Star-Wheel Ramsey Numbers

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Abstract. For given graphs G and H, the Ramsey number $R(G, H)$ is the smallest natural number n such that for every graph F of order n: either F contains G or the complement of F contains H. This paper investigates the Ramsey number $R(S_n, W_m)$ of stars versus wheels. We show that if m is odd, $n \geq 3$ and $m \leq 2n-1$, then $R(S_n, W_m) = 3n-2$. Furthermore, if n is odd and $n \geq 5$ then $R(S_n, W_m) = 3n - \mu$, where $\mu = 4$ if $m = 2n-4$ and $\mu = 6$ if $m = 2n-8$ or $m = 2n-6$.

Keywords : Ramsey numbers, stars, wheels

1 Introduction

For given graphs G and H, the *Ramsey number* $R(G, H)$ is defined as the smallest positive integer n such that for any graph F of order n, either F contains G or $\overline{F}$ contains H, where $\overline{F}$ is the complement of F. Chvátal and Harary [4] established a useful lower bound for finding the exact Ramsey numbers $R(G, H)$, namely $R(G, H) \geq (\chi(G)-1)(C(H)-1)+1$, where $\chi(G)$ is the chromatic number of G and $C(H)$ is the number of vertices of the largest component of H. Since then the Ramsey numbers $R(G, H)$ for many combinations of graphs G and H have been extensively studied by various authours, see a nice survey paper [7]. In particular, the Ramsey numbers for combinations involving stars have also been investigated. Let S_n be a star of n vertices and W_m a wheel with m spokes. Surahmat et al. [8] proved that $R(S_n, W_4) = 2n-1$ for $n \geq 3$ odd, otherwise $R(S_n, W_4) = 2n+1$. They also showed $R(S_n, W_5) = 3n-2$ for $n \geq 3$. Furthermore, it has been shown that if m is odd, $m \geq 5$ and $n \geq 2m-4$, then $R(S_n, W_m) = 3n-2$. This result is strengthened by Chen et al. [3] by showing that this Ramsey number remains the same, even if m (≥ 5) is odd and $n \geq m-1 \geq 2$.

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Additionally, Zhang et al. [10] established $R(S_n, W_6) = 2n+1$ for $n \geq 3$, and $R(S_n, W_8) = 2n + \mu$ for $5 \leq n \leq 10$, where $\mu = 1$ if $n \equiv 1 \pmod 2$ and $\mu = 2$ if $n \equiv 0 \pmod 2$. Recently, Hasmawati showed that for $m \geq 2n-2$ and $n \geq 4$, $R(S_n, W_m) = m+n-2$ if n is odd and m is even, otherwise $R(S_n, W_m) = m+n-1$ [6].

In this paper, we determine the Ramsey numbers $R(S_n, W_m)$ for open cases of n and m. The main results of this paper are the following.

Theorem 1. *If m is odd, $n \geq 3$ and $m \leq 2n-1$, then $R(S_n, W_m) = 3n-2$.*

Theorem 2. *If n is odd and $n \geq 5$ then $R(S_n, W_m) = 3n - \mu$, where $\mu = 4$ if $m = 2n-4$ and $\mu = 6$ if $m = 2n-8$ or $m = 2n-6$.*

Before proving the theorems let us present some notations used in this note. Let $G(V, E)$ be a graph. Let $c(G)$ be the *circumference* of G, that is, the length of a longest cycle, and $g(G)$ be the *girth*, that is, the length of a shortest cycle. For any vertex $v \in V(G)$, the *neighborhood* $N(v)$ is the set of vertices adjacent to v in G. Futhermore we define $N[v] = N(v) \cup \{v\}$. The degree of a vertex v in G is denoted by $d_G(v)$. The minimum (maximum) degree in G is denoted by $\delta(G)$ ($\Delta(G)$). For $S \subseteq V(G)$, $G[S]$ represents the subgraph induced by S in G. A graph on n vertices is *pancyclic* if it contains cycles of every length l, $3 \leq l \leq n$. A graph is *weakly pancyclic* if it contains cycles of length from the girth to the circumference.

2 Some Lemmas

The following lemmas will be useful in proving our results.

Lemma 1. *(Bondy [1]). Let G be a graph of order n. If $\delta(G) \geq \frac{n}{2}$, then either G is pancyclic or n is even and $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Lemma 2. *(Brandt et al. [2]). Every non-bipartite graph G with $\delta(G) \geq \frac{n+2}{3}$ is weakly pancyclic and has girth 3 or 4.*

Lemma 3. *(Dirac [5]). Let G be a 2-connected graph of order $n \geq 3$ with $\delta(G) = \delta$. Then $c(G) \geq \min\{2\delta, n\}$.*

3 The Proofs of Theorems

Proof of Theorem 1. Let F be a graph of order $3n-2$. Suppose F contains no S_n. Let $x \in V(F)$. Since $F \not\supseteq S_n$, then $d_F(x) \leq n-2$. Let $A = V(F) \backslash N[x]$, and $T = F[A]$. So, $|T| \geq 2n-1$. Since for each $v \in T$,

