On the Number of Ones in General Binary Pascal Triangles

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ABSTRACT. This paper answers the question as to whether every natural number n is realizable as the number of ones in the top portion of rows of a general binary Pascal triangle. Moreover, the minimum number $\kappa(n)$ of rows is determined so that n is realizable.

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1. Introduction

Consider generating the first k rows of a general binary Pascal triangle. That is, let $L = (l_1 l_2 \dots l_k)$ and $R = (r_2 r_3 \dots r_k)$ be any lists of zeros and ones placed where normally the 1s of the Pascal triangle are, with l_1 at the top, L down the left diagonal and R down the right. Let the remaining entries be filled in by the Pascal recurrence, modulo 2. Denote the resulting triangle $\Delta_k(L, R)$ and let $\delta_k(L, R)$ be the number of its ones. For example, with $L = (11010)$ and $R = (0011)$ we obtain $\delta_5(L, R) = 8$ (see Figure 1).

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&0&&& \\ &&0&&1&&0&& \\ &1&&1&&1&&1& \\ 0&&0&&0&&0&&1 \end{array}$$

Figure 1. $\Delta_5((11010), (0011))$.

Note that δ_k remains the same if any corner is used (by rotation) as its top corner.

We say that n is realizable if it is possible to find k, L, and R so that $\delta_k(L, R) = n$, and we call $\Delta_k(L, R)$ a realizer of n. The first author asked the following question in [4]: Is every natural number n realizable? We

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answer this affirmatively in Section 2, even with $L = \underline{1}$, the all-ones vector. When the length of the vector is important we will write $\underline{1}^k$ and likewise $\underline{0}^k$ for the all-zeros vector.

It is possible to realize 8 more quickly than in Figure 1. With $L = \underline{1}$ and $R = (110)$ we find $\delta_4(L, R) = 8$. We say that n is k-realizable if it is possible to realize n in k rows, and we denote by $\kappa(n)$ the minimum k such that n is k-realizable. Of course one cannot 3-realize 8 because there are only 6 entries in the first three rows, and thus $\kappa(8) = 4$. We will determine $\kappa(n)$ in Section 4. For this purpose, in Section 3 we will find the maximum number $d(k)$ of ones being possible in a general binary Pascal triangle $\Delta_k(L, R)$.

2. General realizability

Consider $\Delta_k(\underline{1}, \underline{1})$, the standard binary Pascal triangle and write $P^t = \Delta_k(\underline{1}, \underline{1})$ if $k = 2^t$. Then P^t has the well-known recursive structure shown in Figure 2 (see [2,3,5]). Hence, the number of ones in P^t triples when the

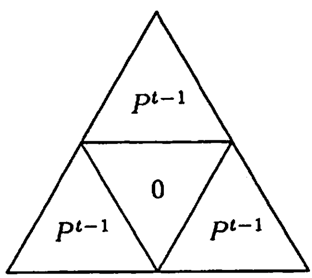


Figure 2. Recursive structure of P^t.

number of rows doubles, and so $\delta(P^t) = 3^t$.

We are now ready to prove that every n is realizable even if $L = \underline{1}$ as in the standard binary Pascal triangle.

Theorem 1. For every natural number $n \leq 3^t$ there are $k \leq 2^t$ and R so that $\Delta_k(\underline{1}, R)$ realizes n.

Proof. An induction base is obvious for $t = 0$. As induction hypothesis every $n \leq 3^{t-1}$ is realizable by $\Delta_k = \Delta_k(\underline{1}, R)$ for some $k = k(n) \leq 2^{t-1}$. Then $\Delta_{2^{t-1}+k(n-3^{t-1})}$ and Δ_{2^t} as in Figures 3 and 4 realize n for $3^{t-1} < n \leq 2 \cdot 3^{t-1}$ and $2 \cdot 3^{t-1} < n \leq 3^t$, respectively. □

3. Maximum number of ones in the first rows

For given k we determine the largest $n = d(k)$ being realizable by $\Delta_k(L, R)$. An earlier completely different proof was given in [1].

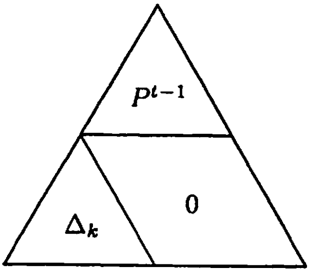


Figure 3. $\Delta_{2^{t-1}+k(n-3^{t-1})}$ for $3^{t-1} < n \le 2\cdot 3^{t-1}$.

