On The Edge-graceful Trees Conjecture

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ABSTRACT

A (p,q)-graph G is said to be **edge graceful** if the edges can be labeled by 1,2,..., q so that the vertex sums are distinct, mod p. It is shown that if a tree T is edge-graceful then its order must be odd. Lee conjectured that all trees of odd orders are edge-graceful. The conjecture is still unsettled. In this paper we give the state of the progress toward this tantalizing conjecture.

Key words and phrases: edge-graceful, super edge-graceful, trees, tree reduction, irreducible, diameter, spider.

AMS 2000 MSC : 05C78, 05C25

1.Introduction. All graphs in this paper are simple graphs. A graph with p vertices and q edges is graceful if there is an injective mapping f from the vertex set V(G) into {0,1,2,...,q} such that the induced map $f^*:E(G)\rightarrow\{1,2,\ldots,q\}$ defined by $f^*(e)=|f(u)-f(v)|$ where e=(u,v), is surjective.

Graceful graph labelings were first introduced by Alex Rosa (around 1967) as means of attacking the problem of cyclically decomposing the complete graph into other graphs. Since Rosa's original article, literally hundreds of papers have been written on graph labelings (see [3]). A well-known conjecture due to Ringel and Kotzig is that all trees are graceful. This notorious conjecture is still unsolved [4]. Rosa [19] showed that all caterpillars are graceful and also that all trees on at most 16 vertices are graceful. In 1998 Aldred and McKay [1] with the aid of computer showed that all trees on at most 24 vertices are graceful.

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Another dual concept of graceful labeling on graphs, edge-graceful labeling, was introduced by S.P. Lo [16] in 1985. G is said to be **edge-graceful** if the edges are labeled by 1,2,3,...,q so that the vertex sums are distinct, mod p.

Figure 1 shows a grid with 12 vertices and 17 edges with two different edge-graceful labelings.

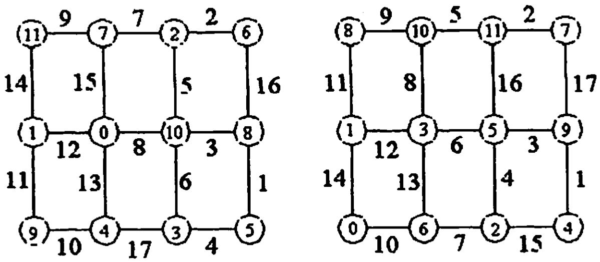


Figure 1.

A necessary condition of edge-gracefulness is (Lo [16])

$$q(q+1)\equiv\frac{p(p-1)}{2}\pmod p \qquad (1)$$

The following tantalizing conjecture is proposed in [8]

<u>**Conjecture 1**</u>: The Lo's condition (2) is sufficient for a connected graph to be edge-graceful.

A sub-conjecture of the above (Lee [7]) has also not yet been proved:

<u>**Conjecture 2**</u>: All odd-order trees are edge-graceful.

In [2,5,8,24] several classes of trees of odd orders are proved to be edge-graceful.

J. Mitchem and A. Simoson [18] introduced the concept of super edge-graceful graphs which is a stronger concept than edge-graceful for some classes of graphs. A graph G=(V,E) of order p and size q is said to be **super edge-graceful** if there exists a bijection

$f: E\to\{0, +1,-1,+2,-2,\dots,(q-1)/2, -(q-1)/2\}$ if q is odd

$f: E\to\{ +1,-1,+2,-2,\dots,(q-1)/2, -(q-1)/2\}$ if q is even

such that the induced vertex labeling f* defined by $f^*(u)= \Sigma f(u,v): (u,v) \in E \}$has the property:

$f^*: V\to\{0,+1,-1,\dots,+(p-1)/2,-(p-1)/2\}$ if p is odd

$f^*: V\to\{+1,-1,\dots,+p/2,-p/2\}$ if p is even

is a bijection. The following figure shows that the path P_5 is super edge-graceful.

Figure 2

Mitchem and A. Simoson [18] showed that

Theorem 1.1. If G is a super-edge-graceful graph and $q \equiv -1 \pmod p$, if q is even or $q \equiv 0 \pmod p$, if q is odd , then G is also edge-graceful.

Thus,

Corollary 1.2. If G is super edge-graceful tree of odd order then it is edge-graceful.

If a tree G is super edge-graceful with labelings pair (f, f*). We define an edge labeling l: $E(G) \to \{1,2,\ldots,p-1\}$ by setting $l(e) = f(e)$ if f(e) is positive and $l(e) = p + f(e)$ if f(e) is negative. Then the edge labeling l is edge-graceful. Figure 3 illustrates how to convert a super edge-graceful tree of order 7 into an edge-graceful tree.

