Exclusive Sum Labelings of Trees

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Abstract. The notions of *sum labeling* and *sum graph* were introduced by Harary in 1990 [3]. In a sum labeling, a vertex is called a *working vertex* if its label is equal to the sum of the labels of a pair of two distinct vertices.
A sum labeling of a graph G is said to be *exclusive* if it is a sum labeling of G such that G contains no working vertex. Any connected graph G will require some additional isolated vertices in order to be labeled exclusively. The smallest number of such isolates is called the *exclusive sum number* of G; it is denoted by $\epsilon(G)$. The number of isolates cannot be less than the maximum number of neighbours of any vertex in the graph, that is, at least equal to $\Delta(G)$, the maximum vertex degree in G. If $\epsilon(G) = \Delta(G)$, then G is said to be a *Δ-optimum summable graph*. An exclusive sum labeling of G using $\Delta(G)$ isolates is called *Δ-optimum exclusive sum labeling* of G.
In this paper we show that some families of trees are Δ-optimum summable graphs. However, this is not true for all trees, and we present an example of a tree which is not Δ-optimum summable graph, giving rise to an open problem.

1 Introduction

All graphs we consider here are finite, simple and undirected. For general terms used in graph theory, please refer to [5].

A *sum labeling* λ of a graph G is a mapping of the vertices of G into distinct positive integers such that $\{u, v\} \in E(G)$ if and only if the sum of the labels assigned to u and v is the label of a vertex w of G. In such a case w is called a *working vertex*. A graph which has a sum labeling is called a *sum graph*.

Any graph G will need some isolated vertices called *isolates*. If a graph lacks some or all such isolates, one can add more isolates until the graph itself, together with the additional isolates, can support a sum labeling. The least number of the additional isolates is called the *sum number* of the graph; it is denoted by $\sigma(G)$. A graph together with its minimum number of isolates, is called an *optimum summable graph*. Moreover, let $\delta(G)$ be the smallest degree of vertices of a graph G. It is obvious that $\sigma(G) \geq \delta(G)$. In case $\sigma(G) = \delta(G)$, the graph G is called δ-*optimum summable* graph.

A sum labeling of a graph $G \cup \overline{K_r}$ for some positive integer r is said to be *exclusive* with respect to G if all of its working vertices are in $\overline{K_r}$; otherwise it is said to be *inclusive*. Every graph can be made to support an exclusive sum labeling, by adding some isolates. The least possible number of isolates that need to be added to a graph G to obtain an exclusive sum labeling is called the *exclusive sum number* of the graph G, denoted by $\epsilon(G)$, and the graph $G \cup \overline{K_{\epsilon(G)}}$ is called an optimum exclusive sum graph of G.

Observation 1 *Let $\Delta(G)$ be the maximum degree of the vertices of a graph G. Then $\epsilon(G) \geq \Delta(G)$.*

In case $\epsilon(G) = \Delta(G)$, the graph G is said to be a Δ-*optimum summable* graph.

We refer to [2] for the notions of tree, caterpillar, and shrub. Furthermore, the following terms will be used in this paper.

1. A *leaf* of a tree is a vertex with degree 1.
2. A *near-leaf* is a non-leaf that has at most one neighbour which is not a leaf.
3. An *inner vertex* is a vertex that has at least two neighbours which are not leaves.

In [2], Ellingham proved that if T is a tree of order at least 2, then $\sigma(T) = 1$. That is, every tree is 1-optimal with respect to sum labeling. The notion of exclusive sum labeling was inroduced by Bergstrand *et al.* [1]. In [7] Miller *et al.* extended the idea to include all graphs. That is, graphs with an inclusive optimal labeling may also bear an exclusive sum labeling.

