a_{\eta(v_p),i} + 1 \text{ for } p = 1,2,\ldots,v \\ \lambda'(e_{i,q}) &= n(\lambda(e_q) - 1) + a_{\eta(e_q),i} + 1 \text{ for } q = 1,2,\ldots,e. \end{aligned} \tag{4}$$

To show that (4) is a VMT labeling we evaluate the sum at each vertex

$$\lambda'(v_{i,p}) + \sum_{e_{i,q} \in N_E(v_{i,p})} \lambda'(e_{i,q}).$$

Expanding the expression we have

$$n\left((\lambda(v_p) - 1) + \sum_{e_q \in N_E(v_p)} (\lambda(e_q) - 1) \right) + (a_{\eta(v_p),i} + 1) + \sum_{e_q \in N_E(v_p)} (a_{\eta(e_q),i} + 1).$$

We can evaluate the expression. Since λ is a VMT labeling with the magic constant h and G is an r-regular graph,

$$\lambda(v_p) - 1 + \sum_{e_q \in N_E(v_p)} (\lambda(e_q) - 1) = h - (r+1).$$

and the second sum runs over the i-th column of the Kotzig array A, so

$$(a_{\eta(v_p),i}+1) + \sum_{e_q \in N_E(v_p)} (a_{\eta(e_q),i}+1) = \frac{1}{2}(n-1)(r+1)+(r+1) = \frac{1}{2}(n+1)(r+1).$$

This means that we have a constant sum $n(h-(r+1)) + \frac{1}{2}n(r+1) = nh + \frac{1}{2}(1-n)(r+1)$ at each vertex in G_i for $i = 1, 2, \ldots, n$.

Now looking at all n vertices $v_{i,p}$ (edges $e_{i,q}$, respectively), which correspond to the vertex v_p (edge e_q) of the original graph G, the labels form the set $\{n\lambda(v_p), n\lambda(v_p)+1, \ldots, n\lambda(v_p)+n-1\}$ ($\{n\lambda(e_q), n\lambda(e_q)+1, \ldots, n\lambda(e_q)+n-1\}$). There are $e+v$ such sets and they are disjoint since λ assigns a different label to every vertex (edge). This means that λ' uses each of the $n(e+v)$ labels exactly once. Thus λ' is a VMT labeling of n copies of G with the magic constant $nh + \frac{1}{2}(1-n)(r+1)$. □

Theorem 2.1 gives a general method for constructing VMT labelings for n copies of certain regular VMT graphs (together with Proof 2 with two different magic constants). The following theorem gives a similar result for copies of certain $2s$-regular supermagic graphs.

Theorem 2.3 *Let s be a positive integer. Let G be a $2s$-regular supermagic graph, which can be factorized into two s-regular factors. Then nG is also a supermagic graph.*

Proof. Let G be a $2s$-regular graph, which can be factorized into two s-regular factors H_1 and H_2. Let G have a supermagic labeling f with the magic constant k. Let v denote the number of vertices of G and e the number of edges of G.

Since f is a supermagic labeling of G, (2) holds for every $v \in V(G)$. We will construct n copies G_i of the graph G for $i = 0, 1, \ldots, n-1$. Each of them will have a supermagic labeling f_i with the same magic constant

$$k_i = nk + s(1-n).$$

Let us denote the vertices of G by $v_1, v_2, \ldots, v_v$ and edges by $e_1, e_2, \ldots, e_e$. In G_i we denote the copies of vertices by $v_{i,1}, v_{i,2}, \ldots, v_{i,v}$ and the copies of edges by $e_{i,1}, e_{i,2}, \ldots, e_{i,e}$.

Consider the following labeling of G_i:

$$f_i(e_{i,k}) = \begin{cases} n(f(e_k)-1) + n - i & \text{if } e_k \in H_1 \\ n(f(e_k)-1) + i + 1 & \text{if } e_k \in H_2 \end{cases} \quad \text{for } i = 0, 1, \ldots, n-1 \quad (5)$$

We show that for every G_i, f_i is a supermagic labeling with the magic constant $k_i = k' = nk + s(1-n)$. The sum at every vertex $v_{i,j}$ of G_i is

$$\sum_{e \in N_E(v_{i,j})} f_i(e_{i,k}).$$

We split the sum into two sums, each over the edges of one of the two s-regular factors H_1 and H_2:

$$\sum_{e \in N_E(v_{i,j}) \cap H_1} f_i(e_{i,k}) + \sum_{e \in N_E(v_{i,j}) \cap H_2} f_i(e_{i,k})$$

and we get

$$\sum_{e \in N_E(v_{i,j}) \cap H_1} n(f(e_k) - 1) + n - i + \sum_{e \in N_E(v_{i,j}) \cap H_2} n(f(e_k) - 1) + i + 1 =$$

$$\sum_{e \in N_E(v_{i,j}) \cap H_1} nf(e_k) - i + \sum_{e \in N_E(v_{i,j}) \cap H_2} nf(e_k) - n + i + 1 =$$

$$n\left(\sum_{e \in N_E(v_{i,j}) \cap H_1} f(e_k) + \sum_{e \in N_E(v_{i,j}) \cap H_2} f(e_k)\right) - \sum_{e \in N_E(v_{i,j}) \cap H_1} i + \sum_{e \in N_E(v_{i,j}) \cap H_2} i - n + 1.$$