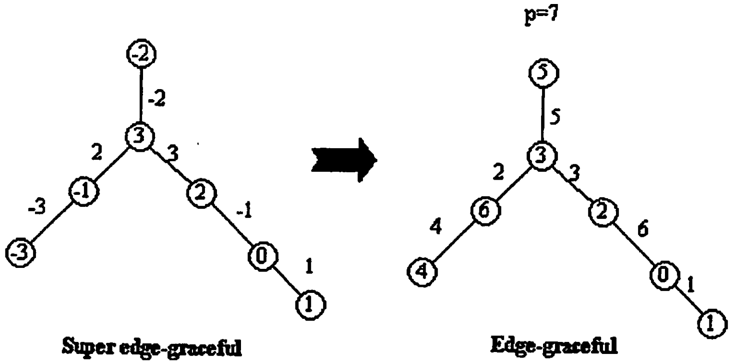


Figure 3

Finding the edge-graceful labelings of graphs is related to solving system of linear Diophantine equations. In general it is difficult to find an edge-graceful labeling of a graph. Several classes of graphs had been shown to be edge-graceful ([9-24]).

Cahit [3] showed that all trees of diameter 3 are graceful. Zhao [26] in 1989 showed that all trees of diameter four are graceful. In section 3, we show that for the class of trees of odd orders, to check a tree T is edge-graceful can be simplified by considering super edge-gracefulness of its irreducible part T. We can show that all trees of odd order of diameter at most four are edge-graceful.

2.Super edge-graceful trees of even orders.

Trees of order 4 and 6 are not super edge-graceful.

However, trees of even order larger than 6 may be super edge-graceful. The following Figure 4 shows that path P_8 of order 8 is super edge-graceful.

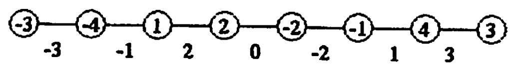


Figure 4.

Theorem 2.1. The star ST(n) is super-edge-graceful if and only if n is even.

Proof. If n is even, say n=2s, we observe any bijection f: E(ST(n))→{1,2,…,s,-1,-2,…,-s} is a super-edge-graceful labeling.

If n is odd, say n=2s+1, we see that any bijection f: E(ST(n))→{0,1,2,…,s,-1,-2,…,-s} is not a super-edge-graceful labeling for there are two vertices have labels 0.

Example 1. The following five trees of order 8 are not super edge-graceful.

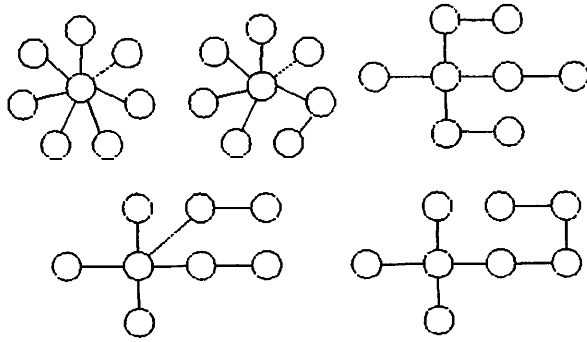
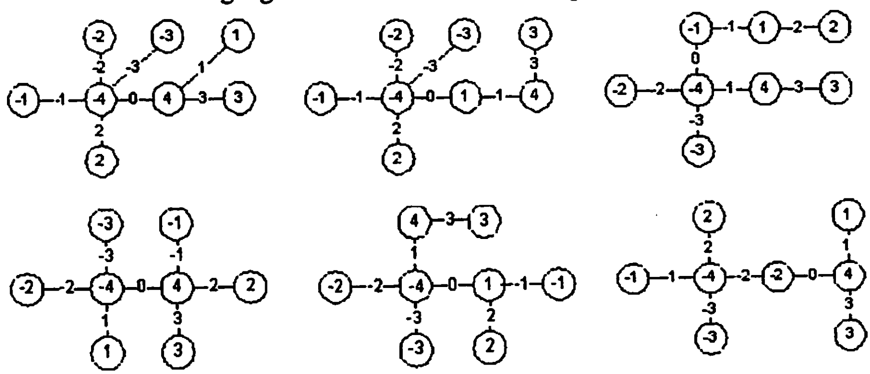


Figure 5

Theorem 2.2. Among twenty three trees of order 8 there are eighteen super edge-graceful.

The following eighteen trees of order 8 are super edge-graceful.