Observation 2 *For any graph G, $\epsilon(G) \geq \sigma(G)$.*

Since exclusive sum labeling was extended to include all graphs it has been a challenge to find the exclusive sum number of trees. Unlike its counterpart problem in sum graph labeling, any attempt to solve this problem so far has been unsuccessful. In this paper we present some new findings in this interesting research area. We will show in particular that some certain classes of trees are Δ-optimum summable graphs and on the other

hand, we will also show that there exist trees which are not Δ-optimum summable graphs. In Section 2 we show that every caterpillar-type tree is a Δ-optimum summable graph. In Section 3, it is shown that every shrub is also a Δ-optimum summable graph. We also include a labelling for two trees that can be classified both as caterpillars and shrubs; stars and double stars. In Section 4 we provide an example of a tree which is not a Δ-optimum summable graph and conclude with some open problems.

2 Exclusive sum labeling of caterpillars

Recall that a caterpillar is a graph which has the property that if we remove all the vertices of degree 1 then what remains is a path. A caterpillar can have more than one longest path. Such a path is called the *spine* of the caterpillar. The two endpoints of a spine are called, respectively, the *tail* and the *head*. Other vertices on the spine are called *internal vertices*. We shall always consider a spine of a caterpillar as oriented in the particular direction from tail to head. The vertices of degree one of a caterpillar, other than tail and head, will be called the *feet*; these vertices are attached to the internal vertices of the spine by edges called the *legs* of the caterpillar.
Let C be a caterpillar with $\Delta(C) = d$.

Labeling 1 (Exclusive sum labeling of a caterpillar)

1. Choose a spine of C and let $P = \{p_1, p_2, ..., p_k\}$ be the set of vertices of the spine. Let

$$
\begin{aligned}
f_i &= deg(p_i) - 2, \quad i = 2, 3, ..., k - 1. \\
&= 0, \quad i = 1, k.
\end{aligned}
$$

For $2 \leq i \leq k - 1$, let $B_i = \{b_{ij} | 1 \leq j \leq f_i, f_i > 0\}$ be the set of feet which are attached to the internal vertex p_i. It is clear that $B_i = N(p_i) \backslash (P \cap N(p_i))$, for $2 \leq i \leq k - 1$. Let $B = \bigcup_{i=2}^{k-1} B_i$ be the set of all feet of C.

2. Label the spine with a mapping L as follows.

$$
\begin{aligned}
L(p_i) &= 1 + 2(i - 1)(d - 2) \quad \text{for odd} \quad i, \\
&= 1 + 4(k - i/2)(d - 2) \quad \text{for even} \quad i.
\end{aligned}
$$

This gives

$$
\begin{aligned}
L(p_i) + L(p_{i+1}) &= 2 + (4k - 4)(d - 2), \quad \text{for odd} \quad i, \\
&= 2 + 4k(d - 2), \quad \text{for even} \quad i.
\end{aligned}
$$

3. Let $T_A = \{t_1^{(a)}, t_2^{(a)}\}$ be two vertices with labels $L(t_1^{(a)}) = 2 + (4k-4)(d-2)$ and $L(t_2^{(a)}) = 2 + 4k(d-2)\}$. Choose $a = 5 + 4k(d-2)$. It is clear that $a \equiv 1 \pmod 4$.

4. Add $d-2$ more vertices $T_B = \{t_i^{(b)} | i = 1, 2, ..., (d-2)\}$ and label with $L(t_i^{(b)}) = (a + L(p_2)) + 4(i-1)$.

5. Let $T = T_A \cup T_B$. For $i = 2, 3, ..., k-1$, label the vertices of B_i as follows. $L(b_{ij}) = L(t_j^{(b)}) - L(p_i), \quad j = 1, 2, ..., f_i$.

For convenience, from now on we will use v instead of $L(v)$ for any $v \in V(C) \cup \overline{K_d}$, and each vertex will be identified by its label under L. Also, now we let $B_i = \{b_{ij} | 1 \le j \le f_i\}$ and $B = \bigcup_{i=2}^{k-1} B_i$ denote the set of the labels of the feet.

