Face antimagic labelings for a special class of plane graphs C_a^b

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Abstract. Suppose $G = (V, E, F)$ is a finite plane graph with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. A bijection $\lambda : V(G) \cup E(G) \cup F(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)| + |F(G)|\}$ is called a labeling of type $(1,1,1)$. The *weight* of a face under a labeling is the sum of the labels (if present) carried by that face and the edges and vertices surrounding it. A labeling of a plane graph G is called *d-antimagic* if for every number $s \geq 3$, the set of s-sided face weights is $W_s = \{a_s + id : 0 \leq i < f_s\}$ for some integers a_s and d ($a_s > 0$, $d \geq 0$), where f_s is the number of s-sided faces. We allow different sets W_s for different s.
In this paper we deal with *d-antimagic* labelings of type $(1,1,1)$ for a special class of plane graphs C_a^b and we show that a C_a^b graph has d-antimagic labeling for $d \in \{a-2, a-1, a+1, a+2\}$.

1 Introduction

All graphs are finite, simple, undirected and plane. The plane graph $G = (V, E, F)$ has vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. Unless otherwise noted, $|V(G)| = v$, $|E(G)| = e$ and $|F(G)| = f$.

A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). If the domain is the vertex set or edge set or the face set, the labelings are called respectively vertex labelings or edge labelings or face labelings. In this paper we deal with the case where the domain is $V \cup E \cup F$, and these are called *labelings of*

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type (1, 1, 1). More formally, a one-to-one map λ from $V(G) \cup E(G) \cup F(G)$ onto the integers $\{1, 2, 3, \ldots, v + e + f\}$ is the labeling of type (1, 1, 1).

The *weight* of a face under a labeling is the sum of the labels (if present) carried by that face and the edges and vertices surrounding it.

A labeling of a plane graph G is called *d-antimagic* if for every number $s \geq 3$, the set of s-sided face weights is $W_s = \{a_s + id : 0 \leq i < f_s\}$ for some integers a_s and d ($a_s > 0$, $d \geq 0$), where f_s is the number of s-sided faces.

Other types of antimagic labelings were investigated by Hartsfield and Ringel [9] and by Bodendiek and Walther [7].

d-antimagic labeling is natural extension of the notion of *magic* labeling (if $d = 0$) introduced by Ko-Wei Lih in [13]. Ko-Wei Lih [13] studied *magic* (0-antimagic) labelings for wheels, friendship graphs and prisms. 0-antimagic labelings of type (1, 1, 1) for m-antiprisms, grid graphs and hexagonal planar maps can be found in [1,2,3]. Consecutive (1-antimagic) labelings for certain classes of plane graphs are given in [10,11,16]. d-antimagic labelings of prisms and generalized Petersen graphs $P(n, 2)$ are shown in [4,14,5]. A general survey of graph labelings is [8]. The reader is directed to Wallis [17] or West [18] for all additional terminology not provided in this paper.

2 Plane graph C_b^a

Let $I = \{1, 2, 3, \ldots, a\}$ and $J = \{1, 2, 3, \ldots, b\}$ be index sets. Let y_1, $y_2, \ldots, y_a$ be the fixed vertices. We connect the vertices y_i and y_{i+1} by means of b internally disjoint paths $P_i^j = \{y_i, x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,i}, y_{i+1}\}$ of length $i + 1$ each, where $i \in I$ and $j \in J$. We make the convention that $y_{a+1} = y_1$ to simplify later notations. The resulting graph embedded in the plane we denote by C_a^b with the vertex set $V(C_a^b) = \{y_i : i \in I\} \bigcup \bigcup_{i \in I} \bigcup_{j \in J} \{x_{i,j,k} : 1 \leq k \leq i\}$ and the edge set $E(C_a^b) = \bigcup_{i \in I} \{y_i x_{i,j,1} : j \in J\} \bigcup \bigcup_{i \in I} \bigcup_{j \in J} \{x_{i,j,k} x_{i,j,k+1} : 1 \leq k \leq i - 1\} \bigcup \bigcup_{i \in I} \{x_{i,j,i} y_{i+1} : j \in J\}$. $v = \frac{ab(a+1)}{2} + a$ and $e = \frac{ab(a+3)}{2}$.