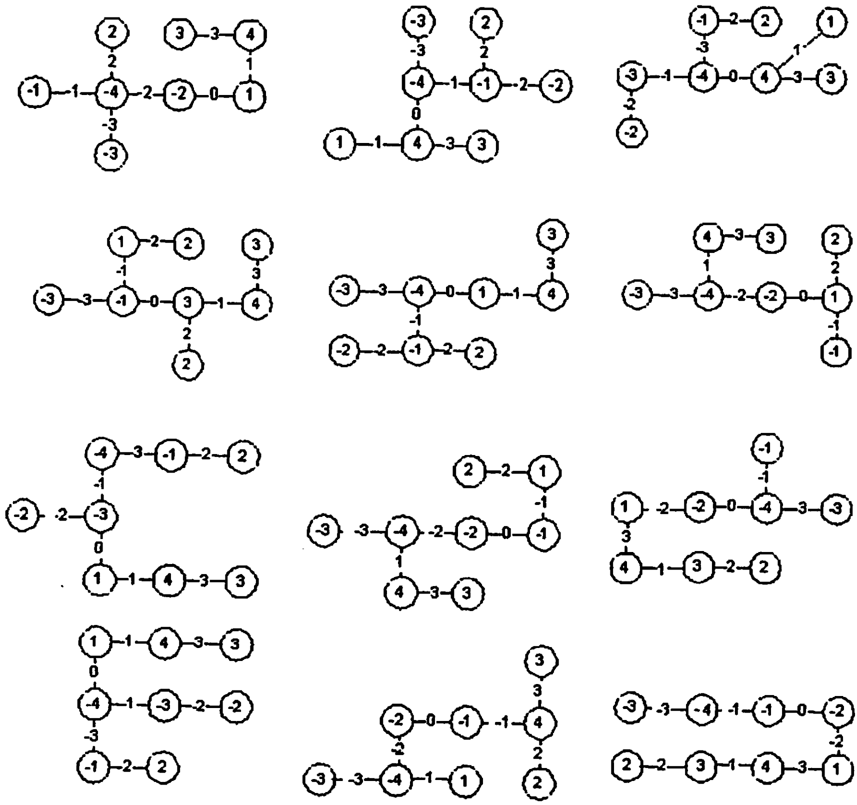


Figure 6.

With the aid of the computer, we show that

Theorem 2.3. Among one hundred and six trees of order 10, seventy four are super edge-graceful.

3. Edge-graceful trees of small orders.

In [18] Mitchem and Simoson provided the following Growing Tree Algorithm: Let T be a super-edge-graceful tree with 2n edges. If any two vertices are added to T such that both are adjacent to a common vertex of T, then the new tree T* is also super-edge-graceful.

If the tree T has odd order p, then the above growing tree algorithm will provide the super edge-graceful labeling of T* directly by assign the two edges with (p+1)/2, -(p+1)/2. However, if the tree T is of even order then we cannot obtain the super edge-graceful labeling of T* directly by this method. Consider the following example:

Example 2. The spider Sp(1,1,3) is super edge-graceful. We see that the super edge-graceful labeling of Sp(1,1,3)* cannot be obtained by assign directly the two new edges with 3, -3.(Figure 7)

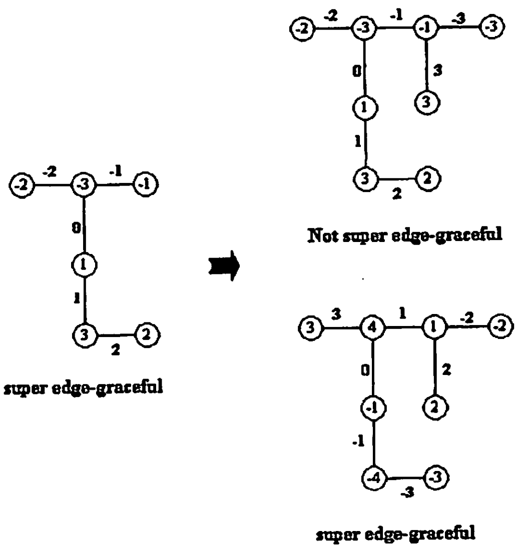


Figure 7.

We reverse the above adding edges process and define a type of reducibility for trees.

Tree Reduction Algorithm: For a tree T we delete all sets of even number of leaves incident with the same vertex and generate a new tree T*. Continue with the deletion process until no such sets of even number of leaves can be found. The final tree is said to be **irreducible part** of T and will denoted by irr(T).

We see that for a tree T of odd order if irr(T) is super edge-graceful then T is super edge-graceful.

Definition 3.1. A tree T is said to be **irreducible** if T = irr(T).

Example 3. Figure 8 illustrates the irreducible tree of order 7 is obtained by the above process for a tree with 15 vertices.

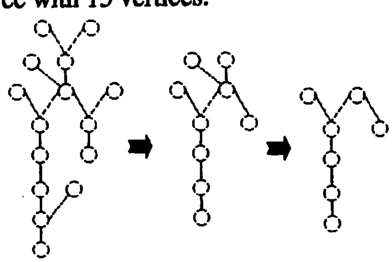


Figure 8.