Remark 1 For $i = 1, 2, ..., k$ let $B_i' = \{t_j^{(b)} - p_i | j = 1, 2, ..., d-2\}$. It is obvious that $B_i \subseteq B_i'$, $i = 1, 2, ..., k$. If $B' = \bigcup_{i=1}^{k} B_i'$ then $B \subseteq B'$.

Before we prove that the labeling L is an optimal exclusive sum labeling of C, we need to consider the following facts:

Observation 3 $max(P) = p_2, min(P) = p_1$.

Observation 4 $max(T_A) = p_2 + p_3, min(T_A) = p_1 + p_2$.

Observation 5 $max(T_B) = a + p_2 + 4(d-3), min(T_B) = a + p_2$.

Observation 6 $max(B') = max(T_B) - min(P) = a + p_2 + 4(d-3) - p_1$, and
$min(B') = min(T_B) - max(P) = a + p_2 - p_2 = a$.

Lemma 1 Let $p \in P$, $t^{(a)} \in T_A$, $b \in B$ and $t^{(b)} \in T_B$, where P, T_A and T_B as in Labeling 1. Then $p < t^{(a)} < b < t^{(b)}$.

Proof. There are three parts to prove.

1. We will show that $p < t^{(a)}$ for all $p \in P$ and for all $t^{(a)} \in T_A$.
Let $p \in P$, $t^{(a)} \in T_A$ then $p \le p_2$ and either $t^{(a)} = p_2 + p_1$ or $t^{(a)} = p_2 + p_3$. In both cases, we have $p_2 < t^{(a)}$.
This gives $p < t^{(a)}$ for all $p \in P$ and for all $t^{(a)} \in T_A$.

2. We will show that $t^{(a)} < b$, for all $t^{(a)} \in T_A$ and for all $b \in B$.
Let $b \in B$ and $t^{(a)} \in T_A$ since,
$B \subseteq B'$ then $b \ge min(B') = a > max(T_A) \ge t^{(a)}$.
Therefore, $b > t^{(a)} \quad \forall t^{(a)} \in T_A$ and $\forall b \in B$.

112

3. We will show that $t^{(b)} > b$, for all $t^{(b)} \in T_B$ and for all $b \in B$.
Let $t^{(b)} \in T_B$ and $b \in B$ then $t^{(b)} \geq \min(T_B) = a + p_2$ and $\max(B') = \max(T_B) - \min(P) = (a + p_2) + 2(d - 3) - p_1$.

$$\begin{aligned}
a + p_2 - \max(B') &= (a + p_2) - (a + p_2) + 2(d - 3) - p_1 \\
&= p_1 - 2(d - 3) \\
&= 2d - 2(d - 3) \\
&> 0
\end{aligned}$$

We have $a + p_2 > \max(B')$, therefore, $t^{(b)} > \max(B) \geq b$.

¿From these three facts we have $p < t^{(a)} < b < t^{(b)}, \forall p \in P, t^{(a)} \in T_A, b \in B$ and $t^{(b)} \in T_B$.

Lemma 2 *Let P, B'_r and B'_s be as in Labeling 1. If $r \neq s$ then $B'_r \cap B'_s = \emptyset$*

Proof. Suppose to the contrary that $B'_r \cap B'_s \neq \emptyset$. Let $x \in B'_r \cap B'_s$.
Then
$x = t_i^{(b)} - p_s$ for some $t_i^{(b)} \in T_B$ and
$x = t_j^{(b)} - p_r$ for some $t_j^{(b)} \in T_B$.
We get $t_i^{(b)} - t_j^{(b)} = p_s - p_r$. However,

$$\begin{aligned}
t_i^{(b)} - t_j^{(b)} &\leq \max\{|t_u^{(b)} - t_v^{(b)}||t_u^{(b)}, t_v^{(b)} \in T_B\} \\
&= \max(T_B) - \min(T_B) \\
&= 4(d - 3)
\end{aligned}$$

On the other hand, $p_s - p_r \geq 4(d - 2)$.
Hence, $t_i^{(b)} - t_j^{(b)} \leq 4(d - 3) < 4(d - 2) \leq p_s - p_r$. This contradicts the fact that $t_i^{(b)} - t_j^{(b)} = p_s - p_r$.
Therefore, we must have $B'_r \cap B'_s = \emptyset$.

