Decompositions of Complete Graphs into Isomorphic Spanning Trees with Given Diameters

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Abstract

We use a new technique for decomposition of complete graphs with even number of vertices based on $2n$-cyclic blended labeling to show that for every $k > 1$ odd, and every d, $3 \le d \le 2^q k - 1$, there exists a spanning tree of diameter d that factorizes $K_{2^q k}$.

1 Introduction

Let H and G be simple graphs. A *G-decomposition* of a graph H on n vertices is a partition of H into pairwise edge disjoint subgraphs $G_0, G_1, \ldots, G_s$ all isomorphic to a given graph G with at most n vertices. If G has exactly n vertices and none of them is isolated, then G is called a *factor* of H and such a G-decomposition is called a *G-factorization* of H. The decomposition is *cyclic* if there exists an ordering $(x_1, x_2, \ldots, x_n)$ of the vertices of H and isomorphism $\phi_i : G_0 \longrightarrow G_i$, $i = 1, 2, \ldots, s$ such that $\phi_i(x_j) = x_{i+j}$ for each $j = 1, 2, \ldots, n$. Subscripts are taken modulo n.

Many papers have been written on graph decompositions. Decompositions of complete graphs and complete bipartite graphs received special attention. However, most of these papers deal with decompositions into isomorphic graphs of smaller order. Not that much is known about decompositions of complete graphs into isomorphic spanning trees. An obvious

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necessary condition for the existence of a G-decomposition of K_n is that the number of edges of G divides the number of edges of K_n. It follows that a factorization of a complete graph with an odd number of vertices into spanning trees is impossible. Therefore, we deal only with complete graphs with an even number of vertices K_{2n}.

It is a well known fact that K_{2n} can be factorized into Hamiltonian paths P_{2n}. It is also easy to observe that a cyclic factorization of K_{2n} into symmetric double stars is possible. First result giving a more general answer is due to P. Eldergill. In his thesis [1] he introduced a method for cyclic decomposition of K_{2n} into symmetric trees. By a symmetric tree he means a tree symmetric with respect to an edge.

Eldergill's method is, similarly as many other methods of decomposition, based on a graph labeling. Two important types of vertex labelings were introduced in 1960's by A. Rosa. In [7] he defined a ρ-labeling and a graceful labeling, which he used for cyclic decompositions of K_{2n+1} into $2n+1$ copies of a graph with n edges. Graceful or ρ-labelings were often used to construct new types of labelings, which in some sense generalize their properties. Among them are: ρ-symmetric graceful labeling introduced in [1] by P. Eldergill, allowing decomposition of K_{2n} into symmetric graphs, or a blended ρ-labeling introduced by D. Frončck [3]. A blended ρ-labeling exists for a wider class of graphs than symmetric trees and guarantees a decomposition of K_{4k+2}. A generalization of the blended ρ-labeling, called $2n$-cyclic blended labeling, was recently developed by Frončck and the author [5].

In this paper we use a method based on $2n$-cyclic blended labeling for decomposition of K_{4k}, where k is not a power of 2. More specifically, we use the method to show that there exists a spanning tree of diameter d that factorizes K_{4k} for any $d, 3 \leq d \leq 4k-1$. This complements an analogous result obtained by Frončck in [2] for the complete graphs K_{4k+2}.

2 Known methods and results

As we already mentioned, A. Rosa defined two fundamental labelings, ρ-labeling and graceful labeling (also called β-valuation).

Definition 2.1 *Let G be a graph with n edges and the vertex set $V(G)$ and let λ be an injection $\lambda : V(G) \longrightarrow S$ where S is a subset of the set $\{0,1,2,\ldots,2n\}$. The* length *of an edge (x,y) is defined as $l(x,y) = \min\{|\lambda(x)-\lambda(y)|, 2n+1-|\lambda(x)-\lambda(y)|\}$. If the set of all lengths of n edges is equal to $\{1,2,\ldots,n\}$ and $S \subseteq \{0,1,2,\ldots,2n\}$, then λ is a ρ-*labeling*; if $S \subseteq \{0,1,2,\ldots,n\}$ instead, then λ is a* graceful labeling.

Every graceful labeling is indeed also a ρ-labeling, and a graph which admits a graceful labeling is called *graceful*.

For our further needs we state here the notions related to decomposition of K_{2n} into symmetric graphs. To simplify our notation we will from now on occasionally unify a vertex with its label. It means that rather than "the vertex x such that $\lambda(x) = i$", we will say just "the vertex i".

