Tables For Constant Composition Codes

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Abstract

A constant composition code of length n over a k-ary alphabet has the property that the numbers of occurrences of the k symbols within a codeword is the same for each codeword. These specialize to constant weight codes in the binary case, and permutation codes in the case that each symbol occurs exactly once. Constant composition codes arise in powerline communication and balanced scheduling, and are used in the construction of permutation codes. Using exhaustive and probabilistic clique search, and by applying theorems and constructions in past literature, we generate tables which summarize the best known lower bounds on constant composition codes for (i) $3 \leq k \leq n \leq 8$, (ii) $k = 3$, $9 \leq n \leq 12$, and (iii) various other interesting parameters with $n \geq 9$.

1 Introduction

Let C be a k-ary code of length n and distance d on the alphabet $\{1,\ldots,k\}$. As usual, the elements of C are *codewords.*

We say C has *constant weight composition* $[n_1,\ldots,n_k]$ if every codeword has n_i occurrences of symbol i for $i=1,\ldots,k$. (Since the alphabet is immaterial to a code, we may view the composition $[n_1,\ldots,n_k]$ as an unordered multiset, and not restrict the alphabet to $\{1,\ldots,k\}$.) Code C is a *constant composition code*, or simply a CCC. Let $A([n_1,\ldots,n_k],d)$ denote the maximum size of such a CCC. Since distinct codewords with the same composition always disagree in at least two positions, it is clear that $A([n_1,\ldots,n_k],2)=n!/n_1!\cdots n_k!$.

When writing compositions, the exponential notation $n_1^{t_1}n_2^{t_2}\cdots n_h^{t_h}$ may be used to abbreviate

$$[\overbrace{n_1,\ldots,n_1}^{t_1},\overbrace{n_2,\ldots,n_2}^{t_2},\ldots,\overbrace{n_h,\ldots,n_h}^{t_h}].$$

Among the most studied CCCs are those with composition 1^n, called *permutation codes.* Permutation codes have been shown in [4] and [5] to have nice application to powerline communication, where changes in amplitude or frequency must respect some overall average.

For later reference, we include some elementary facts concerning upper and lower bounds on $A([n_1,\ldots,n_k],d)$. Proofs can be found in [12].

Lemma 1.1. *If* $d>2(n-n_1)$, *then* $A([n_1,\ldots,n_k],d)=1$. *If* $d=2(n-n_1)$, *then* $A([n_1,\ldots,n_k],d)=\lfloor n/(n-n_1)\rfloor$.

Proposition 1.2. (recursive Johnson bound)

$$A([n_1,n_2,\ldots,n_k],d)\le\frac{n}{n_1}A([n_1-1,n_2,\ldots,n_k],d).$$

Corollary 1.3. $A([n_1,\ldots,n_k],n)=\lfloor n/\max\{n_i\}\rfloor$.

In the references, various other lower bounds and exact values have been obtained for cases such as $d=3,4,2(n-n_k)-1$, $k=2,3$, $n_1=\cdots=n_k$, and n a prime power. There are also useful upper bounds relating to the Plotkin bound; see [3], [7], and [10].

Obviously, the problem of finding lower bounds or exact values for a particular $A([n_1,\ldots,n_k],d)$ can be reduced to clique search in a graph on

$n!/n_1!\cdots n_k!$ vertices. For exhaustive search, we have used an implementation of the algorithm in [8] and [9]. When the graph is too large for exhaustive search, we have used reactive local search, as discussed in [1], which gives an approximate max-clique and hence a lower bound. For still larger graphs, we invoke a greedy search. This finds a clique by simply running through all possible vertices once (say in lexicographic order) and adding a vertex to the current clique if it is adjacent to all others in that clique. When such a graph is still too large for a practical search of any kind, its size can be reduced using automorphism groups; see [4]. For what follows, we briefly mention one special case. A code C of length n admits a *cyclic automorphism* if, whenever $w \in C$, then all cyclic shifts of w belong to C. When using clique search for codes admitting cyclic automorphisms, it suffices to consider vertices whose first entry is a fixed symbol – the least frequent is best – but carefully eliminate those vertices for which nontrivial shifts are too close. If such vertices are excluded, the size of the resulting code is of course n times the size of the clique found.

Constant composition codes with $k = 2$ are known as *constant weight* binary codes and are well-studied. Extensive tables for such codes can be found in [2]. So we restrict our attention in what follows to $k \geq 3$. However, we note that the Johnson bound is used in our tables (and in past results) to reduce the composition $[n_1, n_2, 1]$ to the binary case.