As a consequence,

Lemma 3 *If $r \neq s$ then $B_r \cap B_s = \emptyset$.*

Observation 7 *For all i, $i = 1, 2, ..., k$, $p_i \equiv 1 \pmod 4$ and $b_{ij} \equiv 1 \pmod 4$ for all $j = 1, 2, ..., f_i$. This gives $t \equiv 2 \pmod 4$ $\forall t \in T$.*

We are now ready to prove

Theorem 1 *Let C be a caterpillar with maximum degree $\Delta(C)$. Then $\epsilon(C) = \Delta(C)$, that is, every caterpillar is a Δ-optimum summable graph.*

113

Proof. We will show that Labeling 1 gives an exclusive sum labeling of a caterpillar C.

By Lemma 1, the labeling is surely a bijection from $V(C \cup T)$ onto distinct positive integers in $P \cup B \cup T_A \cup T_B$. Moreover, it is clear that if $\{u, v\} \in E(C)$ then $u + v \in T$. We need to prove that there are no extra edges needed, that is, if $\{u, v\} \notin E(C \cup T)$ then $u + v \notin V(C \cup T) = P \cup B \cup T_A \cup T_B$.

1. Let $x \in B$ and $y \in B, \{x, y\} \notin E(C)$. Obviously, $x + y \notin B$ and $x + y \notin P$. We will show that $x + y \notin T$.

   - $x + y > t, \forall t \in T_A$ due to Lemma 1.
   - Note that $a > p_2 + p_3$.

   $$\begin{aligned} x + y &\geq 2a \\ &> (a + p_2) + p_3 \\ &> (a + p_2) + 1 + 4(d - 2) \\ &> (a + p_2) + 4(d - 3) \\ &= \max(T_B). \end{aligned}$$

   Therefore, $x + y \notin T_B$.

2. Let $x \in B$, $y \in P$ and $\{x, y\} \notin E(C)$. It is obvious that $x + y \notin P$ and $x + y \notin B$. We will show that $x + y \notin T$.
   Let $x = b_{ij}$ and $y = p_k$, where $i \neq k$.

   - By Lemma 1, $b_{ij} > t, \forall t \in T_A$. So $b_{ij} + p_k \notin T_A$
   - $b_{ij} = t - p_i$ for some $t \in T_B$. If $b_{ij} + p_k \in T_B$, then $b_{ij} + p_k = t'$ for some $t' \in T_B$. We get $b_{ij} = t' - p_k$ This says that $b_{ij} \in B'_i \cap B'_k$, which contradicts Lemma 2.

3. Let $\{p_i, p_j\} \notin E(C)$. Clearly, $p_i + p_j \notin P$ and $p_i + p_j \notin B$. We will show that $p_i + p_j \notin T$.

   - If $p_i + p_j = t$, for some $t \in T_A$, then $p_i = t - p_j$, giving $p_i = p_{j-1}$ or $p_i = p_{j+1}$. This of course contradicts the fact that $p_i, p_j \notin E(C)$.
   - If $p_i + p_j = t$, for some $t \in T_B$, then $p_i = t - p_j \in B'_j$. This contradicts Lemma 1 that $p < b$, $\forall p, \forall b$.

By Observation 3, there is no possibility for the occurence of any unwanted edge. Therefore, labeling L is an exclusive labeling of C with $\Delta(C)$ isolates.