Example 4. The following (Figure 9) are three irreducible trees of order 7

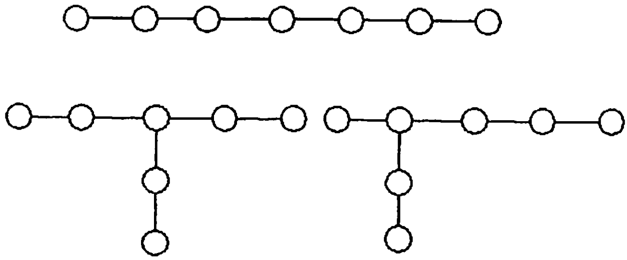


Figure 9.

Theorem 3.1. A tree of order 1 is irreducible. There does not exist irreducible tree of order 3. There are one irreducible tree of order 5, three irreducible trees of order 7, ten irreducible trees of order 9, and thirty nine irreducible trees of order 11.

We observe all odd order trees, with eleven or fewer vertices, reduce to fifty-four irreducible trees.

All fifty-four irreducible trees of odd order, less than or equal to eleven, are super edge-graceful; and therefore, all trees of odd orders less than 13 are edge-graceful. This implies that to prove Lee's conjecture 2 for fixed order is, potentially, simplified.

In the following we list 39 irreducible trees of order 11 and their super edge-graceful labeling. We use the following notation to present the tree and its super edge-graceful labeling. For example "Graph #1 1 2 -5 2 3 -4 2 4 -3 2 5 3 2 6 5 3 7 1 4 8 4 5 9 2 6 10 -2 7 11 -1" indicates labelings for the first tree and "1 2 -5" means the edge, connecting vertices #1 and #2, is labeled by -5. The notation is repeated for each edge in the tree.