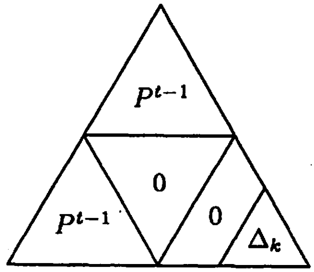


Figure 4. Δ_{2^t} for $2\cdot 3^{t-1} < n \le 3^t$.

Theorem 2. The maximum number of ones in $\Delta_k = \Delta_k(L,R)$ is

$$d(k) = \begin{cases} 1 & \text{if } k = 1, \\ 2 + \dfrac{1}{3}(k^2+k+1) & \text{if } k \equiv 1 \pmod 3, \\ 1 + \dfrac{1}{3}(k^2+k) & \text{if } k \equiv 0,2 \pmod 3, k \ne 8, \\ 27 & \text{if } k = 8. \end{cases}$$

Proof. We obtain lower bounds of $d(k)$ using $L = (1\overline{110})$ and $R = (\overline{110})$ where the triples 110 are repeated (see Figure 5), and where a possible zero in the last positions of L and R is substituted by a one. Then there are $2j$ ones in each of the rows $3j-1$, $3j$, and $3j+1$, $1 \le j \le \lfloor (k+1)/3 \rfloor$. It follows that $d(k) \ge \delta_k(L,R) = \sum_{j=1}^{\lfloor (k+1)/3 \rfloor} 6j$ plus $1 - 4(k+1)/3$, plus $1 - 2(k+1)/3$, and plus 3 for $k \equiv 2, 0$, and 1 (mod 3), respectively. These are the terms of Theorem 2 with the only exception of $k = 8$ where the standard binary Pascal triangle has 2 ones more, that is, $d(8) \ge 3^3$ (see Figure 6).

```
                    1
                  1   1
                1   0   1
              0   1   1   0
            1   1   0   1   1
          1   0   1   1   0   1
        0   1   1   0   1   1   0
      1   1   0   1   1   0   1   1
    1   0   1   1   0   1   1   0   1
  0   1   1   0   1   1   0   1   1   0
1   1   0   1   1   0   1   1   0   1   1
```

Figure 5. $L = (1\overline{110})$ and $R = (\overline{110})$.

```
              1
            1   1
          1   0   1
        1   1   1   1
      1   0   0   0   1
    1   1   0   0   1   1
  1   0   1   0   1   0   1
1   1   1   1   1   1   1   1
```

Figure 6. $d(8) \ge \delta_8(\underline{1}, \underline{1}) = 27$.

To obtain upper bounds of $d(k)$ we notice that in the three corner entries of Δ_k there always have to be ones. Deleting these corner entries from Δ_k leaves Δ'_k, and we have to find upper bounds of the maximum number $f(k) = d(k) - 3$ of ones in Δ'_k.

We use the fact that $x_3 \equiv x_1 + x_2 \pmod 2$ and thus that there are at most 2 ones in each of the sets $\{x_1, x_2, x_3\}$ if they are in one of the positions of Figure 7. Then Δ'_k can be partitioned into

$$\begin{array}{ccc} x_1 & & x_2 \\ & x_3 & \end{array} \qquad \begin{array}{cccc} x_1 & & \bullet & & x_2 \\ & \bullet & & \bullet & \\ & & x_3 & & \end{array}$$

Figure 7. At most 2 ones.

(1) $\dfrac{k^2-1}{8}$ triangles Δ'_3 and one $\Delta'_{(k+1)/2}$ if $k \equiv 1 \pmod 2$

and into

(2) 3 triangles Δ'_3, 3 triangles $\Delta'_{k/2}$, and one $\Delta'_{(k-2)/2}$ if $k \equiv 0 \pmod 2$

as depicted in Figures 8 and 9.

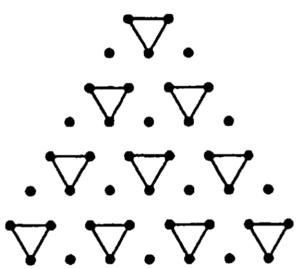


Figure 8. Δ'_k, $k \equiv 1 \pmod 2$.

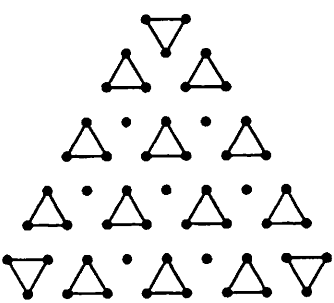
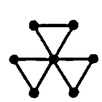


Figure 9. Δ'_k, $k \equiv 0 \pmod 2$.