Let $f_{i,j}$ be $(2i + 2)$-sided face determined by paths P_i^j and P_i^{j+1} for $i \in I$, $j \in J - \{b\}$. The face set $F(C_a^b)$ contains $a(b - 1)$ the $(2i + 2)$-sided faces $f_{i,j}$, one internal $\frac{a(a+3)}{2}$-sided face f_{int} determined by cycle on vertices $\{y_i : i \in I\} \bigcup \bigcup_{i \in I} \{x_{i,b,k} : 1 \leq k \leq i\}$ and one external $\frac{a(a+3)}{2}$-sided face f_{ext} determined by cycle on vertices $\{y_i : i \in I\} \bigcup \bigcup_{i \in I} \{x_{i,1,k} : 1 \leq k \leq i\}$.

If we omit the paths $P_a^j = \{y_a, x_{a,j,1}, x_{a,j,2}, \ldots, x_{a,j,a}, y_1\}$, $j \in J$, we obtain a plane graph defined by Kathiresan and Ganesan in [12] as P_a^b. Kathiresan and Ganesan [12] studied d-antimagic labelings of type (1, 1, 1) for P_a^b and for $d \in \{0, 1, 2, 3, 4, 6\}$. The existence of d-antimagic labelings of type (1, 1, 1) for P_a^b and for many other values of parameter $d \in \{5, 7, a-1, a+2, a+3, \ldots\}$ were investigated by Lin and Sugeng in [15]. d-antimagic labelings of type (1, 1, 1) for C_a^b and for $d \in \{0, 1, 2, 3\}$ are described in [6].

In this paper we show that plane graph C_a^b has d-antimagic labeling for $d \in \{a-2, a-1, a+1, a+2\}$.

3 The weight of faces

Let us denote the weight of the $(2i+2)$-sided face $f_{i,j}$ and the external (internal) $\frac{a(a+3)}{2}$-sided face under a vertex labeling α, an edge labeling β and a face labeling γ as follows:

$$w(f_{i,j}) = \alpha(y_i) + \sum_{k=1}^{i} \alpha(x_{i,j,k}) + \alpha(y_{i+1}) + \sum_{k=1}^{i} \alpha(x_{i,j+1,k}) + \beta(y_i x_{i,j,1}) +$$

$$\sum_{k=1}^{i-1} \beta(x_{i,j,k} x_{i,j,k+1}) + \beta(x_{i,j,i} y_{i+1}) + \beta(y_i x_{i,j+1,1}) +$$

$$\sum_{k=1}^{i-1} \beta(x_{i,j+1,k} x_{i,j+1,k+1}) + \beta(x_{i,j+1,i} y_{i+1}) + \gamma(f_{i,j})$$

for $i \in I$ and $j \in J - \{b\}$,

$$\begin{aligned} w(f_{ext}) = & \sum_{i=1}^{a} \alpha(y_i) + \sum_{i=1}^{a} \sum_{k=1}^{i} \alpha(x_{i,1,k}) + \sum_{i=1}^{a} \beta(y_i x_{i,1,1}) + \\ & \sum_{i=1}^{a} \sum_{k=1}^{i-1} \beta(x_{i,1,k} x_{i,1,k+1}) + \sum_{i=1}^{a} \beta(x_{i,1,i} y_{i+1}) + \gamma(f_{ext}), \\ w(f_{int}) = & \sum_{i=1}^{a} \alpha(y_i) + \sum_{i=1}^{a} \sum_{k=1}^{i} \alpha(x_{i,b,k}) + \sum_{i=1}^{a} \beta(y_i x_{i,b,1}) + \\ & \sum_{i=1}^{a} \sum_{k=1}^{i-1} \beta(x_{i,b,k} x_{i,b,k+1}) + \sum_{i=1}^{a} \beta(x_{i,b,i} y_{i+1}) + \gamma(f_{int}). \end{aligned}$$

4 d-antimagic labeling

In this section we investigate d-antimagic labelings of the plane graph C_a^b for $d \in \{a-2, a-1, a+1, a+2\}$.

Theorem 1. *For $a \geq 3$ and $b \geq 2$, the plane graph C_a^b has an $(a+1)$-antimagic labeling of type* (1, 1, 1).

Proof. We consider three cases.