Definition 2.2 *A connected graph G with an edge (x, y) (called a* bridge*) is* symmetric *if there is an automorphism ψ of G such that $\psi(x) = y$ and $\psi(x) = y$. The isomorphic components of $G - (x, y)$ are called* banks *and denoted by H, H', respectively. A labeling of a symmetric graph G with $2n + 1$ edges and banks H, H' is* ρ-symmetric graceful *if H has a ρ-labeling and $\psi(i) = i + n \pmod{2n}$ for each vertex i in H. A labeling of a symmetric graph G with $2n - 1$ edges is* symmetric graceful *if it is ρ-symmetric graceful and the bank H is moreover graceful. A graph which admits a ρ-symmetric graceful labeling or symmetric graceful labeling is called* ρ-symmetric graceful *or a* symmetric graceful, *respectively.*

The following theorem was proved by Eldergill for symmetric trees. Since the assumption that the graph must be acyclic was never used, the theorem is true for symmetric graphs in general.

Theorem 2.3 (Eldergill) *Let G be a symmetric graph with $2n - 1$ edges. Then there exists a cyclic G-decomposition of K_{2n} if and only if G is ρ-symmetric graceful.*

It is easy to observe how the construction of a ρ-symmetric graceful labeling is based on a ρ-labeling or a graceful labeling defined by A. Rosa. Since any graceful graph with $n-1$ edges yields a symmetric graceful graph with $2n-1$ edges, one can find an infinite class of symmetric graceful graphs whenever an infinite class of graceful graphs is known. It is well known that all caterpillars are graceful. Therefore, it is easy to construct a spanning tree of diameter d for any odd number d. Factorizations into spanning trees with even diameters are slightly more complicated. Eldergill's method is too restrictive, allowing decompositions only into symmetric graphs, which all are of odd diameter. To answer the question about spanning trees with even diameters we need a method which is more general.

To find a more general method, Frončck defined in [3] a blended ρ-labeling, which guarantees a G-decomposition of K_{4k+2}, when G has the labeling. With the blended ρ-labeling (further just blended labeling) available, Frončck found a construction of spanning trees of any diameter between 3 and $4k + 1$ that factorize K_{4k+2} for any $k \geq 1$.

A vertex set of a graph G with a blended labeling can be split into two equal partite sets V_0 and V_1, where $|V_0| = |V_1| = 2k + 1$. Subgraphs of G induced on vertices of V_0 and V_1 are denoted by H_0, H_1 respectively, and H_{01} denotes a bipartite subgraph with partite sets V_0, V_1. If a blended labeling is restricted to these three subgraphs, the labeling on H_0 and H_1 is the usual ρ-labeling which guarantees cyclic decompositions of the complete graph K_{2k+1} into k copies of H_0 or H_1, while the labeling of H_{01} is called bipartite ρ-labeling. Bipartite ρ-labeling of a graph H_{01} with $2k+1$ edges allows a decomposition of the complete bipartite graph $K_{2k+1,2k+1}$ into $2k+1$ isomorphic copies of H_{01}.

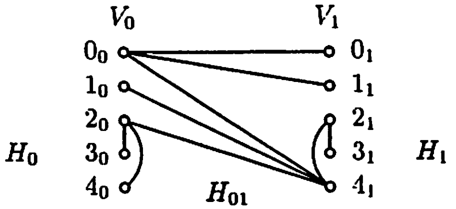

Figure 2.1: *Blended labeling of a tree on* 10 *vertices*

So far, there is no result on decomposition of K_{4k} into spanning trees with all possible diameters. To obtain such a result, a different approach must be used. A cyclic decomposition of each of the partite sets separately as in a method based on blended labeling is not possible when decomposing K_{4k}. It is so because by splitting vertices of K_{4k} into two equal partite sets $V_i, i = 0, 1$, the number of vertices in a partite set is even, namely $|V_i| = 2k$, and a cyclic decomposition of K_{2k} into $2k$ copies of a graph H_i does not exists. The reason is that $2k$ does not divide the number of edges of K_{2k}.

The known method which allows decomposition of K_{4k} into other than symmetric graphs is based on a *switching blended labeling*. This labeling is a modification of the blended labeling and was defined by Fronček and Kubesa in [4]. A switching blended labeling is still too restrictive to answer the question about the diameters. It can be shown that trees with certain diameters do not allow switching blended labeling at all, since this labeling requires a specific "strong" type of automorphism, which does not exist for these diameters.

Therefore, we use a new technique for decompositions of complete graphs with an even number of vertices developed recently by Fronček and the author in [5]. This technique allows decompositions of the complete graphs K_{2qk} where $q, k > 1$ and k is odd. It also gives the complete answer to the question about the diameters of spanning trees of such complete graphs.

3 Decomposition of K_{2nk}, $2n$-cyclic labeling

Here we describe a method of factorization of complete graph on $2nk$ vertices into n isomorphic "locally dense" factors. The method is based on

Eldergill's cyclic factorization of K_{2n} into symmetric trees.