Denoting by $\varphi_k = \varphi_k(L, R)$ the number of ones in $\Delta'_k(L, R)$ we have the possibilities $\varphi_2 = 0$, $\varphi_3 = 0$ or 2, and $\varphi_4 = 0$, 2, 4, or 6. Thus for small $k \geq 2$ we have the maximum values $f(2) = 0$, $f(3) = 2$, and $f(4) = 6$ where the unique example for $f(4) = 6$ occurs for a zero in the central entry (see Figure 10).



$$\begin{array}{ccccc} & 1 & & 1 & \\ 1 & & 0 & & 1 \\ & 1 & & 1 & \end{array}$$

Figure 10. $f(4) = 6$.

By the partitions (1) and (2) it follows inductively that always

$$\varphi_k \equiv 0 \pmod 2. \tag{3}$$

To prove $f(k) \leq (k^2+k-2)/3$ if $k \equiv 1 \pmod 3$ and $f(k) \leq (k^2+k-6)/3$ if $k \equiv 0, 2 \pmod 3$ induction steps from k to $k+6$ using (1) and (2) are successful unless in the case of $k \equiv 2 \pmod 6$. Here $k/2 \equiv 1 \pmod 3$ and $(k-2)/2 \equiv 0 \pmod 3$ imply with (2)

$$f(k) \leq 6+\left(\frac{k^2}{4}+\frac{k}{2}-2\right)+\frac{1}{3}\left(\left(\frac{k-2}{2}\right)^2+\frac{k-2}{2}-6\right)=\frac{1}{3}(k^2+k-6)+4,$$

that is, we obtain a surplus of 4. To manage this missing case we need two lemmas determining for $k \equiv 1 \pmod 3$ the designs of Δ'_k with $f(k)$ ones.

Lemma 1. For the last three rows B_k of Δ'_k consisting of $(k-1)/3$ copies of Δ'_4 with vertical pairs of entries between these copies, the maximum number $b(k)$ of ones in B_k is $b(k) = 2k-2$, and $b(k)$ is attained only if B_k starts at each end with one of the possibilities in Figure 11.

$$\begin{array}{cccccccc} & 1 & 1 & 0 & 1 & 1 & & \\ 1 & 0 & 1 & 1 & 0 & 1 & \cdots & \\ & 1 & 1 & 0 & 1 & 1 & & \end{array} \qquad \begin{array}{cccccccc} & 1 & 1 & 0 & 0 & 1 & & \\ 1 & 0 & 1 & 0 & 1 & \bullet & \cdots \\ & 1 & 1 & 1 & 1 & \bullet & & \end{array}$$

Figure 11. Possible ends of the last three rows B_k of Δ'_k.

Proof of Lemma 1. Consider the vertical pairs of entries between neighboring pairs of Δ'_4s.

If both entries of one of the pairs are zero then $b_k = 2k-2$ is attained inductively only if B_i and B_{k-i+1} on both sides are maximums. In an example every copy of Δ'_4 has 6 ones in the unique design of Figure 10 and all vertical pairs consist of zeros only.

If every vertical pair has at least one entry one then any Δ'_4 with 6 ones force on both sides a zero above, subsequently a one below, and then two zeros in the neighboring Δ'_4s which thus have at most 4 ones. Let t denote in B_k the number of Δ'_4s having 6 ones and x of them occur at both ends, $x = 0, 1$, or 2. Then the remaining Δ'_4s have at most 4 ones, there are $2t-x$ vertical pairs having a zero above and one below, and the remaining vertical pairs have at most two ones. It follows for the number b_k of ones in B_k

$$b_k \leq 6t+4\left(\frac{k-1}{3}-t\right)+2t-x+2\left(\frac{k-1}{3}-1-2t+x\right)=2k-4+x.$$

Thus $b_k = b(k) = 2k-2$ is possible only if $x = 2$, that is, at both ends of B_k are Δ'_4s with 6 ones as in the second possibility of Figure 11. It may be remarked that $k = 16$ is the smallest k with a maximum B_k of this type.□

Lemma 2. The unique Δ_4' with $f(4) = 4$ is shown in Figure 10. All possible Δ_7's with $f(7) = 18$ are shown in Figure 12 where the second Δ_7' can be rotated and where the pair 01 at the corner may be switched to 10. All possible Δ_k's for $k \equiv 1 \pmod 3$, $k \geq 10$, and with $f(k) = (k^2 + k - 2)/3$

$$
\begin{array}{c}
1\quad 1 \\
1\quad 0\quad 1 \\
0\quad 1\quad 1\quad 0 \\
1\quad 1\quad 0\quad 1\quad 1 \\
1\quad 0\quad 1\quad 1\quad 0\quad 1 \\
1\quad 1\quad 0\quad 1\quad 1
\end{array}
\qquad
\begin{array}{c}
0\quad 1 \\
1\quad 1\quad 1 \\
1\quad 0\quad 0\quad 1 \\
1\quad 1\quad 0\quad 1\quad 1 \\
1\quad 0\quad 1\quad 1\quad 0\quad 1 \\
1\quad 1\quad 0\quad 1\quad 1
\end{array}
$$