- *Case* 1. *a is even.*
Construct a vertex labeling α_1, an edge labeling β_1 and a face labeling γ_1 of the plane graph C_a^b in the following way:

$$\alpha_1(y_i) = v - a + e + i + 1, \qquad \text{if } 1 \le i \le a.$$

$$\alpha_1(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + b + \frac{1-j}{2} & \text{if } i \text{ and } j \text{ are odd, } k = 1, \\ \frac{bi(i-1)}{2} + \lfloor \frac{b}{2} \rfloor + \frac{2-j}{2} & \text{if } i \text{ is odd, } j \text{ is even, } k = 1, \\ \frac{bi(i-1)}{2} + b(k-1) + j & \text{if } i \text{ is even, } k \text{ is odd, or} \\ & \text{if } i \text{ is odd, } i \ge 3, k \text{ is even,} \\ \frac{bi(i-1)}{2} + kb + 1 - j & \text{if } i \text{ and } k \text{ are even, or} \\ & \text{if } i \text{ and } k \text{ are odd, } i, k \ge 3, \end{cases}$$

for $i \in I$, $j \in J$ and $1 \le k \le i$.

$$\beta_1(y_i x_{i,j,1}) = \begin{cases} \frac{b(i-1)(i+2)}{2} + j & \text{if } i \text{ is odd,} \\ \frac{b(i-1)(i+2)}{2} + \frac{j+1}{2} & \text{if } i \text{ is even, } j \text{ is odd,} \\ \frac{b(i-1)(i+2)}{2} + \lceil \frac{b}{2} \rceil + \frac{j}{2} & \text{if } i \text{ and } j \text{ are even,} \end{cases}$$

for $i \in I$ and $j \in J$.

$$\beta_1(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{bi(i+3)}{2} + 1 - j & \text{if } i \text{ is odd,} \\ \frac{bi(i+3)}{2} - b + j & \text{if } i \text{ is even,} \end{cases}$$

for $i \in I$ and $j \in J$.

$$\beta_1(x_{i,j,k} x_{i,j,k+1}) = \begin{cases} \frac{bi(i+1)}{2} + kb + 1 - j & \text{if } k \text{ is odd,} \\ \frac{bi(i+1)}{2} + b(k-1) + j & \text{if } k \text{ is even,} \end{cases}$$

for $i \in I$, $j \in J$ and $1 \le k \le i - 1$.

$$\gamma_1(f_{i,j}) = \begin{cases} v + e + f - (j-1)a - i + 1 & \text{if } i \text{ is odd,} \\ v + e + f + (j + 1 - b)a - i + 1 & \text{if } i \text{ is even,} \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$.

$\gamma_1(f_{ext}) = v + e + 2.$
$\gamma_1(f_{int}) = v - a + e + 1.$

Label the vertices, edges and faces of C_a^b by α_1, $v - a + \beta_1$ and γ_1. It can be seen that the resulting labeling is a labeling of type $(1, 1, 1)$ where the weights of the $(2i + 2)$-sided faces form an arithmetic progression with difference $d = a + 1$ and $w(f_{ext}) - w(f_{int}) = a + 1$.

- *Case 2. a and b are odd.*
  Define a vertex labeling α_2, an edge labeling β_2 and a face labeling γ_2 of C_a^b as follows:

$$\alpha_2(y_i) = \begin{cases} v + f - a - 1 + i & \text{if } 1 \le i \le a - 1, \\ v + f & \text{if } i = a. \end{cases}$$

$$\alpha_2(x_{i,j,k}) = \begin{cases} v - a - \frac{j-1}{2} & \text{if } i = 1 \text{ and } j \text{ is odd,} \\ v - a - \frac{b+j-1}{2} & \text{if } i = 1 \text{ and } j \text{ is even,} \\ \frac{bi(i-1)-b+2-j}{2} & \text{if } i \text{ and } j \text{ are odd, } i \ge 3, k = 1, \\ \frac{bi(i-1)-j+2}{2} & \text{if } i \text{ is odd, } j \text{ is even, } i \ge 3, k = 1, \\ \frac{bi(i-1)}{2} + b(k-2) + j & \text{if } i \text{ is even, } k \text{ is odd, or} \\ & \text{if } i \text{ is odd, } i \ge 3, k \text{ is even,} \\ \frac{bi(i-1)}{2} + b(k-1) + 1 - j & \text{if } i \text{ and } k \text{ are even, or} \\ & \text{if } i \text{ and } k \text{ are odd, } i, k \ge 3, \end{cases}$$

for $i \in I$, $j \in J$ and $1 \le k \le i$.