Definition 3.1 *Let T be a symmetric tree on $2n$ vertices with a ρ-symmetric graceful labeling. We define the graph $U(T,s;k)$ with the underlying tree T, where s is the label of any vertex of T, $0 \le s \le n-1$, to have the vertex set*

$$V(U(T,s;k)) = \bigcup_{i=0}^{2n-1} V_i, \ |V_i| = k, \ V_i \cap V_j = \emptyset \text{ for } i \neq j,$$

and the edge set

$$\begin{aligned} E(U(T,s;k)) \ = \ & \{(x,y) | x \in V_i, y \in V_j \wedge (i,j) \in E(T)\} \\ & \cup \{(x,y)|x,y \in V_s\} \cup \{(x,y)|x,y \in V_{s+n}\}. \end{aligned}$$

In other words, the graph $U(T,s;k)$ is a union of $2n-1$ bipartite graphs $K_{k,k}$ on the vertices of the partite sets V_i, V_j whenever i is adjacent to j in T and two complete graphs K_k on the vertices of the vertex sets V_s and V_{s+n} for the chosen vertex with label s in T. Each vertex set V_i is of size k and its index i is the label of the corresponding vertex in T.

It is easy to observe that K_{2nk} can be decomposed into isomorphic copies of $U(T,s;k)$. This was proved in [5] and we therefore just state the relevant result here.

Lemma 3.2 *Let T be a tree on $2n$ vertices with a ρ-symmetric graceful labeling. Then there is a $U(T,s;k)$-factorization of K_{2nk} into n isomorphic copies of $U(T,s;k)$ for any $k \ge 1$.*

A graph $U(T,s;k)$ can be further $2n$-cyclically decomposed into k isomorphic copies of a graph G with $2nk-1$ edges, which consequently gives a G-decomposition of K_{2nk} into nk isomorphic copies of G. A G-decomposition of $U(T,s;k)$ is guaranteed by a $2n$-cyclic blended labeling of a graph G. We state all related notions here.

Definition 3.3 *Let G be a graph with at most $2nk$ vertices such that there exits a G-decomposition, $G_0, G_1, \ldots, G_s$, of a graph U on $2nk$ vertices. We say that the G-decomposition is $2n$-cyclic if there exists an ordering*

$(0_0, 1_0, 2_0, \ldots, (k-1)_0, 0_1, 1_1, 2_1, \ldots, (k-1)_1, \ldots, 0_{2n-1}, 1_{2n-1}, 2_{2n-1}, \ldots, (k-1)_{2n-1})$

of the vertices of U, and an isomorphisms $\phi_t : G_0 \longrightarrow G_r$, where $r = 1, 2, \ldots, s$, such that $\phi_r(x_i) = (x+r)_i \pmod k$ for every $x = 0, 1, \ldots, k-1$ and $i = 0, 1, \ldots, 2n-1$.

Definition 3.4 *Let G be a graph with $2nk-1$ edges, for k odd and $k, n > 1$, and the vertex set $V(G) = \bigcup_{i=0}^{2n-1} V_i$, where $|V_i| = k$ and $V_i \cap V_j = \emptyset$*

for $i \neq j$. *Let* λ *be an injection,* $\lambda : V_i \to \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, *for* $i = 0, 1, \ldots, 2n-1$.

The mixed length *of an edge* (x_i, y_j) *is defined as*

$$l_{i,j}(x_i, y_j) = \lambda(y_j) - \lambda(x_i) \pmod k$$

for $x_i \in V_i$, $y_j \in V_j$ *and the* pure length *of an edge* (x_i, y_i) *with* $x_i, y_i \in V_i$ *as*

$$l_{ii}(x_i, y_i) = \min\{|\lambda(x_i) - \lambda(y_i)|, k - |\lambda(x_i) - \lambda(y_i)|\}.$$

We say that G *has* $2n$-cyclic blended labeling *if there exists a tree* T *on* $2n$ *vertices with a* ρ*-symmetric graceful labeling such that*

(i) for each edge $(i,j) \in E(T)$, *is*
$\{l_{i,j}(x_i, y_j) | (x_i, y_j) \in E(G)\} = \{0, 1, 2, \ldots, k-1\}$

(ii) and for some vertex $s \in T$ *and its symmetric image* $t = s + n \pmod n$
$\{l_{ss}(x_s, y_s) | (x_s, y_s) \in E(G)\} = \{1, 2, \ldots, (k-1)/2\}$, *and*
$\{l_{tt}(x_t, y_t) | (x_t, y_t) \in E(G)\} = \{1, 2, \ldots, (k-1)/2\}$.

The edges of pure length are called *pure edges*, and the edges of mixed length are called *mixed edges*. The labeling is in fact a generalization of the blended labeling. We shall notice that similarly as a graph with a blended labeling, also a graph G with a $2n$-cyclic blended labeling can be split into subgraphs H_s and H_t on vertices of the partite sets V_s and V_t with pure edges, and $2n-1$ subgraphs H_{ij} for each $(i,j) \in E(T)$ with mixed edges. When a $2n$-cyclic blended labeling is restricted to H_s and H_t, we have just the usual ρ-labeling, while when restricted to H_{ij} we obtain a bipartite ρ-labeling.