Figure 12. All Δ_7's with $f(7) = 18$.

are shown in Figure 13 where $L' = R' = (l_2 l_3 l_4 \overline{110} 1 1 l_{k-3} l_{k-2} l_{k-1})$ and where the trapezoids at the corners may be arbitrarily chosen from the depicted two copies with 01 at one corner being switchable.

$$
\begin{array}{c}
\bullet\quad \bullet \\
\bullet\quad \bullet\quad \bullet \\
\bullet\quad \bullet\quad \bullet\quad \bullet \\
1\quad 1\quad 0\quad 1\quad 1 \\
1\quad 0\quad 1\quad 1\quad 0\quad 1 \\
0\quad 1\quad 1\quad 0\quad 1\quad 1\quad 0 \\
\cdots \qquad\qquad \cdots
\end{array}
\qquad
\begin{array}{c}
1\quad 1 \\
1\quad 0\quad 1 \\
0\quad 1\quad 1\quad 0 \\
\text{or} \\
0\quad 1 \\
1\quad 1\quad 1 \\
1\quad 0\quad 0\quad 1
\end{array}
$$

$$
\begin{array}{r}
1\quad 1\quad 0\quad 1 \\
1\quad 0\quad 1\quad 1\quad 0 \\
\bullet\quad 1\quad 1\quad 0\quad 1 \\
\bullet\quad \bullet\quad 0\quad 1\quad 1\quad 0 \\
\bullet\quad \bullet\quad \bullet\quad 1\quad 0\quad 1 \\
\bullet\quad \bullet\quad \bullet\quad 1\quad 1\quad 0
\end{array}
\quad \cdots \quad
\begin{array}{l}
1\quad 0\quad 1\quad 1 \\
1\quad 1\quad 0\quad 1 \\
1\quad 0\quad 1\quad 1\quad \bullet \\
1\quad 1\quad 0\quad \bullet\quad \bullet \\
1\quad 0\quad 1\quad \bullet\quad \bullet\quad \bullet \\
1\quad 1\quad \bullet\quad \bullet\quad \bullet
\end{array}
$$

Figure 13. All Δ_k's for $k \equiv 1 \pmod 3$, $k \geq 10$, and with $f(k) = (k^2 + k - 2)/3$.

Proof of Lemma 2. The uniqueness of Δ_4' in Figure 10 was mentioned above.

For $k \geq 7$ we partition Δ_k' into Δ_{k-3}', B_k, and a horizontal pair of entries (see Figure 14).

For $k \geq 7$ the maximum $f(k) = (k^2+k-2)/3$ cannot be attained if Δ_{k-3}' and B_k both do not have the maximum of ones since (3), Lemma 1, and 2 ones in the horizontal pair imply $\varphi_k \leq f(k-3) - 2 + b(k) - 1 + 2 < f(k)$.

Let $k = 7$. If Δ_4' has 6 ones then Δ_7' can be completed to have 18 ones with 00 and with 01 as the horizontal pair and not with 11. If B_7 has the

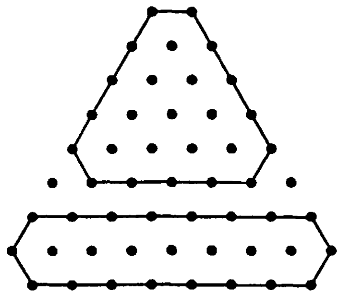


Figure 14. Partition of Δ'_k into Δ'_{k-3}, B_k, and a pair.

maximum of 12 ones then 18 ones for Δ'_7 are possible with the pair 00 and with 11 and not with 01. All possibilities are covered by Figure 12.

Let $k = 10$. If Δ'_7 has $f(7) = 18$ ones then we may rotate Δ'_7 such that 11011 is between the horizontal pair of entries. Then 6 ones are determined in the middle part of B_{10}. For the remaining trapezoids at both sides only those of Figure 13 with 6 ones are possible to obtain Δ'_{10} with $f(10) = 18 + 6 + 2 \cdot 6 = 36$ ones.