Example 1. Let C be a caterpillar with $\Delta(C) = 8$ and a spine of length 5. In this case $d = 8$ and $k = 5$. Label the vertices of the spine with

$$\begin{aligned} p_i &= 12i - 11, &&\text{for odd } i, \\ &= 121 - 12i, &&\text{for even } i. \end{aligned}$$

We get

$$p_i + p_{i+1} = 98, \qquad \text{for odd} \quad i,$$
$$= 122, \qquad \text{for even} \quad i.$$

Let $T_A = \{98, 122\}$ and take $a = 125$.
$T_B = \{(a+p_2)+4(i-1)|i = 1, 2, ..., (d-2)\} = \{222, 226, 230, 234, 238, 242\}$.
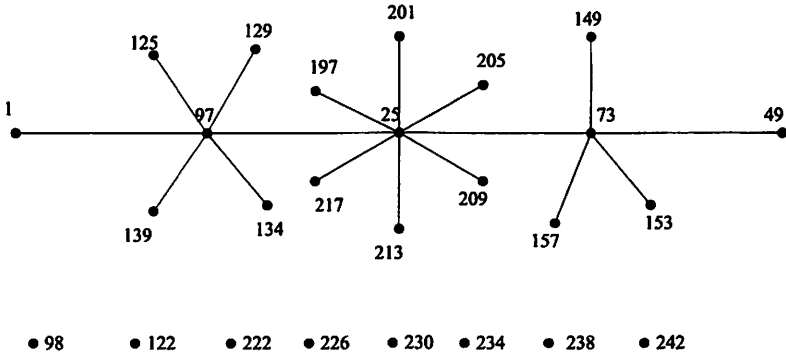Now we have the following figure.

Fig. 1. Exclusive sum labeling of a caterpillar.

3   Exclusive sum labeling of shrubs

A *shrub* is a tree which has at most one inner vertex. This special vertex
is called the *root* of the shrub. All neighbours of this vertex are leaves or
near leaves. In this subsection we will show that if Sh is a shrub, then
$\epsilon(Sh) = \Delta(Sh)$, that is, every shrub is a Δ-optimum summable graph.
In order to construct an exclusive sum labeling to a shrub Sh, we consider
the following sequence

$$u_n = 2^{n+1} - 3, \quad n \geq 1.$$

Observation 8 *For the above sequence, for all n*

1. $u_n > 0$.
2. $u_n > u_{n-1}$.
3. $u_n \equiv 1 (\bmod \ 4)$.

115

4. For i, j, k, l any positive integers, with $i > j$, $u_i - u_j = u_k - u_l$ if and only if $i = k$ and $j = l$, that is, the differences between any two terms are all distinct.

Let Sh be a shrub with root r and the maximum degree of the vertices is Δ. Let $b_1, b_2, ..., b_m$ be vertices adjacent to r. We denote $f_i = deg(b_i) - 1$ and for $i = 1, 2, ..., m$ and $j = 1, 2, ..., f_i, (f_i > 0)$ let c_{ij} be the vertices adjacent to b_i other than r. Let t_i, $i = 1, 2, ..., \Delta$ be isolates, and $U(n) = u_n$ be the sequence as defined in (1).

Observe that $m \leq \Delta$ and for any $i, i = 1, 2...m$, $f_i < \Delta$. Let $Sh' = Sh \cup \overline{K_\Delta}$.

Labeling 2 Exclusive sum labeling of shrubs
Let L be a mapping from the vertices of Sh' to a set of positive integers defined as follows.

$$L(b_i) = u_i, \quad i = 1, 2, ..., m$$
$$L(r) = a \quad \text{where } a = 3(2^{\Delta+1}) - 7$$
$$L(t_i) = a + u_i, \quad i = 1, 2, ..., \Delta.$$

$$L(c_{ij}) = \begin{cases} a + u_j - u_i & \text{for } i \neq j \quad i = 1, 2, ..., m, \quad j = 1, 2, ..., f_i \\ a + u_\Delta - u_i & \text{for } i = j \end{cases}$$

Observation 9

1. $a = 3u_\Delta + 2$
2. $a \equiv 1 \pmod 4$

¿From now on when we mention a vertex, we mean its label under L. Let $C = \{c_{ij} | i = 1, 2, ..., m, \quad j = 1, 2, ..., f_i\}$ and let $T = \{t_i | i = 1, 2, ..., \Delta\}$..
Figure 2 shows an exclusive sum labeling of a shrub using this schema.
Next we will show that the mapping L is an exclusive sum labeling of any shrub-type graph.