Since $|N_E(v_{i,j}) \cap H_1| = |N_E(v_{i,j}) \cap H_2| = s$ and using (2) we get

$$nk - si + s(i - n + 1) = nk + s(1-n).$$

Since the sum is constant the labeling given in (5) is a supermagic labeling of n copies of G. □

Unfortunately we can't use the same nice approach as Wallis in his Theorem 2.1. The idea is based on Vizing's Theorem, which guarantees that there always exists a proper edge coloring with $\Delta + 1$ colors and thus there exists a proper $\Delta + 1$ total coloring (both edges and vertices) of G. In the copies the labels are assigned based on the coloring of G and the rows of Kotzig arrays. The $\Delta + 1$-st color is used for vertex labels. But there are no vertex labels in an SPM graph! Instead of a general statement as in Theorem 2.1 we can obtain a result only for graphs with proper Δ coloring of edges. We have partial results and at the time this paper is published is not clear if it is more general result that in Theorem 2.3 or whether there are classes of graphs that are both SPM and decomposable into two s-regular graphs and do not have a proper edge coloring with Δ colors.

3 Main result

We are almost ready to show the main result of this paper. The proof of the result will make use of the two following easy observations, that adding the same number t to every label of a VMT graph G (or an SPM graph H, respectively) with the magic constant h (or k) yields the same sum $h+(r+1)t$ at every vertex (or $h+rt$). We refer to such a labeling as to a *generalized VMT (or SPM)* since the labels in general are *not* consecutive positive integers starting at 1.

Lemma 3.1 *Let G be an r-regular graph on v vertices, which has a VMT labeling λ with the magic constant h. Let x be an element (a vertex or an edge) of G. Let t be an integer.*

1. *The labeling $\lambda'(x) = \lambda(x) + t$ is a generalized VMT labeling with the magic constant $h + (r+1)t$.*
2. *The labeling $\lambda''(x) = t\lambda(x)$ is a generalized VMT labeling with the magic constant ht.*

Proof. *Part 1* There are $r+1$ terms in the sum of labels at every vertex. Each of them increases the sum by t.
Part 2 Each term in the sum of labels at every vertex is multiplied by t. □

Lemma 3.2 *Let H be an s-regular graph on u vertices, which has an SPM labeling λ with the magic constant k. Let x be an element (a vertex or an edge) of G. Let t be an integer.*

1. *The labeling $\lambda'(x) = \lambda(x) + t$ is a generalized SPM labeling with the magic constant $h + st$.*
2. *The labeling $\lambda'(x) = t\lambda(x)$ is a generalized SPM labeling with the magic constant ht.*

Proof. Proof is similar to the proof of Lemma 3.1. □

Taking a graph G_0 on v vertices, which satisfies the conditions of Theorem 2.1 and taking a graph H_0 on u vertices, which satisfies the conditions of Theorem 2.3 one can get a graph uG_0 on uv vertices, which is VMT and a graph vH_0 on uv vertices, which is SPM. But we can have a VMT (or an SPM) labeling of nG_0 (or nH_0) based on a different construction. In both cases we can combine the graphs to get a VMT graph.

Theorem 3.3 *Let G be an r-regular VMT graph on uv vertices, which consists of u copies of G_0. Let λ_G be a VMT labeling of G with the magic constant h. Let H be an s-regular SPM graph on uv vertices, which consists* *of v copies of H_0. Let λ_H be an SPM labeling of H with the magic constant k. Then there exists a VMT labeling of $G_0 \Box H_0$ with the magic constant $h + k + \frac{1}{2}s(2+r)$.*

Proof. We denote the vertices of G_0 by v_j for $j = 1, 2, \ldots, v$ and the vertices of H_0 by u_i for $i = 1, 2, \ldots, u$. The vertices of G can be seen as copies of v_j. We denote them by $v_{i,j}$, where $i = 1, 2, \ldots, u$. For the same reason we denote the vertices of H by $u_{i,j}$, where $j = 1, 2, \ldots, v$.

It is easy to observe that by identifying vertices $v_{i,j}$ with $u_{i,j}$ we get the graph $G_0 \Box H_0$. We can also say that we are adding correspondent edges of H to G.