$d_T(v) \leq n-2$ then $d_{\overline{T}}(v) \geq |T| - (n-1) \geq \frac{|\overline{T}|}{2}$. By Lemma 1, $\overline{T}$ contains a cycle C_m, where $3 \leq m \leq 2n-1 \leq |\overline{T}|$. With the center x, we obtain a wheel W_m in $\overline{F}$ for all odd m and $3 \leq m \leq 2n-1$. Hence, $R(S_n, W_m) \leq 3n-2$. On the other hand, Since $3K_{n-1}$ does not contains S_n and its complement does not contains W_m, for odd m, then $R(S_n, W_m) \geq 3n-2$. Hence, $R(S_n, W_m) = 3n-2$. □

Proof of Theorem 2. Let n be odd, $n \geq 5$ and $m = 2n-4$. Since $K_{n-1} \cup K_{n-2,n-2}$ has no S_n and its complement contains no W_m, for $m = 2n-4$, then $R(S_n, W_m) \geq 3n-4$. On the other hand, now, let F be a graph of order $3n-4$. Suppose F contains no S_n, and so $d_F(v) \leq n-2$, $\forall v \in F$. Since n is odd, not all vertices of F has degree of $n-2$ (odd). Let $x_0 \in F$ with $d_F(x_0) \leq n-3$. Let $A = V(F)\backslash N[x_0]$, and $T = F[A]$. Since for each $v \in T$, $d_T(v) \leq n-2$ and $|T| \geq 2n-2$, then $d_{\overline{T}}(v) \geq |T| - (n-1) \geq \frac{|\overline{T}|}{2}$. This implies that $\overline{T}$ contains a C_{2n-4} (by Lemma 1). Hence, $\overline{F}$ contains a W_{2n-4}, with the center x_0. Therefore, $R(S_n, W_m) = 3n-4$ for this case.

Now, consider the case of odd n and $m = 2n-8$ or $m = 2n-6$. Graph $K_{n-1} \cup [(\frac{n-3}{2})K_2 + (\frac{n-3}{2})K_2]$ guaranties $R(S_n, W_m) \geq 3n-6$. Now, let F be a graph of order $3n-6$ and suppose $F \not\supseteq S_n$. Hence, for each $x \in F, d_F(x) \leq n-2$. Suppose to the contrary there exists $x_0 \in F, d_F(x_0) \leq n-5$. If $A = V(F)\backslash N[x_0]$ and $T = F[A]$ then $|T| \geq 2n-2$ and $\delta(\overline{T}) \geq |T| - (n-1) \geq \frac{|\overline{T}|}{2}$. By Lemma 1, $\overline{T}$ contains a C_m where $m = 2n-8$ or $m = 2n-6$, and so $\overline{F}$ contains W_m with the center x_0. Therefore, for each $v \in F, n-4 \leq d_F(v) \leq n-2$. Since the order of F is odd, then not all its vertices has odd degree. Hence, there exists $v_0 \in F$ with $d_F(v_0) = n-3$. Let $A = V(F)\backslash N[v_0]$, $T = F[A]$, and so $|T| = 2n-4$. Since for each $v \in T$, $n-4 \leq d_T(v) \leq n-2$, then $2n-5 \geq d_{\overline{T}}(v) \geq n-3$, which implies $\delta(\overline{T}) \geq \frac{|\overline{T}|+2}{3}$, if $n \geq 7$. Now, consider the following two cases.

Case 1. $\overline{T}$ is bipartite.
Let V_1, V_2 be the partite sets of T. Since $2n-5 \geq d_{\overline{T}}(v) \geq n-3$, then $|V_1| = n-3$ and $|V_2| = n-1$, or $|V_1| = n-2$ and $|V_2| = n-2$.

If $|V_1| = n-3$ and $|V_2| = n-1$, then $\overline{T}$ is isomorphic to $=K_{n-1,n-3}$. Hence, $\overline{T}$ contains a C_m, where $m = 2n-8$ or $m = 2n-6$. This cycle together with v_0 form a W_m in $\overline{F}$.

Let $|V_1| = n-2$ and $|V_2| = n-2$. Then, $\overline{T}$ is not isomorphic to $K_{n-2,n-2}$ since otherwise $\overline{F} \supseteq W_m$, where $m = 2n-8$ or $m = 2n-6$. Since $\delta(\overline{T}) \geq 3$, then we can order its vertices so that $v_1, v_2, \cdots, v_r$ $(u_1, u_2, \cdots, u_r)$ are the vertices of V_1 (V_2) that have degree $n-3$ each, where $1 \leq r \leq n-2$. But, now for $j = 3, 4, \cdots, n-2$ we have a cycle $C_{2j} = (u_1, v_j, u_2, v_1, u_3, v_2, \cdots, u_{j-1}, v_{j-2}, u_j, v_{j-1}, u_1)$ in $\overline{T}$ and it implies that $W_m \subseteq \overline{F}$.