For $k \geq 13$ the case that Δ'_{k-3} has the maximum of $f(k-3)$ ones is treated as follows. Here the last row of Δ'_{k-3} starts at both ends with 110, 011, or 111. The main part $11\overline{011}$ of this row determines $6((k-1)/3-4) = 2k-26$ ones in B_k. At both ends of B_k together with an entry of the horizontal pair there can occur only the six designs of Figure 15. The sixth

$$
\begin{array}{ccc}
\begin{array}{l}
\;\;\;0\;\;1\;\;1\;\;0\;\;1\;\;1\\
\;\bullet\;\;1\;\;\overline{0\;\;1\;\;1}\;\;0\\
\bullet\;\;\bullet\;\;1\;\;1\;\;0\;\;1\\
\;\bullet\;\;\bullet\;\;0\;\;1\;\;1\;\;\bullet
\end{array}
&
\begin{array}{l}
\;\;\;1\;\;1\;\;1\;\;0\;\;1\;\;1\\
\;\bullet\;\;0\;\;\overline{0\;\;1\;\;1}\;\;0\\
\bullet\;\;\bullet\;\;0\;\;1\;\;0\;\;1\\
\;\bullet\;\;\bullet\;\;1\;\;1\;\;1\;\;\bullet
\end{array}
&
\begin{array}{l}
\;\;\;0\;\;1\;\;1\;\;1\;\;1\;\;1\\
\;\bullet\;\;1\;\;\overline{0\;\;0\;\;0}\;\;0\\
\bullet\;\;\bullet\;\;1\;\;0\;\;0\;\;0\\
\;\bullet\;\;\bullet\;\;1\;\;0\;\;0\;\;\bullet
\end{array}
\\[2ex]
\begin{array}{l}
\;\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\\
\;\bullet\;\;0\;\;\overline{0\;\;0\;\;0}\;\;0\\
\bullet\;\;\bullet\;\;0\;\;0\;\;0\;\;0\\
\;\bullet\;\;\bullet\;\;0\;\;0\;\;0\;\;\bullet
\end{array}
&
\begin{array}{l}
\;\;\;0\;\;0\;\;1\;\;1\;\;1\;\;1\\
\;\bullet\;\;0\;\;\overline{1\;\;0\;\;0}\;\;0\\
\bullet\;\;\bullet\;\;1\;\;1\;\;0\;\;0\\
\;\bullet\;\;\bullet\;\;0\;\;0\;\;0\;\;\bullet
\end{array}
&
\begin{array}{l}
\;\;\;1\;\;0\;\;1\;\;1\;\;1\;\;1\\
\;\bullet\;\;1\;\;\overline{1\;\;0\;\;0}\;\;0\\
\bullet\;\;\bullet\;\;0\;\;1\;\;0\;\;0\\
\;\bullet\;\;\bullet\;\;1\;\;1\;\;0\;\;\bullet
\end{array}
\end{array}
$$

Figure 15. Possible designs at the ends of B_k, $k \equiv 1 \pmod 3$.

entry in the last row of B_k can be a one only for $k = 13$ if there is a pair 01 or 10 in the middle of the second last row. The 5 entries at the ends contain at most 4 ones each. To attain $f(k)$ ones, two designs of Figure 15 have to have $x = f(k) - f(k-3) - (2k-26) = 24$ ones. This is possible for the first two designs only since each of the other designs has at most 11 ones. All resulting possibilities are covered by Figure 13.

It remains that B_k for $k \geq 10$ has the maximum of $b(k) = 2k-2$ ones and Δ'_{k-3} has less than the maximum $f(k-3)$ of ones. Then $b(k) = 2k-2$

is possible only if Δ'_{k-3} has $f(k-3)-2$ ones and the horizontal pair (see Figure 14) is 11. Then by Lemma 1 the last row of Δ'_{k-3} starts on each end with 0010 or 0001, that are together at most $f(k-3)-4$ ones in Δ'_{k-3} since by induction hypothesis all Δ'_{k-3}s with $f(k-3)$ ones start with 1101, 0111, or 1111 at the ends of the last row. Thus $f(k-3)-2$ cannot be attained. □

Now we are ready to handle the missing case $k \equiv 2 \pmod 6$ in the proof of Theorem 2.

For $k=8$ we have $f(8) \leq 3f(3)+3f(4)+f(3)$ by (2). If at least 2 of the unique Δ'_4s with 6 ones occur in Δ'_8 then the central Δ'_3 is forced to be without ones and thus $f(8) \leq 3\cdot 2+3\cdot 6 = 24$ as asserted. If at most one of the Δ'_4s has 6 ones and the other 2 have at most 4 then $f(8) \leq 3\cdot 2+6+2\cdot 4+2 = 22$. Thus there is the unique solution (see the part of Figure 6) when all Δ'_4s have $f(4)=6$ ones.