Theorem 3.5 (Fronček, Kovářová) *Let a graph* G *with* $2nk-1$ *edges, for* k *odd and* $k, n > 1$, *have a* $2n$*-cyclic blended labeling. Then there exists a* $2n$*-cyclic* G*-decomposition of* $U(T, s; k)$ *into* k *copies of* G *and also a* G*-decomposition of* K_{2nk} *into* nk *copies of* G.

Proof. Was proved in [5]. □

4 Diameters of spanning trees factorizing $K_{2^q k}$ where k is odd

Now we answer the question about diameters of spanning trees. In our attention are complete graphs K_{4m}. The method of decomposition based on $2n$-cyclic blended labeling can be used whenever the number of vertices of K_{4m} is not a power of two. By this condition we are left with complete graphs $K_{2^q k}$, where k is odd and $k, q > 1$. Therefore we construct spanning trees of $K_{2^q k}$ with 2^q-cyclic blended labeling. Because for each such a spanning tree there must be also an underlying tree on 2^q vertices with ρ-symmetric graceful labeling, we first introduce a class of symmetric graceful trees, which will be used in constructions.

All symmetric graceful trees we deal with are caterpillars. A ***caterpillar*** is a path with attached vertices of degree one, and all caterpillars are known to have graceful labeling. A caterpillar on n vertices, which is a star $K_{1,h}$ with a path P_{n-h} attached to its central vertex is called a ***broom*** and denoted by $B(n,h)$. By $X(2n,h)$ we denote the symmetric caterpillar with banks H, H' both isomorphic to $B(n,h)$ and the symmetric edge connecting the endvertices of the paths P_{n-h}, $1 \leq h \leq n-1$. In other words, the tree $X(2n,h)$ is a union of two stars $K_{1,h}$ and the path $P_{2(n-h)}$ connecting their central vertices. To obtain a symmetric graceful labeling of $X(2n,h)$ it is sufficient to find a graceful labeling of one bank $H = B(n,h)$ since labels of the other bank H' are induced by the isomorphism $\psi(i) = i + n \pmod{2n}$ (see Def. 2.2).

There are of course more ways how to assign the labels to the vertices of $B(n,h)$ to obtain a graceful labeling. We will consider the following labeling.

Graceful labeling of a broom $B(n,h)$

- The label 0 is assigned to the central vertex of $K_{1,h}$, the labels $n-1, n-2, \ldots, n-h$ to h attached vertices of degree one. Lengths of the edges are $n-1, n-2, \ldots, n-h$.
- The vertices of the path P_{n-h} receives the labels:

  (i) $0, n-h-1, 1, n-h-2, \ldots, \frac{n-h}{2}-1, \frac{n-h}{2}$ for $n-h$ even,

  (ii) $0, n-h-1, 1, n-h-2, \ldots, \frac{n-h-1}{2}+1, \frac{n-h-1}{2}$ for $n-h$ odd, consecutively.

  The edges of the path have remaining lengths $n-h-1, n-h-2, \ldots, 1$.

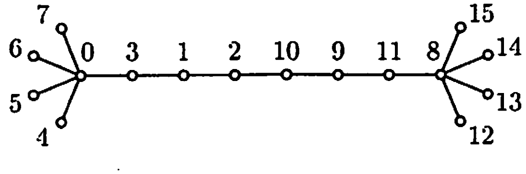

Figure 4.1: *Symmetric graceful labeling of* $X(16,4)$.

Before we state the theorem we also define two types of trees with bipartite ρ-labelings.

Construction of S_I and S_{II}

By S_I and S_{II} we denote double stars with bipartite ρ-labeling on a vertex set $V_i \cup V_j$, $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $V_j = \{0_j, 1_j, 2_j, \ldots, (k-1)_j\}$, where $k = 2m+1$ for $m \geq 1$.

- The double star S_I is constructed as two stars $K_{1,m-1}$ with the central vertices m_i and m_j connected by the edge (m_i, m_j) of the mixed length 0. The vertices of degree one connected to the central vertex m_i are $0_j, 1_j, \ldots, (m-1)_j$. The edges have mixed lengths $m+1, m+2, \ldots, 2m$. The vertices of degree one connected to the central vertex m_j are $0_i, 1_i, \ldots, (m-1)_i$ so that the edges have the missing lengths $1, 2, \ldots, m$.
- The double star S_{II} is isomorphic to S_I so that there is an isomorphism $f : V(S_I) \longrightarrow V(S_{II})$ defined by $f(x_r) = (2m - x)_r$ for every vertex $x_r \in V(S_I)$.

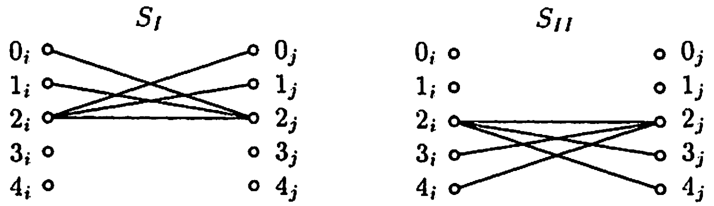

Figure 4.2: *Double stars S_I and S_{II} on partite sets with number of vertices $k = 5$.*

Construction of $C_I(D)$ and $C_{II}(D)$

By $C_I(D)$ and $C_{II}(D)$ we denote the trees with a bipartite ρ-labeling and diameter D on a vertex set $V_i \cup V_j$, $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $V_j = \{0_j, 1_j, 2_j, \ldots, (k-1)_j\}$, where $k = 2m+1$ for $m \geq 1$. The diameter D is odd, ranging from minimum 3 to maximum k. Let $D = 2t+1$, where $1 \leq t \leq m$.