For constructing a VMT labeling of $G_0 \Box H_0$ we can use the VMT labeling λ_G of G and the SPM labeling λ_H of H. Consider the labeling

$$\begin{aligned} \lambda(x) &= \lambda(x) \quad \forall x \in V(G) \cup E(G) \\ \lambda(y) &= |E(G) + V(G)| + \lambda(y) \quad \forall y \in E(H). \end{aligned} \tag{6}$$

The sum of the labels $\lambda(v_{i,j}) + \sum_{e \in N_E(v_{i,j})} \lambda(e)$ at every vertex $v_{i,j}$ now consists of the sum of the labels in G and the sum of the labels at $u_{i,j}$ in H. We get

$$\lambda(v_{i,j}) + \sum_{e \in N_E(v_{i,j})} \lambda(e) = \lambda(v_{i,j}) + \sum_{e \in N_E(v_{i,j}) \cap E(G)} \lambda(e) + \sum_{e \in N_E(v_{i,j}) \cap E(H)} \lambda(e).$$

The first sum corresponds to the VMT labeling of G.

$$\lambda(v_{i,j}) + \sum_{e \in N_E(v_{i,j}) \cap E(G)} \lambda(e) = h$$

The second sum

$$\sum_{e \in N_E(v_{i,j}) \cap E(H)} \lambda(e) = k + s(|V(G) + E(G)|) = k + \frac{1}{2}suv(2+r)$$

follows immediately from Lemma 3.2 since we add $|V(G) + E(G)|$ to every label of the SPM labeling λ_H. Every label is used exactly once since we use the first $|V(G) + E(G)|$ labels from the labeling λ_G and the following $|E(H)|$ labels from λ_H. λ is a VMT labeling of $G_0 \Box H_0$ with the magic constant $h + k + \frac{1}{2}suv(2+r)$. □

Note 3.4 There is a natural modification of the construction given in the proof of Theorem 3.3. We can keep the labels given by λ_H and increase the labels arising from λ_G by $|E(H)|$. The labeling

$$\begin{aligned} \lambda(x) &= |E(H)| + \lambda(x) \quad \forall x \in V(G) \cup E(G) \\ \lambda(y) &= \lambda(y) \quad \forall y \in E(H) \end{aligned} \tag{7}$$

is a VMT labeling of $G_0 \square H_0$ with the magic constant $h+k+(r+1)|E(H)| = h + k + \frac{1}{2}(r+1)uvr$.

4 Conclusion

The construction given in the proof of Theorem 3.3 allows to build products for several families of graphs. For G in the theorem we can take any graph with a VMT labeling, which satisfies Theorem 2.1, e.g., K_n, $K_{n,n}$, C_n, Petersen graph $P(n,k)$, prisms $P_2 \times C_n$ if the necessary condition of being even-regular or odd regular with even number of copies is satisfied. Also for H we have a variety of graph classes to choose from. Among graphs, which are proven to have an SPM labeling and are also factorable into two s-regular factors are e.g. K_n for $n \not\equiv 0 \pmod 4$, $K_{n,n}$, Q_n for n even.

We also want to point out that the $G \square H$ graph is a VMT graph, which again satisfies the conditions of Theorem 3.3. This means that also *repeated products* are VMT graphs, e.g., $(K_{2n} \square K_{2m,2m} \square Q_{2p} \square \ldots)$ etc.

References

[1] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R. Simanjuntak, and Slamin, Vertex-antimagic total labelings of graphs, *Discussiones Mathematicae. Graph Theory*, **1**, (2003)

[2] M. Bača, Mirka Miller and Slamin, Every generalized Petersen graphs has a vertex-magic total labeling, *preprint* (2002).

[3] J.A. MacDougall, M. Miller, Slamin, W. D. Wallis, Vertex-Magic total labelings of graphs, *Util. Math*, **61** (1990) pp. 3–21.

[4] D. Frončck, P. Kovář, T. Kovářová, Vertex magic total labeling of Cartesian products of cycles, *Proceedings of Conference MIGHTY XXXVI*, Oshkosh (2003).

[5] D. Frončck, P. Kovář, T. Kovářová, Vertex magic total labeling of Cartesian products of some vertex magic total regular graphs and odd cycles, *Proceedings of Conference MIGHTY XXXVI*, Oshkosh (2003).

[6] D. Frončck, P. Kovář, T. Kovářová, Vertex magic total labeling of products of cycles, *submitted*.

[7] Joseph A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics*, **6** (2002).

[8] P. Kovář, T. Kovářová, Funding for project to include figures, tables and index to Dynamic Survey, summer project under prof. Gallian, *University of Minnesota Duluth*, (2003).

[9] P. Kovář, Vertex magic total labeling of Cartesian products of certain regular VMT regular graphs and even cycles, *Proceedings of Conference MIGHTY XXXVII*, Valparaiso (2003).

[10] A. Kotzig, A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), pp. 451–461.

[11] Y. Lin, M. Miller, Vertex-magic total labelings of complete graphs, *Bull. Inst. Combin. Appl*, (2002).

[12] Y. Lin, M. Miller, R. Simanjuntak and Slamin, Magic and antimagic labelings of wheels,

[13] J, Sedláček, Problem 27. *Theory of graphs and its applications*, Publ. House Czechoslovak Acad. Sci., Prague (1964) pp. 163–164

[14] W. D. Wallis, Magic Graphs, *Birkhäuser* (2001).

[15] W. D. Wallis, Vertex Magic Labelings of Multiple Graphs, *Vongressus Numeratntium 152* (2001). pp. 81–83.