Lemma 4 Let Sh be a shrub with maximum vertex degree Δ. Let $Sh' = Sh \cup \overline{K_\Delta}$. If L is a mapping as defined by Labeling 2, then $L(u) \neq L(v)$ for any two different vertices u and v in $V(Sh')$.

Proof.

1. By definition of L, it is clear that $r \neq b_i$ for all $i = 1, 2, ..., m$.
2. We claim that $r \neq c_{ij}$. Suppose there exist i and j such that $r = c_{ij}$. Then if $i \neq j$ we have $a = a + u_j - u_i$ which gives $u_j = u_i$ or if $i = j$ we have $a = a + u_\Delta - u_i$ which gives $u_\Delta = u_i$. These are impossible by the definition of the sequence U.
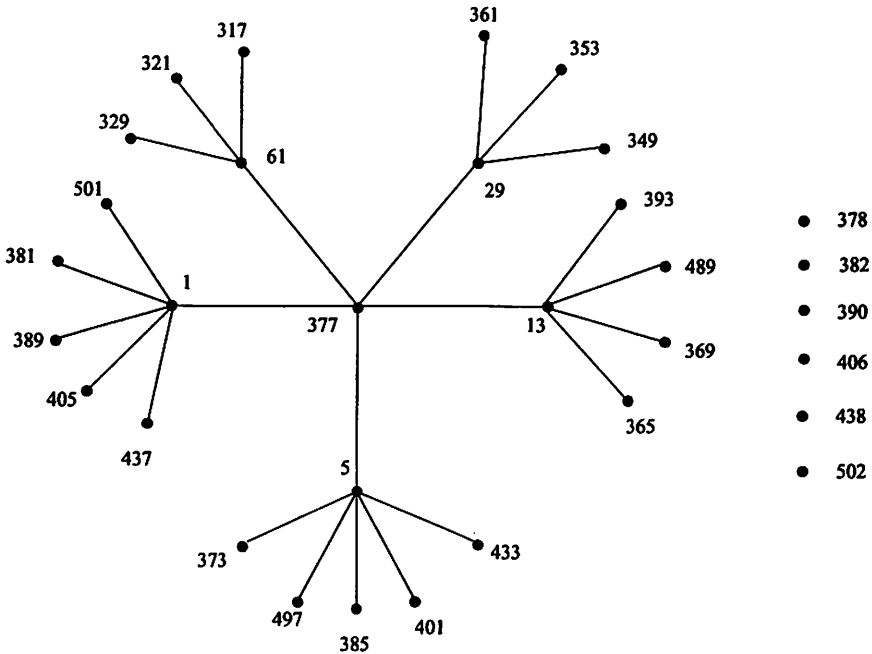
116

Fig. 2. An optimum exclusive sum labeling of a shrub.

3. Obviously $r \neq t_i$ for all $i = 1, 2, ..., \Delta$.
4. We claim that there will be no i, j, k such that $b_i = c_{jk}$. Suppose to the contrary that for some i, j, k, $b_i = c_{jk}$. Then $u_i = a + u_k - u_j$. This gives $a = u_i + u_j - u_k$. But this contradicts $a > 3u_\Delta$.
5. Also, by the definition of a, there will be no i and j such that $b_i = t_j$.
6. Let $i \neq k$. We claim that $c_{ij} \neq c_{kl}$. Otherwise, $a + u_j - u_i = a + u_l - u_k$ which gives $u_i - u_j = u_k - u_l$, which is in conflict with Observation 8.
7. Finally, suppose $c_{ij} = t_k$ then $a + u_j - u_i = a + u_k$. This gives $u_j = u_i + u_k$ which is impossible since all the terms of U are congruent to 1 (mod 4).