For $k=14$ we consider (2). If at least 2 of the Δ'_7s have $f(7)=18$ ones then by Lemma 2 (see Figure 12) for Δ'_{14} at most 8 ones occur in Δ'_6 (see Figure 16) so that $f(14) \leq 3f(3)+3f(7)+8 = 68$. If at most one Δ'_7 has 18

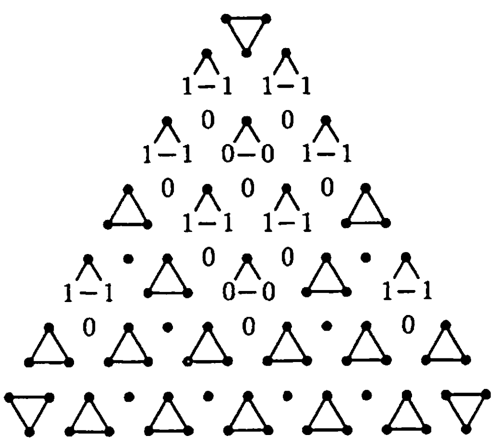


Figure 16. Δ'_{14} with 2 maximum Δ'_7s.

ones then by (3) we obtain $f(14) \leq 3f(3)+f(7)+2(f(7)-2)+f(6) = 68$.

For $k \equiv 2 \pmod 6$, $k \geq 20$, at least 2 of the $\Delta'_{k/2}$s with the maximum of ones force most entries of $\Delta'_{(k-2)/2}$ to be zero. Using Lemma 2, there are at most 6 ones at each corner. Since $18 \leq f((k-2)/2)-4$ for $k \geq 20$, we obtain from (2) that $f(k) \leq 3f(3)+3f(k/2)+f((k-2)/2)-4 = f(k)$. If at most one $\Delta'_{k/2}$ has $f(k/2)$ ones then, because of (3), we also can subtract 4 on the right part of (2).

Thus the induction step now works for all residue classes modulo 6. However, due to the exceptional case $k=8$ the induction bases remain unsolved for $k = 15$, 16, and 18.

If $k = 15$ then from (1) we have $f(15) \leq 28f(3) + f(8)$. From another interpretation of the partition as in Figure 8 we get $f(15) \leq 3f(3)+3f(7)+f(8)$. If Δ_8' has at most 22 ones it follows $f(15) \leq 56+22 = 78$ as asserted. Otherwise, there exists the unique Δ_8' with 24 ones. If all three Δ_7's have at most $f(7) - 2$ ones then $f(15) \leq 6 + 3 \cdot 16 + 24 = 78$. It remains that at least one Δ_7' has 18 ones. Because of Lemma 2 we can rotate Δ_{15}' such that Δ_8' and Δ_7' determine the 5^{th} row from the bottom of Δ_{15}' to be 11110001111 or 11110000111. Then 2 of the Δ_3's are without ones and $f(15) \leq (28-2)f(3) + f(8) = 76$. This proves $f(15) = 78$.

If $k = 16$ then from (2) we have $f(16) \leq 3f(3)+3f(8)+f(7)$. If all Δ_8's have at most 22 ones then $f(16) \leq 6+66+18 = 90$. If at least 2 of the Δ_8's have 24 ones then Δ_7' has zeros only and we get $f(16) \leq 6 + 3 \cdot 24 = 78$. If exactly one Δ_8' has 24 ones then with $f(7) - 2$ ones in Δ_7' it follows $f(16) \leq 6 + 24 + 44 + 16 = 90$. Otherwise, we have $f(7) = 18$ ones in Δ_7' and because of Lemma 2 and Figure 12 we can insert digits of Δ_7' into Δ_{16}' as in Figure 17. If the unique Δ_8' (see the part of Figure 6) is

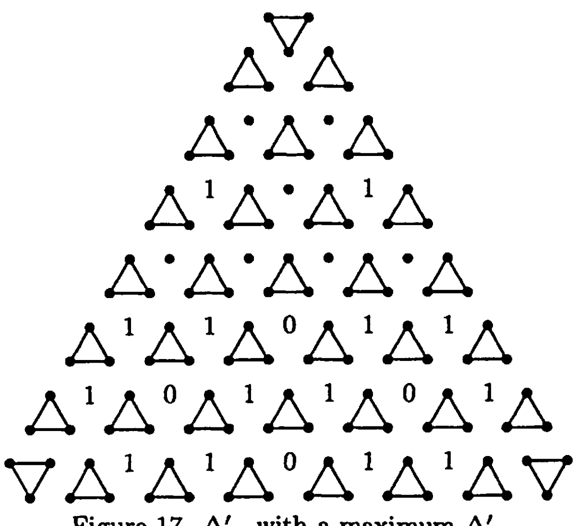


Figure 17. Δ_{16}' with a maximum Δ_7'.

inserted in both possibilities then at least 13 zeros are forced in one of the 2 other Δ_8's so that one Δ_8' has at most $33 - 13 = 20$ ones which implies $f(16) \leq 6+24+22+20+18 = 90$ which completes the proof of $f(16) = 90$.