$\beta_2(y_i x_{i,j,1}) = \beta_1(y_i x_{i,j,1})$ and
$\beta_2(x_{i,j,i} y_{i+1}) = \beta_1(x_{i,j,i} y_{i+1})$ for $i \in I$ and $j \in J$.
$\beta_2(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$ for $i \in I$, $j \in J$ and $1 \le k \le i - 1$.

$$\gamma_2(f_{i,j}) = \begin{cases} v + e + f - (j+1)a + i - 2 & \text{if } i \text{ is odd,} \\ v + e + f + (j - 1 - b)a + i - 2 & \text{if } i \text{ is even,} \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$.
$\gamma_2(f_{ext}) = v + e + f - a - 1$.
$\gamma_2(f_{int}) = v + e + f - 1$.

If we combine the labelings $e + \alpha_2$, β_2 and γ_2 we obtain labeling of type (1, 1, 1) such that the weights of the $(2i+2)$-sided faces, for each $i \in I$, constitute an arithmetic progression of difference $a+1$ and $w(f_{int}) - w(f_{ext}) = a - \lfloor \frac{b}{2} \rfloor$. If we swap the vertex label $\alpha_2(x_{1,b,1}) = v + e - a - \frac{b-1}{2}$ with the face label $\gamma_2(f_{1,b-1}) = v + e + f - ba - 1$ then the face weight of $f_{1,b-1}$ will remain the same, but the face weight of f_{int} will be increased by $\lfloor \frac{b}{2} \rfloor + 1$. Thus, after swaping $w(f_{int}) - w(f_{ext}) = a + 1$.

- *Case 3. a is odd and b is even.*
  Define a vertex labeling α_3 and an edge labeling β_3 for C_a^b in the following way:

$$\alpha_3(y_i) = \begin{cases} v + e - a + 1 + i & \text{if } 1 \le i \le a - 1, \\ v + e + f & \text{if } i = a. \end{cases}$$

$\alpha_3(x_{i,j,k}) = \alpha_1(x_{i,j,k})$ for $i \in I$, $j \in J$ and $1 \le k \le i$.
$\beta_3(y_i x_{i,j,1}) = \beta_1(y_i x_{i,j,1})$ for $i \in I$ and $j \in J$.
$\beta_3(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$ for $i \in I$, $j \in J$ and $1 \le k \le i-1$.

$$\beta_3(x_{i,j,i} y_{i+1}) = \begin{cases} \frac{bi(i+3)}{2} + 1 - j & \text{if } i \text{ is odd, } i < a, \\ \frac{ba(a+3)}{2} + 2 - j & \text{if } i = a, \\ \frac{bi(i+3)}{2} - b + j & \text{if } i \text{ is even,} \end{cases}$$

for $i \in I$ and $j \in J$.

If we label the vertices and the edges of C_a^b by α_3 and $v - a + \beta_3$ and complete the values for faces $f_{i,j}$ by

$\gamma_3(f_{i,j}) = \gamma_1(f_{i,j}) - 1$ for $i \in I$ and $j \in J - \{b\}$,

we can observe that these labelings combine to a resulting labeling which uses each integer $1, 2, 3, \ldots, v + e + f$ exactly once and that the set of the $(2i+2)$-sided face weights, for each $i \in I$, consists of arithmetic progression with difference $a + 1$.

The difference of the sum of all values carried by the vertices and edges surrounding the external face f_{ext} and the sum of all values carried by the vertices and edges surrounding the internal face f_{int} is $b - 1$, i.e.

$$\Big(\sum_{i=1}^{a} \alpha_3(y_i) + \sum_{i=1}^{a}\sum_{k=1}^{i} \alpha_3(x_{i,1,k}) + \sum_{i=1}^{a} [v - a + \beta_3(y_i x_{i,1,1})] + \sum_{i=1}^{a}\sum_{k=1}^{i-1} [v - a + \beta_3(x_{i,1,k} x_{i,1,k+1})] + \sum_{i=1}^{a} [v - a + \beta_3(x_{i,1,i} y_{i+1})]\Big) - \Big(\sum_{i=1}^{a} \alpha_3(y_i) + \sum_{i=1}^{a}\sum_{k=1}^{i} \alpha_3(x_{i,b,k}) + \sum_{i=1}^{a} [v - a + \beta_3(y_i x_{i,b,1})] + \sum_{i=1}^{a}\sum_{k=1}^{i-1} [v - a + \beta_3(x_{i,b,k} x_{i,b,k+1})] + \sum_{i=1}^{a} [v - a + \beta_3(x_{i,b,i} y_{i+1})]\Big) = b - 1.$$