Graph #1 1 2 -5 2 3 -4 2 4 -3 2 5 3 2 6 5 3 7 1 4 8 4 5 9 2 6 10 -2 7 11 -1

Graph #2 10 2 -5 2 3 -4 2 4 -2 2 5 3 2 6 4 3 7 2 4 8 -3 5 9 1 6 1 -1 10 11 5

Graph #3 1 2 -5 2 3 -4 2 4 1 2 5 4 3 6 3 3 7 5 4 8 -1 5 9 -3 6 10 2 7 11 -2

Graph #4 1 2 -5 2 3 -4 2 4 1 2 5 4 3 6 -1 3 7 3 4 8 -2 5 9 -3 6 10 5 8 11 2

Graph #5 1 2 -5 2 3 -4 2 4 1 2 5 4 3 6 -1 3 7 5 4 8 3 5 9 -2 6 10 2 10 11 -3

Graph #6 10 2 -5 2 3 -4 2 4 -1 2 5 5 3 6 2 3 7 -2 4 8 1 5 9 -3 6 1 3 10 11 4

Graph #7 1 2 -5 2 3 -4 2 4 1 2 5 4 3 6 3 4 7 2 5 8 -3 6 9 -1 6 10 -2 9 11 5

Graph #8 1 2 -5 2 3 -4 2 4 1 2 5 4 3 6 2 4 7 -1 5 8 -2 6 9 -3 7 10 5 8 11 3

Graph #9 1 2 -5 2 3 -4 2 4 1 2 5 4 3 6 2 4 7 3 5 8 -3 6 9 -2 7 10 -1 9 11 5

Graph #10 10 2 -5 2 3 -4 2 4 1 2 5 3 3 6 5 4 7 -1 5 8 2 6 9 -2 7 1 -3 10 11 4

Graph #11 1 2 -5 2 3 -4 2 4 1 2 5 5 3 6 3 4 7 4 5 8 -2 6 9 -3 9 10 -1 10 11 2

Graph #12 10 2 -5 2 3 -4 2 4 -1 2 5 5 3 6 3 4 7 4 5 8 -3 6 9 2 9 1 -2 10 11 1

Graph #13 1 2 -5 2 3 -4 2 4 5 3 5 -2 3 6 4 4 7 -3 4 8 2 5 9 -1 6 10 1 7 11 3

Graph #14 1 2 -5 2 3 -4 2 4 5 3 5 -3 3 6 4 4 7 -2 4 8 2 5 9 1 7 10 3 9 11 -1

Graph #15 10 2 -5 2 3 -4 2 4 4 3 5 -3 3 6 3 4 7 -2 4 8 2 5 9 1 7 1 -1 10 11 5

Graph #16 1 2 -5 2 3 -4 2 4 5 3 5 -3 3 6 4 4 7 -1 5 8 2 6 9 -2 7 10 1 8 11 3

Graph #17 1 2 -5 2 3 -4 2 4 5 3 5 -1 3 6 3 4 7 -2 5 8 2 6 9 -3 7 10 1 10 11 4

Graph #18 10 2 -5 2 3 -4 2 4 4 3 5 -3 3 6 3 4 7 -1 5 8 2 6 9 1 7 1 -2 10 11 5

Graph #19 1 2 -5 2 3 -4 2 4 5 3 5 -3 3 6 4 4 7 -1 5 8 3 6 9 1 8 10 -2 9 11 2

Graph #20 1 2 -5 2 3 -4 2 4 5 3 5 -3 3 6 4 4 7 -1 5 8 3 6 9 -2 8 10 2 10 11 1

Graph #21 1 2 -5 2 3 -4 2 4 5 3 5 -3 3 6 4 4 7 -2 5 8 1 7 9 3 7 10 -1 9 11 2

Graph #22 1 2 -5 2 3 -4 2 4 5 3 5 -3 3 6 4 4 7 -2 5 8 1 7 9 3 8 10 -1 9 11 2

Graph #23 1 2 -5 2 3 -4 2 4 5 3 5 -3 3 6 4 4 7 -2 5 8 1 7 9 2 9 10 3 10 11 -1

Graph #24 1 2 -5 2 3 -4 2 4 5 3 5 2 4 6 -3 5 7 -2 5 8 1 6 9 3 7 10 -1 8 11 4

Graph #25 1 2 -5 2 3 -4 2 4 5 3 5 2 4 6 -2 5 7 1 5 8 -3 6 9 3 7 10 4 9 11 -1

Graph #26 1 2 -5 2 3 -4 2 4 5 3 5 1 4 6 -3 5 7 -2 5 8 -1 6 9 4 7 10 2 10 11 3

Graph #27 10 2 -5 2 3 -4 2 4 4 3 5 1 4 6 -3 5 7 -2 5 8 5 6 9 -1 7 1 2 10 11 3

Graph #28 10 2 -5 2 3 -4 2 4 4 3 5 3 4 6 -2 5 7 -1 5 8 2 7 9 -3 8 1 1 10 11 5

Graph #29 1 2 -5 2 3 -4 2 4 5 3 5 1 4 6 -3 5 7 -1 6 8 4 7 9 2 7 10 -2 9 11 3

Graph #30 1 2 -5 2 3 -4 2 4 5 3 5 1 4 6 -3 5 7 3 6 8 4 7 9 2 8 10 -1 9 11 -2

Graph #31 10 2 -5 2 3 -4 2 4 4 3 5 1 4 6 -3 5 7 -2 6 8 -1 7 9 2 8 1 5 10 11 3

Graph #32 1 2 -5 2 3 -4 2 4 5 3 5 1 4 6 -2 5 7 -3 6 8 4 7 9 3 9 10 2 10 11 -1
Graph #33 10 2 -5 2 3 -4 2 4 4 3 5 5 4 6 1 5 7 -1 6 8 -2 7 9 -3 9 1 3 10 11 2
Graph #34 9 2 -5 2 3 -4 2 4 4 3 5 1 4 6 -3 5 7 3 6 8 -1 7 1 2 9 10 5 10 11 -2
Graph #35 1 2 -5 2 3 -4 2 4 5 3 5 1 4 6 -2 5 7 -1 7 8 -3 7 9 3 8 10 4 9 11 2
Graph #36 1 2 -5 2 3 -4 2 4 5 3 5 1 4 6 -1 5 7 -3 7 8 4 8 9 -2 8 10 3 9 11 2
Graph #37 1 2 -5 2 3 -4 2 4 5 3 5 1 4 6 -2 5 7 -1 7 8 3 8 9 2 9 10 -3 10 11 4
Graph #38 10 2 -5 2 3 -4 2 4 4 3 5 3 4 6 -2 5 7 1 7 8 2 8 9 -1 9 1 -3 10 11 5
Graph #39 1 2 -5 2 3 1 3 4 -4 4 5 2 5 6 -3 6 7 3 7 8 -2 8 9 4 9 10 -1 10 11 5

__Theorem 3.2.__ All trees of odd order at most 17 are super edge-graceful.

The listing of super edge-graceful labelings for 48629 trees of order 17 will be over four thousand pages in printing. We posted the file on the website at http://www.cs.sjsu.edu/faculty/lee, and interested readers can check from there.

4. Edge-graceful Trees of Diameter at most four

Let T be a super edge-graceful irreducible tree, we denote the class of all trees H such that under the super edge-graceful reducibility algorithm it transforms to T by ***Core(T)***.

We have the following result

__Theorem 4.1.__ All odd trees with only one vertex of even order is super edge-graceful reducible.

Proof. A tree of odd order with only one vertex of even order is in Core (K_1) which is super edge-graceful reducible.