If $k = 18$ from (2) we have $f(18) \leq 3f(3) + 3f(9) + f(8)$. If Δ_8' has at most 22 ones we get $f(18) \leq 112$ as asserted. Thus Δ_8' is unique with 24 ones. From a variation of (1) (see Figure 8) we obtain $f(9) \leq 3f(3) + 3f(4) + f(5)$ and thus $f(18) \leq 12f(3) + 9f(4) + 3f(5) + f(8) \leq 72 + 9f(4)$. In Figure 18 with the unique Δ_8' there are 21 small triangles hosting the 9 different Δ'_4s. The 3 triangles in the central region of Δ'_{18} are forced to have

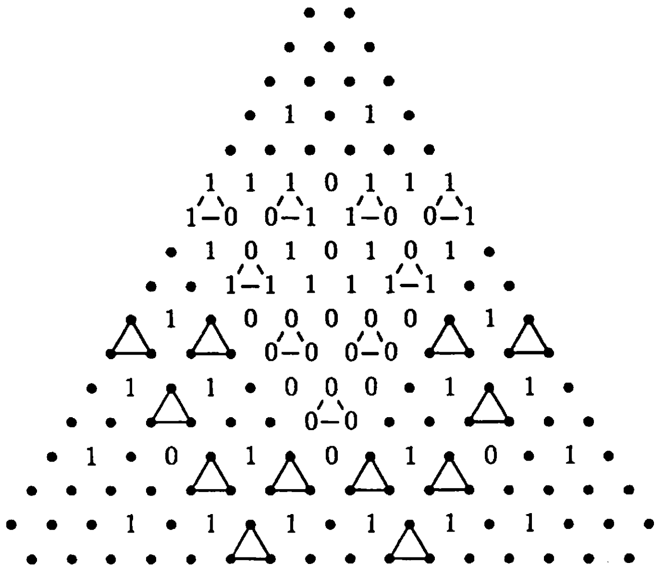


Figure 18. Δ'_{18} with a maximum Δ'_8.

only ones or only zeros. All other 18 triangles are forced to have 1 one or 2 ones. At least one of the 3 central triangles has only zeros since otherwise every Δ'_4 has at most 4 ones yielding $f(18) \leq 72+9\cdot4 = 108$. If at least 2 of the 18 triangles have exactly 1 one then $f(18) \leq 72+2\cdot3+16\cdot2+2\cdot1 = 112$. Otherwise, say the upper 6 triangles of the 18 triangles in Figure 18 have to have 2 ones each. This is possible in the unique way of Figure 18 and it forces the 3 central triangles to have zeros only so that $f(18) \leq 72+18\cdot2 = 108$. This completes the proof of $f(18) = 112$ and the proof of Theorem 2.

□

4. Minimum number of rows to realize a number

Of course, the value $n = d(k)$ is k-realizable. The values of n between $d(k-1)$ and $d(k)$ are not $(k-1)$-realizable. However, are all these values k-realizable?

Theorem 3. For the minimum number $\kappa(n)$ of rows of Δ_k such that n is k-realizable we have

$$\kappa(1) = 1,\ \kappa(2) = \kappa(3) = 2,\ \kappa(26) = \kappa(27) = 8,$$

$$\kappa(n) = \left\lceil \sqrt{3n-7} - \frac{1}{2} \right\rceil \quad \text{if } n = \frac{1}{3}(k^2+k+7),\ k \equiv 1 \pmod 3,$$

$$\kappa(n) = \left\lceil \sqrt{3n-4} - \frac{1}{2} \right\rceil \quad \text{otherwise.}$$

Proof. The values of $\kappa(n)$ for $n \le 3$, $n = 26$, and $n = 27$ are obvious (see Figure 6 where a one at a corner can be replaced by a zero). In general, $d(k) - i$ for $i \le 3$ is k-realizable since the ones at the 3 corners of Δ_k can be replaced by zeros.

For $r \ge 4$, $r \equiv 0 \pmod 2$, we consider for $k > r$ the upper triangle $\Delta_r(L, R)$ of $\Delta_k(L, R)$ with $d(k)$ ones and so that $L = (1\overline{110})$ and $R = (\overline{110})$. Then Δ_r has $y(r) = d(r) - 2$ ones for $r \equiv 1 \pmod 3$ and $y(r) = d(r)$ ones otherwise ($k \ne 8$). We will construct an interval of consecutive numbers $y(r + i)$ so that all $n = d(k) - y(r) - i$, $i \ge 0$, are k-realizable. If these intervals overlap one another then $n = d(k) - j$, $0 \le j \le y(r)$ is k-realizable so that because of $d(k) - d(k-1) \le (2k+1)/3 \le y(r)$ all values of n between $d(k-1)$ and $d(k)$ are k-realizable for $k \le 3y(r)/2$.

