On the Non-Existence of Some Orthogonal Arrays

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Abstract

In this paper we consider the problem of the non-existence of some orthogonal arrays (O-arrays) of strength four with two levels, the number of constraints k satisfying $4 \leq k \leq 32$ and index set λ where $1 \leq \lambda \leq 64$.

1. Introduction and Preliminaries.

For ease of reference we present some basic results and definitions on balanced arrays (B-arrays).

Definition. A balanced array (B-array) T of strength t with two levels, k constraints, and N runs is merely a matrix T with two elements (say, 0 and 1), k rows, and N columns having the following combinatorial structure:

In every ($t \times N$; $t \leq k$) submatrix T^* of T, every ($t \times 1$) vector $\underline{\alpha}$ of weight $i (0 \leq i \leq t)$, (the weight of a vector is defined as the number of 1's in it) appears with the same frequency λ_i (say). The vector $(\lambda_0, \lambda_1, .., \lambda_t)$ is called the index set of the B-array.
Definition: If $\lambda_i = \lambda$ for each i, then the B-array is called an orthogonal array (O-array) with index λ. We will also use OA(N,k,s,t), $N = \lambda s^t$, to denote an O-array with parameters N,k,s,t. We specialize here to the case when $t = 4$.

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Definition: A B-array T with index set $(\lambda-1, \lambda, \lambda, \lambda, \lambda-1)$ is called a near O-array.
Remark: It is quite obvious that if we attach to a near O-array with k rows, two vectors, one of weight 0 and another of weight k, we obtain an O-array with k rows and index set λ.

The following result on O-arrays is due to Rao [11]:
Result 1: For O-arrays with $t \geq 2$ and with s levels, k must satisfy the following inequality: For $t = 2u$, $\lambda s^t - 1 \geq \binom{k}{1}(s-1) + \binom{k}{2}(s-1)^2 + \ldots + \binom{k}{u}(s-1)^u$. For $t = 4$ and $s = 2$, the above is reduced to

$$32\lambda - 2 \geq k^2 + k \tag{1.1}$$

The next result on near O-arrays is from Chopra and Dios [5].

Result 2: Consider a near O-array $T(k \times N)$ with $t = 4$ and index set $(\lambda-1, \lambda, \lambda, \lambda, \lambda-1)$. Then the following inequality is true:

$$L_2 L_4 \geq L_2^3 + L_3^2 \tag{1.2}$$

where $L_2 = ND_2 - D_1^2, L_3 = N^2 D_3 - 3ND_2 D_1 + 2D_1^3$,

$L_4 = N^3 D_4 - 4N^2 D_3 + 6ND_2 D_1^2 - 3D_1^4$ with $D_l = \sum j^l x_j, 0 \leq l \leq 4$. For a given λ, the D_l's are polynomial functions in k and are given as follows:

$D_0 = N = 2C_1, D_1 = kC_1, D_2 = k_2 B_2 + kC_1,$
$D_3 = k_3 A_3 + 3k_2 B_2 + kC_1, D_4 = k_4(\lambda-1) + 6k_3 A_3 + 7k_2 B_2 + kC_1$
where A_3, B_2, and C_1 are given by $2\lambda-1, 4\lambda-1$, and $8\lambda-1$ respectively and k_r is defined by $k_r = k(k-1)(k-2)\ldots(k-r+1)$.
Remark: It is quite obvious that (1.2) is merely a polynomial inequality involving the parameters k and λ of the array T with $t = 4$ and $s = 2$. For a given λ, we obtain a polynomial inequality in k (the number of constraints).

B-arrays and O-arrays have been extensively used to construct symmetrical as well as asymmetrical factorial experiments. The rows of the arrays will correspond to factors and columns to treatment combinations. They are also generalizations of other areas of combinatorial mathematics (e.g., balanced incomplete block designs form a sub-class of B-arrays with strength two). To gain further insight into the importance of B-arrays and O-arrays to combinatorics and design of experiments, the interested reader may consult the list of references (by no means an exhaustive one) at the end of this paper, as well as further references listed therein.

2. Main Results with Discussion.

There are two important problems arising in the study of orthogonal arrays: (*i*) to determine the maximal number of factors k in any OA with fixed values of λ, s, t; (*ii*) to determine the minimal value of λ (meaning thereby minimal number of runs) for given values of k, s, and t.

Hedayat, Sloane, and Stufkin [6] in their book, Orthogonal Arrays, address both of these problems at great length, and is an excellent source book for O-arrays. They list numerous open problems on O-arrays throughout the book. Tables 12.1-12.3 provide minimal possible index of O-arrays with s, k, and t satisfying $(s = 2, 4 \leq k \leq 32, 2 \leq t \leq 10)$, $(s = 3, 4 \leq k \leq 32, 2 \leq t \leq 10)$, and $(s = 4, 4 \leq k \leq 32, 2 \leq t \leq 10)$ respectively. In this paper, we are concerned with Table 12.1 for the case $t = 4$. The construction of these tables has been carried out by using, as well as other results, the following trivial observations:

(a) The nonexistence of an OA $(\lambda s^t, k, s, t)$ implies the nonexistence of an OA $(\lambda s^t, k+1, s, t)$.

(b) An OA (N, k, s, t) of index λ implies the existence of an OA $(N, k-1, s, t)$ of index λ, an OA $(\frac{N}{s}, k-1, s, t-1)$ of index λ, and an OA $(sN, k+1, s, t)$ of index $s\lambda$.