When we complete values
$\gamma_3(f_{ext}) = v + e - a - b + 1$ and
$\gamma_3(f_{int}) = v + e + 1$
we have $w(f_{int}) - w(f_{ext}) = a + 1$. We achieve at the desired result. ■

Theorem 2. *If $a \ge 3$ and $b \ge 2$, then the plane graph C_a^b has an $(a-1)$-antimagic labeling of type $(1,1,1)$.*

Proof. Let us distinguish three cases.

– *Case* 1. *a is even.*

Construct the functions α_4 and γ_4 as follows:

$$\alpha_4(y_i) = \begin{cases} v - a + e + i + 1 & \text{if } 1 \le i \le a - 2, \\ v - a + e + i + 2 & \text{if } a - 1 \le i \le a. \end{cases}$$

$$\alpha_4(x_{i,j,k}) = \alpha_1(x_{i,j,k}) \qquad \text{for } i \in I,\ j \in J \text{ and } 1 \le k \le i.$$

$$\gamma_4(f_{i,j}) = \begin{cases} v + e + f + (j - b + 1)a - i + 1 & \text{if } i \text{ is odd}, \\ v + e + f - (j - 1)a - i + 1 & \text{if } i \text{ is even}, \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$.

$\gamma_4(f_{ext}) = v + e.$
$\gamma_4(f_{int}) = v - a + e + 1.$

Label the vertices, edges and faces of C_a^b by α_4, $v - a + \beta_1$ and γ_4. We obtain the resulting labeling of type $(1, 1, 1)$ in which the weights of the $(2i + 2)$-sided faces constitute an arithmetic progression with difference $d = a - 1$ and $w(f_{int})$ is $a - 1$ less than $w(f_{ext})$.

- *Case* 2. *a and b are odd.*
  Consider the following vertex labeling α_5 where

$$\alpha_5(y_i) = \begin{cases} v + f - a - 1 + i & \text{if } 1 \le i \le a - 2, \\ v + f - a + i & \text{if } a - 1 \le i \le a. \end{cases}$$

$$\alpha_5(x_{i,j,k}) = \begin{cases} v - a - \frac{b+j-2}{2} & \text{if } i = 1 \text{ and } j \text{ is odd}, \\ v + f - a - \frac{j}{2} & \text{if } i = 1 \text{ and } j \text{ is even}, \\ \frac{bi(i-1)-b+2-j}{2} & \text{if } i \text{ and } j \text{ are odd}, i \ge 3, k = 1, \\ \frac{bi(i-1)-j+2}{2} & \text{if } i \text{ is odd}, j \text{ is even}, i \ge 3, k = 1, \\ \frac{bi(i-1)}{2} + b(k-2) + j & \text{if } i \text{ is even}, k \text{ is odd, or} \\ & \text{if } i \text{ is odd}, i \ge 3, k \text{ is even}, \\ \frac{bi(i-1)}{2} + b(k-1) + 1 - j & \text{if } i \text{ and } k \text{ are even, or} \\ & \text{if } i \text{ and } k \text{ are odd}, i, k \ge 3, \end{cases}$$

for $i \in I$, $j \in J$ and $1 \le k \le i$.

It can be seen that if the vertices of C_a^b receive values $e + \alpha_5$, the edges receive values β_1 and the faces $f_{i,j}$ receive values

$$\gamma_5(f_{i,j}) = \begin{cases} v + e + f - \frac{b-1}{2} + (j - b)a - i - 1 & \text{if } i \text{ is odd}, , \\ v + e + f - \frac{b-1}{2} - ja - i - 1 & \text{if } i \text{ is even}, \end{cases}$$

for $i \in I$ and $j \in J - \{b\}$,

then the weights of the $(2i+2)$-sided faces form an arithmetic progression with difference $a-1$. The sum of all values received by the vertices and edges surrounding the face f_{ext} is $\frac{b-1}{2}$ greater than the sum of all values received by the vertices and edges surrounding the face f_{int}.