2 Tables

Following are four tables which summarize the current status of lower bounds and known exact values on $A([n_1, \ldots, n_k], d)$ for $k \geq 3$. Table 1 considers all (non-binary) compositions and distances with $n \leq 8$. Table 2 concerns all ternary compositions and all distances with $9 \leq n \leq 12$. Tables 3 and 4 provide a selection of lower bounds for various compositions with $n \geq 9$. A variety of algebraic, combinatorial, and computational constructions by the same authors in [3] and [4] provides several new entries in these tables, particularly for $n \geq 13$, where we have merely illustrated some of these constructions with selected entries. In addition, some exact values in Table 4 appear in the recent paper [7] and many of the bounds on ternary compositions for $n \leq 10$ are found in [12]. We note that the bound $A([6, 2, 2], 4) \geq 80$ (found by clique search) agrees with that stated in [12]; however, the code given in that reference is cyclic containing a "short orbit" 2100021000.

Subscripts on entries in each table point to justification for the claimed code, given in the legend preceding the tables. Exact values (where an upper bound is known to meet a given lower bound) are shown in bold. Superscripts on these values indicate, also according to the legend, the (simplest) reason for the stated upper bound. If either a superscript or subscript is missing, it indicates a reference to one of the easy facts in the introduction: Lemma 1.1, Proposition 1.2, or Corollary 1.3.

Legend

	subscripts for lower bounds
a	[4]
b	[3]
c	clique computation
d	algebraic construction in [6]
e	simple direct construction
g	greedy search as in [3]
h	exact value obtained in [7]
o	from larger d or smaller n_i
s	[12]
t	[11]
y	cyclic codes via cyclotomy as in [3]

	superscripts for upper bounds
n	nonrecursive Johnson bound in [7]
p	(strengthened) Plotkin bound [10]
s	[12]
v	exhaustive search referenced in [6]
x	exhaustive clique search

comp	$d=3$	4	5	6	7	8
1^3	3					
1^4	12_d	4				
$1^2 2^1$	4_e	2				
1^5	60_d	20_d	5			
$1^3 2^1$	18^x	6^x	2			
$1^1 2^2$	10_s	5_e	2			
$1^2 3^1$	5_e	3	1			
1^6	360_d	120_d	18^v_d	6		
$1^4 2^1$	108_b	36_c	9^x	3		
$1^2 2^2$	48^x	18_c	6_e	3		
2^3	30_s	15_s	3^p_o	3		
$1^3 3^1$	28^x	11^x	4_c	2		
$1^1 2^1 3^1$	12_ι	6_e	3_c	2		
$1^2 4^1$	6_e	3	1	1		
1^7	2720_d	349_c	77_c	42_d	7	
$1^5 2^1$	720_b	170_c	41_c	15_c	3	
$1^3 2^2$	296_c	90_c	25_c	8^x	3	
$1^4 3^1$	196_c	60_c	21_c	7^x	2	
$1^1 2^3$	168_c	48_c	21_c	7^p_b	3	
$1^2 2^1 3^1$	84_c	33_c	10^x	4^x	2	
$2^2 3^1$	42_s	21_s	7_s	3^s_o	3	
$1^3 4^1$	42_c	16^x	7_e	2	1	
$1^1 3^2$	28_s	14_s	7_e	2^s_o	2	
$1^1 2^1 4^1$	21_ι	9^s_s	3^c_s	2	1	
$1^2 5^1$	7_c	3	1	1	1	
1^8	20160_d	2688_a	560_a	336_d	56_d	8
$1^6 2^1$	5760_b	1056_c	200_c	56_c	16_g	4
$1^4 2^2$	2208_c	544_c	112_c	32_c	11_g	4
$1^5 3^1$	1296_c	400_c	80_c	24_c	8_e	2
$1^2 2^3$	1104_c	281_c	66_c	23_c	8^x	2
2^4	672_c	184_c	39_c	28_c	5^p_b	4
$1^3 2^1 3^1$	603_c	187_c	56_c	16_c	5^x	2
$1^4 4^1$	336_c	112_c	31_c	10^x	2^s_o	2
$1^1 2^2 3^1$	336_c	120_c	30_c	12^x	4^x	2
$1^2 3^2$	224_c	80_c	23_c	8^x	3^x	2
$1^2 2^1 4^1$	152_c	56_c	18_c	8_c	2^s_o	2
$2^1 3^2$	104_s	56_s	16^s_s	8_e	3^s_c	2
$2^2 4^1$	84_s	36_s	12_s	5^s_s	2^s_o	2
$1^1 3^1 4^1$	56_s	24_s	8_e	4_c	2^s_o	2
$1^3 5^1$	56_c	23^x	8^x	2	1	1
$1^1 2^1 5^1$	24_ι	12_s	4^s_c	2	1	1
$1^2 6^1$	8_e	4	1	1	1	1