Case 2. $\overline{T}$ is nonbipartite.
Let $\kappa(\overline{T}) = 0$. Then, $\overline{T}$ is disconnected. The constraint of the degree of each vertex in $\overline{T}$ forces $\overline{T}$ to be isomorphic to $2K_{n-2}$. Since $\Delta(F) = n-2$, then no vertices of T are adjacent to any vertex of $N[x_0]$ in F. This means that every vertex in $N[x_0]$ is adjacent to all vertices of $\overline{T}$ in $\overline{F}$. Therefore, $N[x_0]$ together with the vertices of one component K_{n-2} of $\overline{T}$ form a wheel W_m with any vertex of K_{n-2} as the center, where $m = 2n-8$ or $m = 2n-6$.

Let $\kappa(\overline{T}) = 1$. Let G_1 and G_2 be the components of $\overline{T}\backslash\{u\}$, for a cut vertex $u \in \overline{T}$. Since $2n-5 \geq d_{\overline{T}}(v) \geq n-3$, then $|G_1| = n-2$ and G_2 must be isomorphic to K_{n-3}, where vertex u is adjacent to all vertices of G_2, and adjacent to at least one vertex of G_1.

Let $B = \{x \in G_1 | (x,u) \in E(\overline{T})\}$. Since $\delta(\overline{T}) \geq n-3$ and $|G_1| = n-2$, each vertex $x \in G_1\backslash B$ must be adjacent to all other vertices of G_1 in $\overline{T}$. As

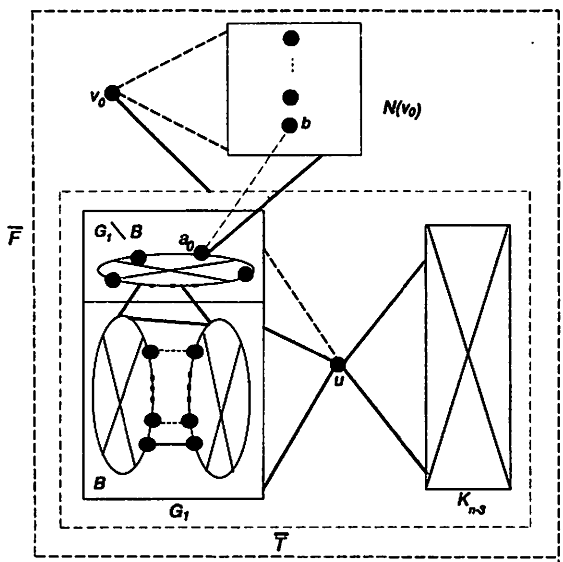

Fig. 1. The proof of Theorem 2 for $\kappa(\overline{T}) = 1$.

a consequence, if there exist two vertices x, y of G_1 that are not adjacent in $\overline{T}$, then x and y are both in B. Furthermore, since $\delta(\overline{T}) \geq n-3$ then for each $x \in B$ there is at most one vertex of B which is not adjacent to x. Therefore, all edges in $\overline{G_1}$ (if they exist) will form a matching. This matching does not cover all vertices of G_1 since $|G_1|$ is odd. Thus, there exists a vertex $a_0 \in G_1$ that is adjacent to all vertices of G_1 in $\overline{T}$. Since each vertex x of G_1 is adjacent to all vertices of G_2 in F and $\Delta(F) \leq n-2$, then vertex x is adjacent to at most one vertex of $N(v_0)$ in F, see Fig.1. Therefore, there exists a wheel W_m in $\overline{F}$ with the center a_0 and the rim consists of v_0, the vertices in $N(v_0)\backslash\{b\}$ where $(a_0, b) \in E(F)$ and other vertices of G_1, for $m = 2n-8$ or $m = 2n-6$.

Let $\kappa(\overline{T}) \geq 2$. Then $\overline{T}$ is 2-connected. By Lemma 3, $c(\overline{T}) \geq \min\{2(n-3), 2n-4\}$. Not that $\delta(\overline{T}) \geq \frac{|\overline{T}|+2}{3}$, and $\overline{T}$ is nonbipartite. By Lemma 2, $\overline{T}$ is *weakly pancyclic*. Thus, $\overline{T}$ contains all cycles C_m, $g(\overline{T}) \leq m \leq 2n-6 \leq c(\overline{T})$, where $g(\overline{T})$ is 3 or 4. Hence, $\overline{F}$ contains W_m, with the center v_0 and for $m = 2n-8$ or $m = 2n-6$. □

4 Open Problems

As a final remark, let us present the following open problems to work on.

Problem 1. Find the Ramsey number $R(S_n, W_m)$ for even $n \geq 4$ and all even m, $n+1 \leq m \leq 2n-4$.

Problem 2. Find the Ramsey number $R(S_n, W_m)$ for odd $n \geq 5$ and even m , $n+1 \leq m \leq 2n-10$.

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