Lemma 5 *Let Sh be a shrub with maximum vertex degree Δ. Let $Sh' = Sh \cup \overline{K_\Delta}$. If L is a mapping as defined by Labeling 2 then L is a sum labeling of Sh'.*

Proof. Lemma 4 shows that L is an injection from $V(Sh')$ to a set of positive integers. It is clear that the sum of any two adjacent vertices will be one of the isolates.

We need to show that the sum of any two non adjacent vertices is not in $V(Sh')$. Since we are using the numbers congruent to 1 (mod 4) for the

117

labels of $V(Sh)$ and numbers congruent to 2 (mod 4) for the isolates, then all we need to do is to show that the sum of any two non adjacent vertices of $V(Sh)$ is not in T.

1. We claim that $r + c_{ij} \notin T_1$ Otherwise suppose that $r + c_{ij} = t_k$ then $a + (a + u_j - u_i) = a + u_k$. This gives $a = u_k + u_i - u_j$. This is in conflict with our choice of a.
2. The sum of two vertices in B is not in T. Suppose to the contrary that $b_i + b_j = t_k$. Again, we get $a = u_i + u_j - u_k$ which conflicts with the choice of a.
3. If $i \neq j$ then $b_i + c_{jk} \notin T$. Suppose $b_i + c_{jk} = t_l$ for some $i \neq j$. If $j \neq k$ we have $b_i + (a + b_k - b_j) = a + b_l$ or if $j = k$, we have $b_i + (a + u_\Delta - u_k) = a + u_l$. These give $b_j - b_i = b_k - b_l$ or $u_\Delta - u_k = u_l - u_i$, which is impossible by Observation 8.
4. The sum of two vertices in C is not in T. Observe that $\min(C) = t_1 - b_m = a + u_1 - u_m$. Since $a = 3u_\Delta + 2$ then

$$
\begin{aligned}
2 \times \min C &= 2a + 2u_1 - 2u_m \\
&\geq 2a + 2u_1 - 2u_\Delta \\
&= 4u_\Delta + 4 + 2u_1 \\
&> a + u_\Delta \qquad \text{we have} \\
2 \times \min(C) &> t_\Delta = \max(T)
\end{aligned}
$$

This gives, for any i, j, k, l, that $c_{ij} + c_{kl} \geq 2 \min (C) > t_\Delta = \max (T)$ and hence, $c_{ij} + c_{kl} \notin T$.

Theorem 2 *Let Sh be a shrub with vertex maximum degree Δ, then $\epsilon(Sh) = \Delta$*

Proof. Let Sh be a shrub with maximum vertex degree Δ. Let $Sh' = Sh \cup \overline{K_\Delta}$. If L is a mapping as defined in Labeling 2 then Lemma 4 and Lemma 5 imply that L is a sum labeling of Sh using Δ isolates. Obviously, Sh itself contains no working vertices. Therefore L is an exclusive sum labeling of Sh, using Δ isolates. Combined with Observation 1, it can be concluded that $\epsilon(Sh) = \Delta$.

3.1 Graphs that are both Caterpillars and Shrubs

The methods developed for labeling caterpillars and shrubs can be applied to star and double star since they are both caterpillars and shrubs. We then have