Table 1: All compositions with $3 \le k \le n \le 8$.

comp	$d=3$	4	5	6	7	8	9	10	11	12
[3,3,3]	312_s	168_s	36^s_s	24_s	6^s_s	3^p_o	3			
[4,3,2]	216_b	108_s	24^x	15_s	5^s_s	3^p_e	2			
[4,4,1]	126_s	54_s	18_s	9_s	3^s_s	2_o	2			
[5,2,2]	108^s_s	49_b	18_s	9_s	3^s_s	2	1			
[5,3,1]	72^s_b	33^s_s	10^s_s	6_s	3^s_s	2	1			
[6,2,1]	36_t	18_s	4^s_e	3	1	1	1			
[7,1,1]	9_e	4	1	1	1	1	1			
[4,3,3]	690_b	360^s_s	60_s	50_s	10^s_s	5^s_s	3^s_e	2		
[4,4,2]	532_b	270^s_s	50_s	25^s_s	10_s	5^s_s	2^s_o	2		
[5,3,2]	327_b	140_s	40_s	30_s	7^s_s	4^s_s	2^s_o	2		
[5,4,1]	168_b	76_b	21^s_s	10^s_s	5^s_s	2^s_o	2^s_o	2		
[6,2,2]	180^s_s	80_c	20_s	15_s	5^s_s	2	1	1		
[6,3,1]	116_b	50_s	13_s	10_s	3^s_s	2	1	1		
[7,2,1]	40_t	20_t	5^s_e	3	1	1	1	1		
[8,1,1]	10_e	5	1	1	1	1	1	1		
[4,4,3]	1804_c	990_c	110_c	55_c	22_c	11^p_c	4^x	3^p_e	2	
[5,3,3]	1100_c	396_c	110_o	110_c	17_c	8^x	4^x	2^s_o	2	
[5,4,2]	792_c	319_c	110_c	55_c	13_c	6^x	3	2^s_o	2	
[5,5,1]	343_c	137_c	37_c	22_c	11_c	4_c	2^x	2^n_o	2	
[6,3,2]	539_c	209_c	56_c	55_c	11_c	4^x	2^s_o	2_e	1	
[6,4,1]	287_c	121_c	32_c	15_c	6^x	3_c	2^s_o	2_e	1	
[7,2,2]	218_c	101_c	26_c	15^x	5_s	2	1	1	1	
[7,3,1]	143_c	69_c	17_c	11_c	3_e	2	1	1	1	
[8,2,1]	55_t	25^t_c	5^s_e	3	1	1	1	1	1	
[9,1,1]	11_e	5	1	1	1	1	1	1	1	
[4,4,4]	3222_g	1010_g	242_g	84_g	35_g	16_g	9_h	4^p_e	3^p_o	3
[5,4,3]	2456_g	990_o	198_g	110_o	28_g	13_g	7^n_g	4^p_g	3^p_e	2
[5,5,2]	1716_c	744_c	144_c	132_c	24_c	12_g	5^x	3^x	2^x	2
[6,3,3]	1848_c	684_c	144_c	84_c	24_c	10_g	5^x	4^p_g	2^s_o	2
[6,4,2]	1428_c	528_c	120_c	60_c	18_g	12_g	4^x	3^n_g	2^s_o	2
[6,5,1]	588_c	252_c	60_c	24_c	12_c	6^n_c	3^x	2^x	2_o	2
[7,3,2]	780_c	324_c	72_c	55_o	13_c	6_c	3_e	2	1	1
[7,4,1]	397_c	182_c	45_c	20_c	9_c	4^x	3_e	2	1	1
[8,2,2]	308_c	150_c	30_c	18_c	6^s_s	3	1	1	1	1
[8,3,1]	203_c	94_c	20_c	12_c	4_e	3	1	1	1	1
[9,2,1]	60_t	30_c	6^s_e	4	1	1	1	1	1	1
[10,1,1]	12_e	6	1	1	1	1	1	1	1	1