- The tree $C_I(D)$, for t odd, has the diametrical bipartite path $P_{D+1} = m_i, 0_j, (m-1)_i, 1_j, \ldots, (m-\frac{t-1}{2})_i, (\frac{t-1}{2})_j, (\frac{t-1}{2})_i, (m-\frac{t-1}{2})_j, \ldots, 1_i, (m-1)_j, 0_i, m_j$.

  For t even, $P_{D+1} = m_i, 0_j, (m-1)_i, 1_j, \ldots, (\frac{t}{2}-1)_j, (m-\frac{t}{2})_i, (m-\frac{t}{2})_j, (\frac{t}{2}-1)_i, \ldots, 1_i, (m-1)_j, 0_i, m_j$.

  The edges on the path have the mixed lengths $m+1, m+2, \ldots, m+t, 0, m-t+1, m-t+2, \ldots, m-1, m$, and the missing lengths are $1, 2, \ldots, m-t$ and $m+t+1, m+t+2, \ldots, 2m$.

We obtain the edges of the missing lengths by adding two stars $K_{1,m-t}$ with the central vertices on the path P_{D+1}. When t is odd, the central vertices are $\left(\frac{t-1}{2}\right)_i$ and $\left(\frac{t-1}{2}\right)_j$. The vertices of degree one are in the other partite set than the central vertex. They are $\frac{t-1}{2}+1, \frac{t-1}{2}+2, \ldots, m-\frac{t-1}{2}-1$. When t is even, the central vertices are $\left(\frac{t}{2}-1\right)_i$ and $\left(\frac{t}{2}-1\right)_j$. The vertices of degree one in the opposite partite set are $\frac{t}{2}, \frac{t}{2}+1, \ldots, m-\frac{t}{2}-1$.

- Te tree $C_{II}(D)$ is isomorphic to $C_I(D)$ by the isomorphism $f: V(C_I(D)) \longrightarrow V(C_{II}(D))$ defined as $f(x_r) = (2m-x)_r$ for every vertex $x_r \in V(C_I(D))$.

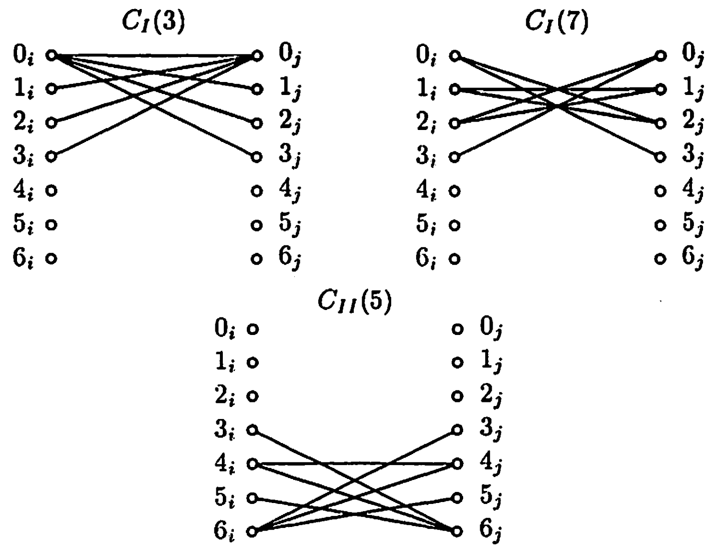

Figure 4.3: *Examples of trees $C_I(D)$ and $C_{II}(D)$ on partite sets with number of vertices $k=7$.*

Theorem 4.1 *For any d, $3 \le d \le 2^q k - 1$, there exists a tree T with the diameter d such that there is a T-factorization of the complete graph $K_{2^q k}$, where $k = 2m+1$ and $q, m > 1$.*

Proof. To obtain a spanning tree of $K_{2^q k}$ with any odd diameter is easy. We can take for instance $X(2^q, h)$, which cyclically factorizes K_{2^q} and has the diameter $d = 2(2^{q-1} - h) + 1$, where $1 \le h \le 2^{q-1} - 1$. If $h = 2^{q-1} - 1$, the caterpillar $X(2^q, h)$ is a double star with the diameter $d = 3$, which is the smallest possible. The only spanning tree of K_{2n} with smaller diameter $d = 2$ is the star, and a factorization into stars does not exists. If $h = 1$,

$X(2^q, h)$ is path P_{2^q}, and the diameter is the longest possible $d = 2^q - 1$. Further we will concentrate only on spanning trees with even diameter.