In particular, we have

__Corollary 4.2 .__ All complete k-ary trees of even k are super edge-graceful reducible.

The double star D(m,n) is a tree of diameter three such that there are m appended edges on one ends of P_2 and n appended edges on another end.(Figure 10)

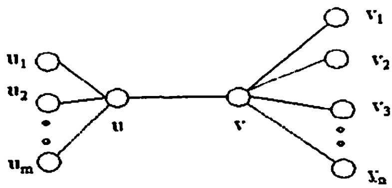


Figure 10.

<u>**Theorem 4.3.**</u> A double star D(m,n) is super edge-graceful if and only if (m,n) $\neq$ (1,k) where k is odd $\geq$ 1.

Proof. Suppose a double star D(m,n) is given. We want to show that if m =1 and n is odd then D(1,n) is not super edge-graceful. Since |E| is odd and |V| is even, 0 must be the label for some edge, but cannot be the label for any vertex. Therefore, if the graph is super edge-graceful, then 0 must be the label for the edge (u,v). Let the label for the edge (u_1,u) be x. Then both the vertices u_1 and u must have labels x. This labeling cannot be super edge-graceful.

If m and n are even, then its irreducible part D(m,n)* is P_2 which is super edge-graceful. Thus D(m,n) is super edge-graceful.

If only one of m and n is odd, then its irreducible part D(m,n)* is P_3 which is super edge-graceful. Hence D(m,n) is super edge-graceful.

If m,n are odd and (m,n) $\neq$ (1,k) then m $\geq$ 3 and n $\geq$ 3. We can apply the Tree reduction algorithm to reduce the tree to the form D(3,3). We observe that D(3,3) is in Core (P_4). If we label the edge of P_2 by 0, the three edges append to one side of P_2 by {-1,2,3} and another three edges of other side by {1,-2,-3} then it is clear that this labeling of D(3,3) is super edge-graceful. Hence D(m,n) is super edge-graceful.

<u>**Remark.**</u> There exits super edge-graceful trees of even orders using the Tree Reduction Algorithm the final trees are not super edge-graceful. D(3,3) has irreducible part which is P_4 and is not super edge-graceful.

<u>**Example 5.**</u> Figure 11 illustrates a tree of order 10 such that its super edge-graceful labeling can not be derived from a tree of order 8.

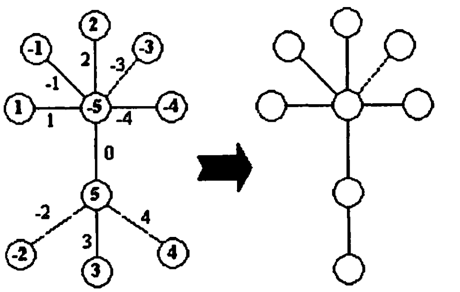


Figure 11.

A tree is called a **spider** if it has a center vertex c of degree k >1 and all the other vertex is either a leaf or with degree 2. Thus a spider is an amalgamation of k paths with various lengths. If it has x1's path of length a1, x2's path of length x2,..., we shall denote the spider by $SP(a_1^{x1}a_2^{x2},,,a_m^{xm})$ where $a_1 < a_2 < \ldots < a_m$ and x1+x2+..+xm=k. (see Figure 12).

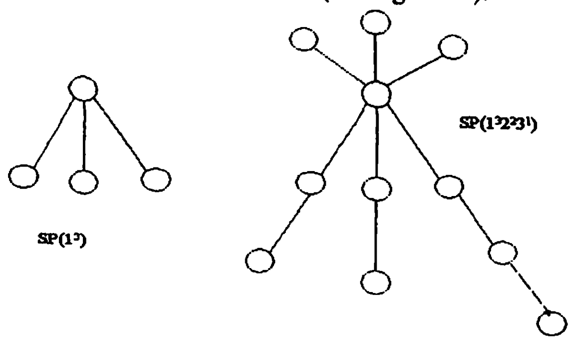


Figure 12

J. Keene and A. Simoson [4] showed that
Theorem 4.4 All three legged spiders of odd order are edge-graceful.

Theorem 4.5. Any four-legged spider of odd order, with two legs of equal length, is edge-graceful.

J. Mitchem and A. Simoson [18] also showed that
Theorem 4.6. Every regular spider of odd order is super-edge –graceful .

For each spider describe in the above theorems, the class Core(T) consists of infinite many edge-graceful trees.

By Theorem 4.3., we know that all trees of odd order with diameter three are edge-graceful. In the following part, we want to demonstrate the following result

Theorem 4.7.. All trees of odd orders and of diameter at most 4 are edge-graceful.

We need the following

Lemma 4.8. Any irreducible tree T of odd order and of diameter 4 is of the form $Sp(2^k)$ for some integer $k \geq 2$.