The construction starts in row r of Δ_k by choosing in row $r - j + 1$ a one at the position $r/2 - j$ for $1 \le j \le r/2 + 1$. This one together with row $r - j + 2$ determines the whole row $r - j + 1$. From row $r/2 + 1$ upwards, zeros are chosen in position 1 of every row (see Figure 19). Then

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&0&&&&&&&&\\
&&&&&&&0&&0&&&&&&&\\
&&&&&&0&&0&&0&&&&&&\\
&&&&&0&&0&&0&&0&&&&&\\
&&&&0&&0&&0&&0&&0&&&&\\
&&&1&&0&&0&&0&&0&&1&&&\\
&&1&&1&&0&&0&&0&&1&&1&&\\
&0&&0&&1&&0&&0&&1&&0&&0&\\
1&&0&&1&&1&&0&&1&&1&&0&&1
\end{array}$$

Figure 19. The construction for $r = 8$.

$y(r) - 2z(r)$ ones are subtracted from $d(k)$ in Δ_k, where $z(r)$ denotes the number $\delta_{(r-2)/2}$ of ones in one of the two identical triangles $\Delta_{(r-2)/2}(L, R)$ at both lower corners of Δ_r and with $L = \underline{1}$, $R = (\overline{001})$ if the corner inside the row r of Δ_k is interpreted as the top of $\Delta_{(r-2)/2}$.

Now the top triangle $\Delta_{(r+2)/2}$ consists of zeros only and we will use the construction in the proof of Theorem 1 to realize all numbers j for $0 \le j \le 3^t$ in the first 2^t rows of $\Delta_{(r+2)/2}$. In the proof of Theorem 1 we have realized all these values of j with $L = \underline{1}$ and now we rotate the triangles so that $L = \underline{1}$ becomes a row of $\Delta_{(r+2)/2}$ having ones only and followed by a row of zeros. Thus we can subtract from $d(k)$ the numbers

$$\begin{aligned}
&y(r) - 2z(r) - j \quad \text{for } 0 \le j \le 3^t \quad \text{if } 2^t \le \tfrac{r+2}{2} < 2^t + 2^{t-1}, \\
(4)\qquad &y(r) - 2z(r) - j \quad \text{for } 0 \le j \le 3^t + 2 \cdot 3^{t-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 2^t + 2^{t-1} \le \tfrac{r+2}{2} < 2^{t+1}.
\end{aligned}$$

An overlapping of the intervals is guaranteed if

$$(5)\quad y(r+2)-y(r)-2(z(r+2)-z(r)) \leq \begin{cases} 3^t & \text{for } r < 2^{t+1}+2^t-4, \\ 3^t+2\cdot 3^{t-1} & \\ & \text{for } r = 2^{t+1}+2^t-4, \\ 3^t+2\cdot 3^{t-1} & \\ & \text{for } r < 2^{t+2}-4, \\ 3^{t+1} & \text{for } r = 2^{t+2}-4. \end{cases}$$

Using $z(r+2)-z(r) \geq 1$ and $y(r+2)-y(r) \leq (4r+8)/3$ the inequalities (5) are valid for $t \geq 3$ and the last two also for $t = 2$. In the remaining cases for $r \leq 10$ we can check that $y(r) = 53$, 37, 25, 15, and 7 and that $z(r) = 10$, 7, 4, 2, and 1 for $r = 12$, 10, 8, 6, and 4, respectively, and that the corresponding intervals are overlapping one another.

Thus it is proved that all values n between $d(k-1)$ and $d(k)$ are k-realizable.

Now it follows from Theorem 2 that

$$\kappa(n) = \left\lceil \left(\sqrt{12n-11}-1\right)/2 \right\rceil$$

for $(k^2+k+3)/3 < n \leq (k^2+3k+5)/3$ if $k \equiv 0 \pmod 3$ and also for $(k^2-k+7)/3 < n \leq (k^2+3k+5)/3$ if $k \equiv 2 \pmod 3$ and that

$$\kappa(n) = \left\lceil \left(\sqrt{12n-27}-1\right)/2 \right\rceil$$

for $(k^2-k+7)/3 < n \leq (k^2+3k+5)/3$ if $k \equiv 1 \pmod 3$. Because of the ceiling function this may be written as in Theorem 3. □

Several further problems remain open. For example, what is the number $\delta_k(L, R)$ of ones in Δ_k for $L = \underline{1}$ and $R = (\overline{101})$? What is the maximum number of ones in $\Delta_k(L, R)$ for $L = \underline{1}$? How many pairs L and R determine a Δ_k with equal numbers of ones and zeros?

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