We will complete the proof in three steps, constructing spanning trees of even diameters with a 2^q-cyclic blended labeling. We always consider a spanning tree T with the vertex set $V(U_{T,k}) = \bigcup_{i=0}^{2^q-1} V_i$, where $V_i \cap V_j = \emptyset$ for $i \neq j$ and $V_i = \{0_i, 1_i, 2_i, \ldots, (k-1)_i\}$, $i = 0, 1, 2, \ldots, 2^q - 1$.

(i) Stretching the underlying double star into Hamiltonian paths (diameters: $4 \le d \le 2^q$)

As the underlying tree, we consider $X(2^q, h)$ with the symmetric graceful labeling given above. We will construct subgraphs H_{ij} with mixed edges for each $(i, j) \in E(X(2^q, h))$ and subgraphs H_0 and H_{2^q-1} with pure edges separately.

We construct each H_{ij} corresponding to an edge (i, j) on the path $P_{2(2^{q-1}-h)}$ as a double star. More precisely, we alternate double stars S_I and S_{II}.

When $2^{q-1} - h$ is even, $H_{ij} = S_I$ for

$$(i, j) \in \{(x, 2^{q-1} - h - 1 - x),\ (x + 2^{q-1}, 2^q - h - 1 - x)\},$$

where $0 \le x \le \frac{2^{q-1}-h}{2} - 1$.

$H_{ij} = S_{II}$ for

$$(i, j) \in \{(2^{q-1} - h - x, x),\ (2^q - h - x, x + 2^{q-1})\} \cup \{(\tfrac{2^{q-1}-h}{2}, \tfrac{2^{q-1}-h}{2} + 2^{q-1})\},$$

where $1 \le x \le \frac{2^{q-1}-h}{2} - 1$.

When $2^{q-1} - h$ is odd, $H_{ij} = S_I$ for

$$(i, j) \in \{(x, 2^{q-1} - h - 1 - x),\ (x + 2^{q-1}, 2^q - h - 1 - x)\} \cup \{(\tfrac{2^{q-1}-h-1}{2}, \tfrac{2^{q-1}-h-1}{2} + 2^{q-1})\},$$

where $0 \le x \le \frac{2^{q-1}-h-1}{2} - 1$.

$H_{ij} = S_{II}$ for

$$(i, j) \in \{(2^{q-1} - h - x, x),\ (2^q - h - x, x + 2^{q-1})\},$$

where $1 \le x \le \frac{2^{q-1}-h-1}{2}$.

The subgraphs H_{ij} corresponding to the $2h$ vertices of degree one in $X(2^q, h)$ are constructed as the stars $K_{1,2m+1}$.

For $(i, j) \in \{(0, 2^{q-1} - 1), (0, 2^{q-1} - 2), \ldots, (0, 2^{q-1} - h)\}$,

the star $K_{1,2m+1}$ has the central vertex $(m+1)_0$ and the attached vertices of degree one are all $2m + 1$ vertices of V_j.

For $(i, j) \in \{(2^{q-1}, 2^q - 1), (2^{q-1}, 2^q - 2), \ldots, (2^{q-1}, 2^q - h)\}$, the star $K_{1,2m+1}$ has the central vertex $m_{2^{q-1}}$ and again $2m+1$ leaves in V_j.

Obviously, in each star $K_{1,2m+1}$ we have $2m+1$ edges, one of each mixed length $0, 1, \ldots, 2m$.

To obtain H_0 and $H_{2^{q-1}}$ we add the star $K_{1,m}$ on vertices of V_i for $i \in \{0, 2^{q-1}\}$. The central vertex of $K_{1,m}$ is m_i and the leaves are $(m+1)_i, m+2)_i, \ldots, (k-1)_i$, so that we have all required edges of pure lengths $1, 2, \ldots, m$.

Now if we "glue" all subgraphs H_{ij}, H_0, and $H_{2^{q-1}}$ together, we obtain a tree T with 2^q-cyclic blended labeling which guarantees a 2^q-cyclic T-factorization of $U(X(2^q, h), 0, k)$ and consequently a T-factorization of $K_{2^q k}$. To the diameter d of T each of the $2^q - 2h - 1$ double stars S_I, S_{II} contributes by 1. All stars $K_{1,m}$ and $K_{1,2m+1}$ contribute all together by 3. Then such a spanning tree T has the diameter $d = 2^q - 2h + 2$. For h ranging from 1 to $2^{q-1} - 1$ we get spanning trees with all even diameters d from the interval $4 \le d \le 2^q$.