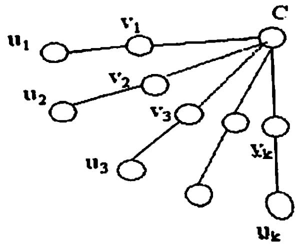


Figure 13.

Proof. Note that the following irreducible tree T is of even order, and that if C has more than one leaf child, then a pair of them can be deleted. Thus it is not irreducible.

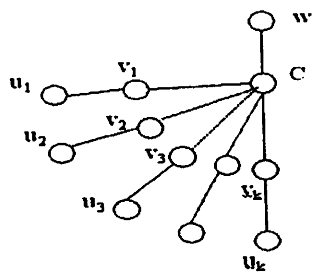


Figure 14.

<u>**Lemma 4.9**</u>. The irreducible tree Sp(2^k) is super edge-graceful if k =2n.
Proof. When k=2n, the following labeling is super edge-graceful:

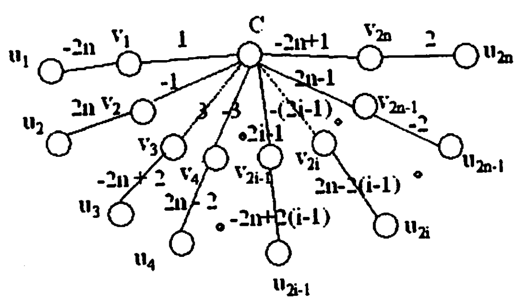


Figure 15.

Let $L_1 = \{2i-1,-(2i-1): i=1,2,\ldots,2n\}$ and $L_2 = \{-2n+2(i-1),2n-2(i-1): i=1,2, \ldots, 2n\}$. Then the set of edge labels is $L_1 \cup L_2$, which is $\{i,-i: i=1,2, \ldots, 2n\}$ Noting that $(2i-1)+(-2n+2(i-1)) = -2(n-i)-3$ and

$$-(2i-1)+(2n-2(i-1)) = 2(n-i)+3$$

we see that the set of induced vertex labels is

$$\{0\} \cup L_2 \cup \{-2(n-i)-3,2(n-i)+3: i=1,2,\ldots,2n\}$$

which is $\{0,i,-i: i=1,2,\ldots,2n\}$.Therefore, the above labeling is super edge-graceful.

<u>**Lemma 4.10**</u>. The irreducible tree $Sp(2^k)$ is super edge-graceful if $k=2n+1$.
Proof. When $k=2n+1$, the following labeling is super edge-graceful:

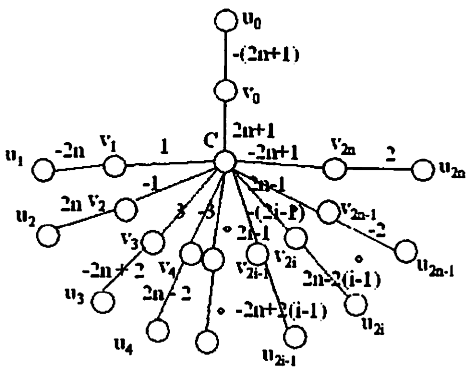


Figure 16.

Note that the number of edges is $4n+3$ and the vertex label $L(C) = 2n+1$, $L(v_0) = 0$, $L(u_0) = -(2n+1)$. The rest of the proof is similar to the proof of Lemma 4.9 .

Now we observe that every tree of odd order T of diameter 4 can be super edge-graceful reducible to some $Sp(2^k)$. By Lemma 4.8.4.9 and 4.10, we observe that T is super edge-graceful.

<u>Examples 6.</u> Figure 17 shows the super edge-graceful labelings of $SP(2^6)$ and $SP(2^5)$ respectively.

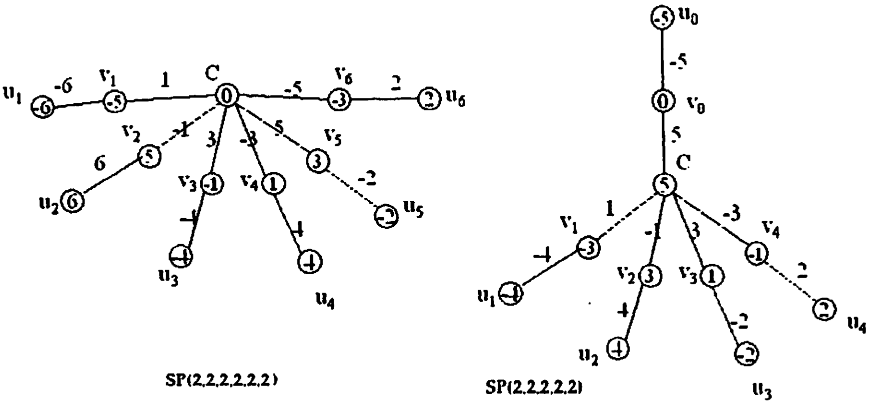


Figure 17.