If we complete the values

$\gamma_5(f_{ext}) = v+e+f-\frac{b+1}{2}-a$ and
$\gamma_5(f_{int}) = v+e+f-2$

then we get $w(f_{int})-w(f_{ext}) = a-1$.

Moreover, it is matter of routine checking to see that labelings $e+\alpha_5$, β_1 and γ_5 combine to a labeling of type $(1,1,1)$.

- *Case* 3. *a is odd and b is even.*
  Define a vertex labeling α_6 and a face labeling γ_6 in the following way:

$$\alpha_6(y_i) = \begin{cases} v+e-a+1+i & \text{if } 1 \le i \le a-3, \\ v+e-a+2+i & \text{if } a-2 \le i \le a. \end{cases}$$

$\alpha_6(x_{i,j,k}) = \alpha_1(x_{i,j,k})$ for $i \in I$, $j \in J$ and $1 \le k \le i$.
$\gamma_6(f_{i,j}) = \gamma_4(f_{i,j})$ for $i \in I$ and $j \in J-\{b\}$.
$\gamma_6(f_{ext}) = v+e-a-b+1$.
$\gamma_6(f_{int}) = v+e-1$.

Now, label the vertices of C_a^b by the labeling α_6, the edges by the labeling $v-a+\beta_3$ and the faces by the labeling γ_6. The labelings α_6, $v-a+\beta_3$ and γ_6 together describe a labeling of type $(1,1,1)$. It is tedious but not difficult to check that the weights of all $(2i+2)$-sided faces, for each $i \in I$, constitute an arithmetic progression with difference $a-1$ and that $w(f_{int})-w(f_{ext}) = a-1$. ∎

Theorem 3. *For $a \ge 3$ and $b \ge 2$ and $d \in \{a-2, a+2\}$, the plane graph C_a^b has a d-antimagic labeling of type* $(1,1,1)$.

Proof. Suppose that $a \ge 3$ and $b \ge 2$. Then the next two cases will be considered.

- *Case* 1. *a is even.*
  Define a vertex labeling δ_1 and an edge labeling σ_1 so that

$\delta_1(y_i) = v+f-a-1+i$ if $1 \le i \le a$.

$$\delta_1(x_{i,j,k}) = \begin{cases} \frac{bi(i-1)}{2} + b(k-1) + j & \text{if } i \text{ is even, } k \text{ is odd, or} \\ & \text{if } i \text{ is odd, } i \geq 3, k \text{ is even,} \\ \frac{bi(i-1)}{2} + kb + 1 - j & \text{if } i \text{ and } k \text{ are even, or} \\ & \text{if } i \text{ and } k \text{ are odd,} \end{cases}$$

for $i \in I$, $j \in J$ and $1 \leq k \leq i$.

$\sigma_1(y_i x_{i,j,1}) = \frac{b(i-1)(i+2)}{2} + j$ for $i \in I$ and $j \in J$.
$\sigma_1(x_{i,j,i} y_{i+1}) = \beta_1(x_{i,j,i} y_{i+1})$ for $i \in I$ and $j \in J$.
$\sigma_1(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$ for $i \in I$, $j \in J$ and $1 \leq k \leq i-1$.

If we label the vertices of C_a^b by $e + \delta_1$, the edges by σ_1 and the faces by ρ_1 where

$$\rho_1(f_{i,j}) = \begin{cases} v + e + f - ja - i - 1 & \text{if } i \text{ is odd,} \\ v + e + f + (j-b)a - i - 1 & \text{if } i \text{ is even,} \end{cases}$$

for $i \in I$, $j \in J - \{b\}$,

$\rho_1(f_{ext}) = v + e + f$,
$\rho_1(f_{int}) = v + e + f - a - 1$,

then we obtain the resulting labeling with labels from the set $\{1, 2, 3, \ldots, v + e + f\}$ in which the weights of $(2i+2)$-sided faces, for each $i \in I$, form an arithmetic progression with difference $a + 2$. At the same time $w(f_{ext}) - w(f_{int}) = a + 1$.

Swaping the vertex label $e + \delta_1(x_{a,1,a}) = e + \frac{ab(a+1)}{2}$ with the face label $\rho(f_{a,1}) = e + \frac{ab(a+1)}{2} + 1$ does not change the face weight of $f_{a,1}$, but the weight of f_{ext} is increased by one. Thus $w(f_{ext}) - w(f_{int}) = a + 2$.