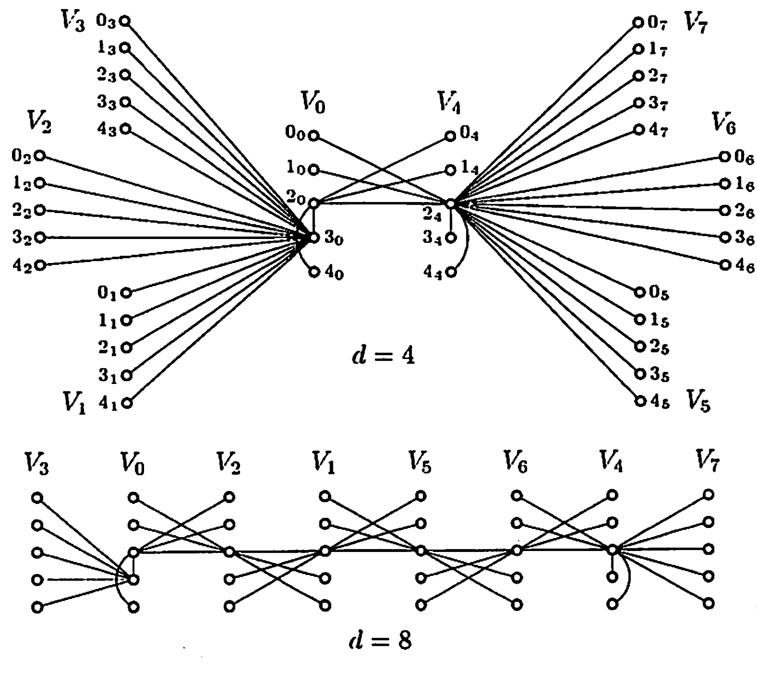

Figure 4.4: *Spanning trees of K_{40} with 8-cyclic blended labeling and diameters $d = 4$ and $d = 8$.*

(ii) Stretching the bipartite paths (diameters: $2^q + 2 \leq d \leq 2^q k - k + 1$)

The longest diameter of previous case we obtained for $h = 1$ when the underlying tree was the path $X(2^q, 1) = P_{2^q}$. The underlying tree cannot be stretched anymore, and to obtain longer diameter than $d = 2^q$ we have to increase the diameters of the subgraphs H_{ij}.

Suppose the underlying tree is $X(2^q, 1) = P_{2^q}$ again with the symmetric graceful labeling given above. We start with a spanning tree T of the odd diameter $d = 2^q - 1$. Again each subgraph H_{ij} corresponding to the edge $(i, j) \in E(P_{2^q})$ is a double star S_I or S_{II}.

$$\text{For} \quad (i, j) \in \{(2^{q-1} - 1 - x, x),\ (2^q - 1 - x, x + 2^{q-1})\},$$

where $0 \leq x \leq 2^{q-2} - 1$, the subgraph H_{ij} is constructed as S_I.

$$\text{For} \quad (i, j) \in \{(x, 2^{q-1} - 2 - x,),\ (x + 2^{q-1}, 2^q - 2 - x,)\} \cup \{(2^{q-2} - 1, 3 \cdot 2^{q-2} - 1)\},$$

where $0 \leq x \leq 2^{q-2} - 2$, the subgraph H_{ij} is constructed as S_{II}.

We choose the endvertices of P_{2^q}, which are $2^{q-1} - 1$ and $2^q - 1$, to construct two subgraphs $H_{2^{q-1}-1}$, H_{2^q-1} with pure edges. The subgraph $H_{2^{q-1}-1}$ is the star $K_{1,m}$ with the central vertex 0_i and m vertices of degree one $(m+1)_i, (m+2)_i, \ldots, (2m)_i$, where $i = 2^{q-1} - 1$. The subgraph H_{2^q-1} is also the star $K_{1,m}$ with the central vertex m_i and m vertices of degree one $(m + 1)_i, (m + 2)_i, \ldots, (2m)_i$, where $i = 2^q - 1$.

By gluing all subgraphs H_{2^q-1}, $H_{2^{q-1}-1}$, and H_{ij} we get the spanning tree T of $U(P_{2^q}, 2^{q-1} - 1; k)$ with a 2^q-cyclic labeling. We can choose a diametrical path of T so that the subgraphs $H_{ij} = S_I$ corresponding to the first and the last edge on P_{2^q} contribute to the diameter d by 2 and all the other $2^q - 3$ subgraphs H_{ij} by 1. Two stars $K_{1,m}$ do not extend diameter, and so $d = 2^q + 1$.

Now we can replace the first double star S_I corresponding to the first edge on P_{2^q} by a tree $C_I(D)$. Diameter D of $C_I(D)$ is odd, ranging from 3 to k, which extends the diameter d of the spanning tree always by 2 from $2^q + 2$ to $2^q - 1 + k$.

Similarly we replace stepwise all $2^q - 1$ double stars S_I and S_{II} by trees $C_I(D)$ and $C_{II}(D)$, respectively. When one of the stars is replaced and D is changed gradually we obtain spanning trees with the next $\frac{k-1}{2}$ even diameters. The longest diameter is $d = 2^q - 1 + k + (2^q - 2)(k - 1) = 2^q k - k + 1$. Overall we obtain spanning trees with even diameters $2^q + 2 \leq d \leq 2^q k - k + 1$.