Corollary 1 *Let S_n be a star, then $\epsilon(S_n) = n$.*

Corollary 2 *Let $S_{m,n}$ be a double star then $\epsilon(S_{m,n}) = mn$.*

As examples Figures 3 and 4 show optimum exclusive sum labelings of a star and a double star using the methods developed for caterpillars.
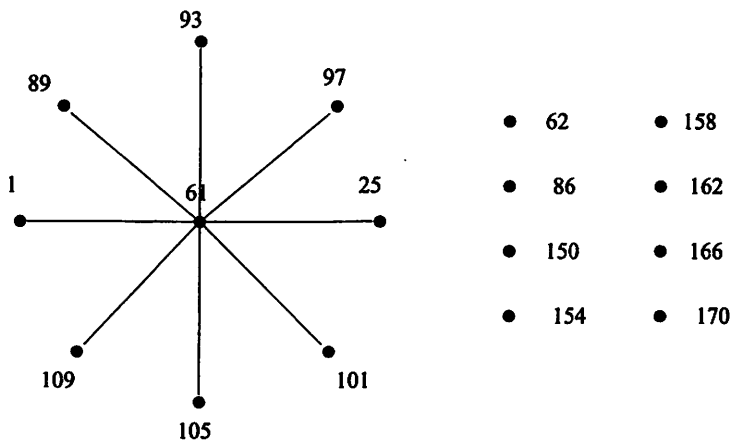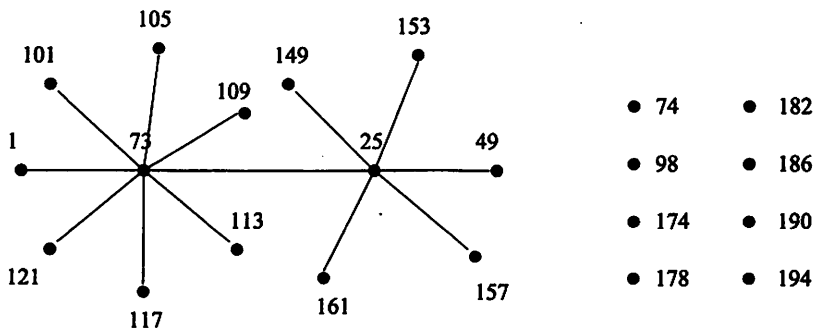
Fig. 3. Exclusive sum labeling of a star (S_8).

Fig. 4. Exclusive sum labeling of a double star.

4 Trees which are not Δ-optimum summable graphs

While there are many trees which are Δ-optimum summable graphs, in general trees are not Δ-optimum summable graph. For example, let T be

119

a tree as depicted in Figure 5. We will show that this tree can not be a Δ-optimum summable graph. Suppose T is a Δ-optimum summable graph. We
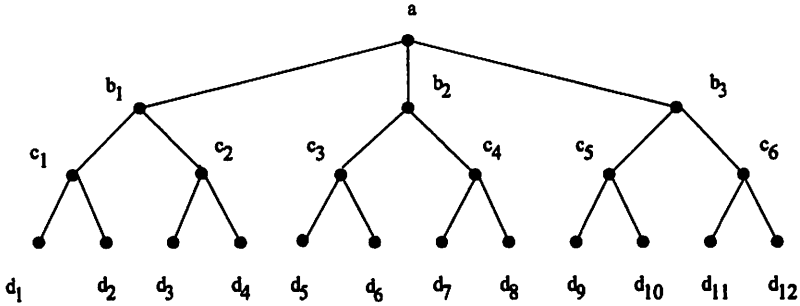

Fig. 5. A tree with $\Delta = 3$ which is not a 3-optimum summable graph.

need 3 isolates, labeled by $a+b_1, a+b_2, a+b_3$ respectively. As a consequence, $c_j = a + b_m - b_l$, where $j = 1, 2, ..., 6$, with $l, m = 1, 2, 3 \; l \neq m$, and

$$\begin{aligned}
d_i &= (a + b_k) - c_j \\
&= (a + b_k) - (a + b_m - b_l) \\
&= b_k + b_l - b_m
\end{aligned}$$

There are only 9 integers available for labeling $d_i, i = 1, 2, ..., 12$, which is impossible. We conclude that T is not a Δ-optimum summable graph. We then have the following.

Open Problem 1 *Classify trees that are Δ-optimal summable.*

Open Problem 2 *Find general exclusive sum graph labeling for trees.*

Open Problem 3 *Find a general upper bound for the exclusive sum number for trees.*

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