Now, we construct a vertex labeling δ_2 and a face labeling ρ_2 as follows:

$$\delta_2(y_i) = \begin{cases} v + f - a - 1 + i & \text{if } 1 \leq i \leq a-3, \\ v + f - a + i & \text{if } a-2 \leq i \leq a. \end{cases}$$

$\delta_2(x_{i,j,k}) = \delta_1(x_{i,j,k})$ for $i \in I$, $j \in J$ and $1 \leq k \leq i$.

$$\rho_2(f_{i,j}) = \begin{cases} v + e + f + (j-b)a - i - 1 & \text{if } i \text{ is odd,} \\ v + e + f - ja - i - 1 & \text{if } i \text{ is even,} \end{cases}$$

for $i \in I$, $j \in J - \{b\}$.

$\rho_2(f_{ext}) = v + e + f - 3.$
$\rho_2(f_{int}) = v + e + f - a - 1.$

We can observe that the labelings $e + \delta_2$, σ_1 and ρ_2 combine to an $(a-2)$-antimagic labeling of type $(1,1,1)$.

- *Case 2. a is odd.*
  Construct a vertex labeling δ_3 and a face labeling ρ_3 in the following way:

  $\delta_3(y_i) = v + e + f - a - 1 + i$ if $1 \le i \le a$.
  $\delta_3(x_{i,j,k}) = \delta_1(x_{i,j,k})$ for $i \in I$, $j \in J$ and $1 \le k \le i$.
  $\rho_3(f_{i,j}) = \rho_1(f_{i,j})$ for $i \in I$, $j \in J - \{b\}$.
  $\rho_3(f_{ext}) = v + e + f - a - 1.$
  $\rho_3(f_{int}) = v + e + f.$

  Label the vertices, edges and faces of C_a^b by δ_3, $v-a+\sigma_1$ and ρ_3. Again it is easy to verify that the labelings δ_3, $v-a+\sigma_1$ and ρ_3 combine to a labeling of type $(1,1,1)$ where the weights of the $(2i+2)$-sided faces, for each $i \in I$, form an arithmetic progression with difference $a+2$ but $w(f_{int}) - w(f_{ext}) = a+2-b$. Therefore we swap the edge label $v-a+\sigma_1(x_{a,b,a}y_1) = v+e-a+1-b$ with the face label $\rho_3(f_{a,b-1}) = v+e+1-a$. This swaping does not change the face weight of the face $f_{a,b-1}$ but weight of f_{int} will be increased by b, so difference between the resulting weight of f_{ext} and the resulting weight of f_{int} will be $a+2$.

  Finally, we have the following labelings for vertices, edges and faces of C_a^b.

  $$\delta_4(y_i) = \begin{cases} v + e - a + 1 + i & \text{if } 1 \le i \le a-4, \\ v + e - a + 2 + i & \text{if } a-3 \le i \le a. \end{cases}$$

  $\delta_4(x_{i,j,k}) = \delta_1(x_{i,j,k})$ for $i \in I$, $j \in J$ and $1 \le k \le i$.
  $\sigma_2(y_i x_{i,j,1}) = \sigma_1(y_i x_{i,j,1})$ for $i \in I$ and $j \in J$.
  $\sigma_2(x_{i,j,k} x_{i,j,k+1}) = \beta_1(x_{i,j,k} x_{i,j,k+1})$ for $i \in I$, $j \in J$ and $1 \le k \le i$.
  $\sigma_2(x_{i,j,i} y_{i+1}) = \beta_3(x_{i,j,i} y_{i+1})$ for $i \in I$ and $j \in J$.
  $\rho_4(f_{i,j}) = \gamma_4(f_{i,j})$ for $i \in I$ and $j \in J - \{b\}$.
  $\rho_4(f_{ext}) = v + e - b - a + 1.$
  $\rho_4(f_{int}) = v + e - 2.$

  If we combine the labelings δ_4, $v-a+\sigma_2$ and ρ_4, we obtain a labeling of type $(1,1,1)$ such that the weights of the $(2i+2)$-sided faces, for each $i \in I$, form an arithmetic progression with difference $a-2$. Likewise, $w(f_{int}) - w(f_{ext}) = a-2$. This completes the proof. ■

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