5. Conjecture.

We propose here a conjecture which is stronger than the edge-graceful trees conjecture.

Conjecture. All trees of odd orders are super edge-graceful.

References

[1] R.E. Aldred and B.D.McKay, Graceful and harmonious labellings of trees, *Bulletin of the Institute of Combinatorics and its Applications*, **23**, 69-72, 1998.

[2] S. Cabannis, J. Mitchem and R. Low, On edge-graceful regular graphs and trees., *Ars Combin.* **34,** 129-142, 1992.

[3] I. . Cahit, Status of graceful tree conjecture in 1989, *Topics in Combinatorics and Graph Theory* (ed. R. Bodendick and R. Henn), Physica Verlag, Heidelberg , 1990.

[4]J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin.* # DS6, 1-79, 2001.

[5] Jonathan Keene and Andrew Simoson, Balanced strands for asymmetric, edge-graceful spiders, *Ars Combinatoria* **42**,49-64,1996

[6] Q. Kuan, Sin-Min Lee, J. Mitchem, and A.K. Wang, On edge-graceful unicyclic graphs, *Congressus Numerantium* **61**, 65-74, 1988.

[7]Li Min Lee, Sin Min Lee, and G. Murty, On edge-graceful labelings of complete graphs - solutions of Lo's conjecture, *Congressum Numerantum* **62**, 225-233, 1988.

[8] Sin-Min Lee, A conjecture on edge-graceful trees, *Scientia*, Ser. A, vol.3, 45-57, 1989.

[9] Sin- Min Lee, New Directions in the Theory of Edge-Graceful Graphs, *Proceedings of the 6th Caribbean Conference on Combinatorics & Computing*, pp 216-231, 1991.

[10] Sin-Min Lee, Peining Ma, Linda Valdes, and Siu-Ming Tong, On the edge-graceful grids, *Congressus Numerantium* **154**,61-77, 2002.

[11] Sin-Min Lee and Eric Seah, Edge-graceful labelings of regular complete k-partite graphs, *Congressus Numerantium* **75**, 41-50, 1990.

[12] Sin Min Lee and Eric Seah, On edge-gracefulness of the composition of step graphs with null graphs, *Combinatorics, Algorithms, and Applications in Society for Industrial and Applied Mathematics,*, 326-330, 1991.

[13]Sin Min Lee and Eric Seah, On the edge-graceful (n,kn)-multigraphs conjecture, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **9**,141-147, 1991.

[14]Sin-Min Lee, E. Seah and S.P. Lo , On edge-graceful 2-regular graphs, *The Journal of Combinatoric Mathematics and Combinatoric Computing*, **12**, 109-117,1992.

[15] Sin-Min Lee, E. Seah, Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congressus Numerantium* **141**, 37-48 ,1999

[16]S.P. Lo, On edge-graceful labelings of graphs, *Congressus Numerantium*, **50**, 231-241, 1985.

[17] Peng Jin. and Li W., Edge-gracefulness of C_mx C_n, in *Proceedings of the Sixth Conference of Operations Research Society of China*, (Hong Kong: Global-Link Publishing Company), Changsha, October 10-15, p.942-948, 2000.

[18] J. Mitchem and A. Simoson, On edge-graceful and super-edge-graceful graphs. *Ars Combin.* **37**, 97-111 (1994).

[19] A. Rosa, On Certain Valuations of the Vertices of a Graph. In *Theory of Graphs, International Symposium, Rome, July 1966.* New York: Gordon and Breach, pp. 349-355, 1967.

[20]A.Riskin and S. Wilson, Edge graceful labellings of disjoint unions of cycles. *Bulletin of the Institute of Combinatorics and its Applications* **22**: 53-58,1998.

[21]Karl Schaffer and Sin Min Lee, Edge-graceful and edge-magic labelings of Cartesian products of graphs, *Congressus Numerantium* **141**, 119-134, 1999.

[22] W.C.Shiu, P.C.B. Lam and H.L. Cheng, Edge-gracefulness of composition of paths with null graphs, *Discrete Math.*, 253, 63-76, 2002

[23]W.C.Shiu, Sin-Min Lee and K. Schaffer, Some k-fold edge-graceful labelings of (p,p-1)-graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing* **38**, 81-95, 2001.

[24] D. Small, Regular (even) spider graphs are edge-graceful, *Congresssus Numerantium*, 74 ,247-254, 1990,

[25]S. Wilson and A. Riskin, Edge-graceful labellings of odd cycles and their products, *Bulletin of the Institute of Combinatorics and its Applications*, **24**, 57-64, 1998.

[26]Shi-Lin Zhao, All trees of diameter four are graceful, *Annals New York Academy of Science*, **576**,700-706,1989.