Table 2: All ternary ($k=3$) compositions with $9 \leq n \leq 12$.

n	comp	d	size
9	1^9	4	18144_a
9	1^9	5	1944_a
9	1^9	6	1512_a
9	$1^7 2^1$	3	50400_b
9	$1^1 2^4$	7	36_b
9	$1^1 2^4$	8	9_b^p
10	1^{10}	4	86400_a
10	1^{10}	5	13680_a
10	1^{10}	6	4320_a
10	1^{10}	9	35_a
10	$1^8 2^1$	3	504000_b
10	2^5	9	7_b^p
11	1^{11}	4	950400_a
11	1^{11}	8	$\mathbf{7920}_d$
11	1^{11}	9	154_a
11	$1^9 2^1$	3	5443200_b
11	$1^1 2^5$	4	1320_y
11	$1^1 2^5$	6	660_y
11	$1^1 2^5$	7	220_y
11	$1^1 2^5$	9	55_y
11	$1^1 2^5$	10	11_b^p
11	$3^1 2^4$	4	1320_y
11	$3^1 2^4$	5	660_y
11	$3^1 2^4$	6	286_y
11	$3^1 2^4$	7	77_y
11	$3^1 2^4$	8	55_y
12	1^{12}	4	11404800_a
12	1^{12}	8	$\mathbf{95040}_d$
12	2^6	11	8_b

n	comp	d	size
13	1^{13}	6	271908_a
13	1^{13}	9	3588_a
13	$1^1 2^6$	4	9360_y
13	$1^1 2^6$	6	4680_y
13	$1^1 2^6$	7	1560_y
13	$1^1 2^6$	8	936_y
13	$1^1 2^6$	9	208_y
13	$1^1 2^6$	10	117_y
13	$1^1 2^6$	11	78_y
13	$1^1 2^6$	12	$\mathbf{13}_b^p$
13	[6, 4, 3]	9	$\mathbf{13}_b$
13	[6, 6, 1]	7	26_y
13	[5, 4, 4]	8	39_y
13	$4^1 3^3$	6	312_y
13	$4^1 3^3$	7	156_y
13	$4^1 3^3$	8	78_y
13	$4^1 3^3$	9	26_y
13	$4^3 1^1$	7	78_y
13	$4^3 1^1$	9	39_y
13	$3^4 1^1$	6	312_y
13	$3^4 1^1$	8	156_y
13	$3^4 1^1$	10	52_y
13	$3^1 2^5$	4	9360_y
13	$3^1 2^5$	6	4680_y
13	$3^1 2^5$	7	884_y
13	$3^1 2^5$	8	494_y
13	$3^1 2^5$	9	156_y
13	$3^1 2^5$	10	78_y
14	1^{14}	10	6552_a

Table 3: Various compositions and distances with $9 \leq n \leq 14$.

n	comp	d	size
15	1^{15}	13	84_a
15	$6^1 3^3$	12	$\mathbf{10}_h$
15	$4^3 3^1$	12	$\mathbf{15}_h$
16	4^4	9	403_a
16	4^4	12	60_b
17	$4^3 5^1$	12	68_b
20	5^4	16	$\mathbf{16}_h$
21	$[10, 6, 5]$	14	21_b
24	$5^4 4^1$	20	$\mathbf{24}_h$
26	$[9, 8, 8]$	26	$\mathbf{26}_h$
30	6^5	25	$\mathbf{25}_h$
30	$[12, 12, 6]$	20	$\mathbf{25}_h$
31	$[15, 10, 6]$	20	31_b
32	1^{32}	28	372992_a
37	$9^3 10^1$	24	444_b
39	13^3	27	$\mathbf{27}_h$
48	$7^6 6^1$	42	$\mathbf{48}_h$
56	8^7	49	$\mathbf{49}_h$
56	$[24, 24, 8]$	35	$\mathbf{49}_h$
60	$[36, 12, 12]$	35	$\mathbf{25}_h$
63	$8^7 7^1$	56	$\mathbf{63}_h$
72	9^8	64	$\mathbf{64}_h$
72	$[27, 27, 18]$	48	$\mathbf{64}_h$
80	$[27, 27, 26]$	54	$\mathbf{80}_h$
84	21^4	64	$\mathbf{64}_h$
90	$[50, 20, 20]$	54	$\mathbf{81}_h$

Table 4: Various compositions and distances with $n \geq 15$.

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