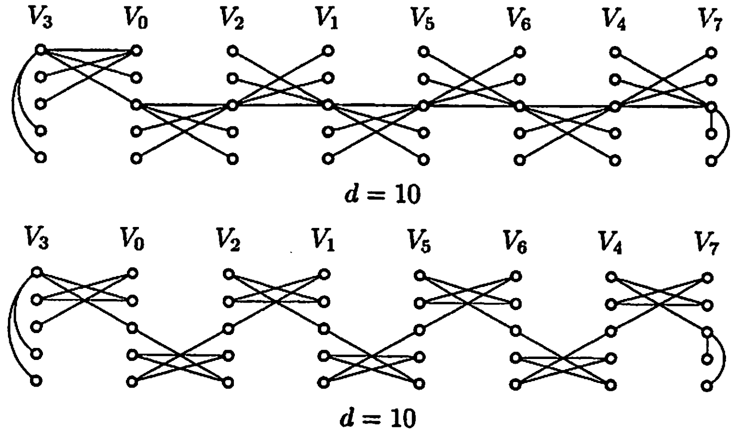

Figure 4.5: *Spanning trees of K_{40} with 8-cyclic blended labeling and diameters $d = 10$ and $d = 36$.*

(iii) Stretching subgraphs with pure edges (diameters: $2^q k - k + 2 \leq d \leq 2^q k - 1$)

In this case the underlying tree is of course again the path P_{2^q}. The subgraphs H_{ij}, for each $(i, j) \in E(P_{2^q})$ are constructed as for the longest diameter in the previous case. It means that they alternate between the graphs $C_I(k)$ and $C_{II}(k)$. The only way how to increase the diameter d of the spanning tree T is to extend the diameter of the subgraphs $H_{2^{q-1}-1}$ and H_{2^q-1} with pure edges.

We start with the odd diameter $d = 2^q k - k + 2$ which is obtained if both subgraphs $H_{2^{q-1}-1}$ and H_{2^q-1} are the stars $K_{1,m}$ with the central vertices m_i, where $i \in \{2^{q-1} - 1, 2^q - 1\}$.

Then we convert one of the stars, say in partite set V_i for $i = 2^{q-1} - 1$, to a broom $B(m + 1, s)$, where $1 \leq s \leq m - 1$. If $m + 1 - s = 2r$, the vertices of the path P_{m+1-s} are $m_i, 2m_i, (m + 1)_i, \ldots, (m + r - 1)_i, (2m + 1 - r)_i$, and the star $K_{1,s}$ has the central vertex $(2m+1-r)_i$ with attached vertices of degree one, $(m+r)_i, (m+r+1)_i, \ldots, (2m - r)_i$. If $m + 1 - s = 2r + 1$, the path P_{m+1-s} has the vertices $m_i, 2m_i, (m + 1)_i, \ldots, (2m + 1 - r)_i, (m + r)_i$, and the star $K_{1,s}$ has the central vertex $(m + r)_i$. The vertices of degree one are $(m + r + 1)_i, (m + r + 2)_i, \ldots, (2m - r)_i$. The edges have in both cases pure lengths $m, m - 1, \ldots, 1$. The diameter of each broom $B(m+1, s)$ is $m + 1 - s$.

When s is changing from $m - 1$ to 1, we obtain the spanning trees with even and odd diameters $2^q k - k + 3, 2^q k - k + 4, \ldots, 2^q k - k + m + 1 =$

$2^q k - \frac{k+1}{2} + 1$. We can repeat the procedure with the brooms in the partite set V_{2^q-1} to obtain the spanning trees with the missing diameters $2^q k - \frac{k+1}{2} + 2, 2^q k - \frac{k+1}{2} + 3, \ldots, 2^q k - \frac{k+1}{2} + m = 2^q k - 1$.

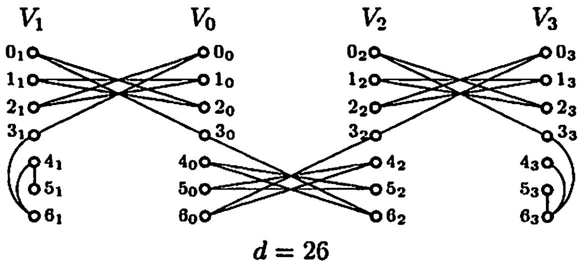

Figure 4.6: *Spanning tree of K_{28} with 4-cyclic blended labeling and diameter $d = 26$.*

Now we constructed spanning trees of all diameters $3 \leq d \leq 2^q k - 1$ and so the proof is complete. □

It remains to solve the problem for the case when the number of vertices of K_{4n} is a power of two, for K_{2^q}. In this case $2n$-cyclic blended labeling cannot be used, since the assumption that the number of vertices of the complete graph is divisible by some odd k is not satisfied. So far we have only partial results based on decomposition of K_{2^q} into 2^{q-2} copies of $U(T, s; 4)$, where $q \geq 3$.

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