(c) An OA $(s^t, t+1, s, t)$ with $\lambda = 1$ always exists. Many of the minimal entries for λ in Table 12.1 are given in the form of an interval $\lambda_1 - \lambda_2$ indicating an O-array for $\lambda = \lambda_2$ is known to exist but nothing is known about the existence of O-arrays for other values of λ in the interval.

3. Comments with Illustrations on Tables I-IV.

The four tables presented here lead us to the non-existence of O-arrays. Table I is merely extracted from Table 12.1 of Hedayat, Sloane, and Stufkin [6]. According to this table, if we pick, for example, $k = 30$, then the minimal value of λ is any integer satisfying 33-64, thus amounting to 32 values of λ. Of these 32 values, the existence of an O-array with $\lambda = 64$ is known but each of the remaining 31 values is an open problem.

Table II is obtained using (1.2). For each λ satisfying $1 \leq \lambda \leq 64$, we obtain a polynomial inequality in k by using (1.2). In this inequality, we substitute values of k starting at $k = 4$ and stopping as soon as we obtain the first condraction (e.g., if the first condraction occurred at $k = k^*$(say), then the maximum value of k is $k^* - 1$). As an illustration, if we pick the entry 30 in the k column, we have four values of λ satisfying 55-58. It means that (1.2) was contradicted for each of these four λ values at $k = 31$.

Table III is merely another way of writing Table I so that a valid comparison can be made with Table II. For the sake of illustration, let us pick $\lambda = 57$ in the interval 55-58. If we trace $\lambda = 57$ in Table I, we find it in every interval of λ values starting at $k = 24$ and ending with $k = 32$. Using this argument, we go from Table I to Table III. In Table III, we also list the k values of Table II in column 3.

Finally, we focus on Table IV. For $k = 30$, the earlier interval from Table I is 33-64, while the new interval from Table III is 55-58. Since O-arrays with ($k = 31$, $\lambda = 59 - 62$) and ($k = 32$, $\lambda = 63 - 64$) are possible, then the interval 55-58 is modified to 55-64. Thus the original interval 33-64 is replaced by new interval of 55-64, thereby showing the non-existence of O-arrays with ($k = 30, 33 \leq \lambda \leq 54$). Observing the entries in Table IV, there is a significant improvement on earlier results.

Table I

Minimal possible index λ of orthogonal arrays with 2 levels, k factors and strength 4.

k	λ	k	λ
4	1	19	14-16
5	1	20	15-32
6	2	21	17-32
7	4	22	20-32
8	4	23	22-32
9	6-8	24	22-64
10	6-8	25	23-64
11	6-8	26	26-64
12	7-8	27	29-64
13	8	28	29-64
14	8	29	29-64
15	8	30	33-64
16	10-16	31	37-64
17	12-16	32	37-64
18	13-16		

Chopra and Dios [5] provide a table for maximal k for λ satisfying $1 \leq \lambda \leq 20$. This is extended to $1 \leq \lambda \leq 64$ and presented as Table 2.

Table II

Maximum possible k (the number of factors/constraints) for orthogonal arrays of strength 4 with $1 \leq \lambda \leq 64$.

λ	$k \leq$	λ	$k \leq$
1	5	20-21	18
2	6	22-23	19
3	7	24-26	20
4	8	27-28	21
5	9	29-31	22
6	10	32-34	23
7	10	35-37	24
8	11	38-40	25
9	12	41-43	26
10	13	44-47	27
11	13	48-50	28
12	14	51-54	29
13	14	55-58	30
14-15	15	59-62	31
16-17	16	63-64	32
18-19	17		

Table III

Rewriting Table I in terms of bounds on k for a given λ for comparison with Table II.

λ	k	Table II k values	λ	k	Table II k values
1	4,5	5	24	20-25	20
2	6	6	25	20-25	20
3	6	7	26	20-26	20
4	7,8	8	27	20-26	21
5	6	9	28	20-26	21
6	9-11	10	29	20-29	22
7	9-12	10	30	20-29	22
8	9-15	11	31	20-29	22
9	15	12	32	20-29	23
10	16	13	33	24-30	23
11	16	13	34	24-30	23
12	16,17	14	35	24-30	24
13	16-18	14	36	24-30	24
14	16-19	15	37	24-32	24
15	16-20	15	38	24-32	25
16	16-20	16	39-40	24-32	25
17	20-21	16	41-43	24-32	26
18	20-21	17	44-47	24-32	27
19	20-21	17	48-50	24-32	28
20	20-22	18	51-54	24-32	29
21	20-22	18	55-58	24-32	30
22	20-24	19	59-62	24-32	31
23	20-24	19	63-64	24-32	32

Table IV

Revised table for the minimal possible index λ with 2 levels, k factors and strength 4.

k	Table I λ	Revised λ	k	Table I λ	Revised λ
4	1	1	19	14-16	16
5	1	1	20	15-32	24-32
6	2	2	21	17-32	27-32
7	4	4	22	20-32	29-32
8	4	4	23	22-32	32
9	6-8	6-8	24	22-64	35-64
10	6-8	6-8	25	23-64	38-64
11	6-8	8	26	26-64	41-64
12	7-8	8	27	29-64	44-64
13	8	8	28	29-64	48-64
14	8	8	29	29-64	51-64
15	8	8	30	33-64	55-64
16	10-16	16	31	37-64	59-64
17	12-16	16	32	37-64	63-64
18	13-16	16			

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