Total Vertex Irregular Labeling of Complete Bipartite Graphs

Kristiana Wijaya[1], Slamin[2], Surahmat[3], and Stanislav Jendrol'[4]

[1] Department of Mathematics, Universitas Jember,
Jalan Kalimantan Jember, Indonesia,
kristiana_wijaya@yahoo.com

[2] Department of Mathematics Education, Universitas Jember,
Jalan Kalimantan Jember, Indonesia,
slamin@unej.ac.id

[3] Department of Mathematics Education, Universitas Islam Malang,
Jalan M.T. Haryono 193 Malang, Indonesia,
caksurahmat@yahoo.com

[4] Institute of Mathematics, P.J.Šafárik University, Jesenná 5
041 54 Košice, Slovak Republic
jendrol@kosice.upsj.sk

Abstract. A *total vertex irregular labeling* of a graph G with v vertices and e edges is an assignment of integer labels to both vertices and edges so that the weights calculated at vertices are distinct. The *total vertex irregularity strength* of G, denoted by $tvs(G)$, is the minimum value of the largest label over all such irregular assignments. In this paper, we consider the total vertex irregular labeling of complete bipartite graphs $K_{m,n}$ and prove that

$$tvs(K_{m,n}) \geq \max\left\{\left\lceil\frac{m+n}{m+1}\right\rceil, \left\lceil\frac{2m+n-1}{n}\right\rceil\right\} \text{ if } (m,n) \neq (2,2).$$

1 Introduction

In this paper all graph are finite, simple, undirected, and connected. The graph G has v vertices and e edges. A *total vertex irregular labeling* on a graph G with v vertices and e edges is an assignment of integer labels to both vertices and edges so that the weights calculated at vertices are distinct. The *weight* of a vertex v in G is defined as the sum of the label of v and the labels of all the edges incident with v, that is,

$$wt(v) = \lambda(v) + \sum_{uv \in E} \lambda(uv)$$

The notion of a total vertex irregular labeling was introduced by Bača, et al.[1]. The *total vertex irregularity strength* of G, denoted by $tvs(G)$, is the minimum value of the largest label over all such irregular assignments.

JCMCC 55 (2005), pp. 129-136

Bača et al.[1] proved that for tree T with n pendant vertices and no vertices of degree 2, $\lceil \frac{n+1}{2} \rceil \leq tvs(T) \leq n$.

In the same paper, Bača et al.[1] gave the lower bound and upper bound on total vertex irregular strength of any graph with minimum degree δ and maximum degree Δ as described in the following theorem.

Theorem 1. [1] *For a graph G with minimum degree δ and maximum degree Δ, then* $\left\lceil \frac{|V|+\delta}{\Delta+1} \right\rceil \leq tvs(G) \leq |V| + \Delta - 2\delta - 1$. ■

So if G is r-regular then $\left\lceil \frac{|V|+r}{r+1} \right\rceil \leq tvs(G) \leq |V| - r - 1$. Hence the total vertex irregularity strength of cycles C_n, $tvs(C_n) = \lceil \frac{n+2}{3} \rceil$. Because $C_4 \simeq K_{2,2}$, we have $tvs(K_{2,2}) = 2$. And also if G is regular hamiltonian graph then $tvs(G) \leq \left\lceil \frac{|V|+2}{3} \right\rceil$. Bača et al.[1] also proved that $tvs(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$ and $tvs(K_n) = 2$ for all $n \geq 2$ and asked to find a total vertex irregularity strength of $K_{m,n}$.

In this paper we give the total vertex irregularity strength of the complete bipartite graph $K_{m,n} (m \leq n)$. In what follows, the graph $K_{m,n}$ has vertex set $V = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, ..., u_m\}$ and $V_2 = \{v_1, v_2, ..., v_n\}$, and the edge set $E = \{e_{ij} = (u_i, v_j) | \ i = 1, 2, \cdots, m \text{ and } j = 1, 2, \cdots n\}$.

2 Main Result

In this section, we first present the total vertex irregular strength of special case of complete bipartite graph that is $tvs(K_{2,n})$, $tvs(K_{n,n})$, $tvs(K_{n,n+1})$, $tvs(K_{n,n+2})$ and $tvs(K_{n,an})$. Then we give the lower bound on the total vertex irregular strength of $K_{m,n}$ for $m \leq n$.

Theorem 2. *The total vertex irregularity strength of $K_{2,n}$, $tvs(K_{2,n}) = \lceil \frac{n+2}{3} \rceil$, for $n > 3$.*

Proof. There are n vertices of degree 2. The weight of each vertex is the sum of three numbers, then the smallest weight must be 3 and the largest weight at least $n + 2$. Then $tvs(K_{2,n}) \geq \lceil \frac{n+2}{3} \rceil$.

To show that $tvs(K_{2,n}) \leq \lceil \frac{n+2}{3} \rceil$, let set of bipartition has two vertices x_1, x_2 and the second partition has vertices $v_1, v_2, \cdots, v_n$. Let the labeling λ is as follows:

$$\lambda(v_i) = i + 2 - \lceil \tfrac{i+2}{3} \rceil - \lfloor \tfrac{i+2}{3} \rfloor,$$
$$\lambda(x_1 v_i) = \lceil \tfrac{i+2}{3} \rceil,$$
$$\lambda(x_2 v_i) = \lfloor \tfrac{i+2}{3} \rfloor,$$
$$\lambda(x_1) = \lceil \tfrac{n+2}{3} \rceil,$$
$$\lambda(x_2) = \lfloor \tfrac{n+2}{3} \rfloor.$$

Thus the weights are as follows:

$$wt(v_i) = i + 2,$$
$$wt(x_1) = \sum_{i=1}^{n} \lceil \tfrac{i+2}{3} \rceil + \lceil \tfrac{n+2}{3} \rceil,$$
$$wt(x_2) = \sum_{i=1}^{n} \lfloor \tfrac{i+2}{3} \rfloor + \lfloor \tfrac{n+2}{3} \rfloor.$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(K_{2,n}) = \lceil \frac{n+2}{3} \rceil$ for $n > 3$. ∎

Theorem 3. *The total vertex irregularity strength of $K_{n,n}$, $tvs(K_{n,n}) = 3$, for $n \geq 3$.*

Proof. $K_{n,n}$ is a regular graph of degree n. Then the weight of each vertex is the sum of $n+1$ numbers. The smallest weight must be $n+1$. Since there are $2n$ vertices in $K_{n,n}$, then the largest weight is at least $3n$. Therefore $tvs(K_{n,n}) \geq \left\lceil \frac{3n}{n+1} \right\rceil = \left\lceil 3 - \frac{3}{n+1} \right\rceil = 3$ for $n \geq 3$.

To show that $tvs(K_{n,n}) \leq 3$, we label the vertices and edges of the graph $K_{n,n}$ in the following way.

$$\lambda(u_i) = \begin{cases} 1 & \text{for} \quad i = 1, \\ 2 & \text{for} \quad i = 2, 3, \cdots, n; \end{cases}$$
$$\lambda(v_j) = \begin{cases} 2 & \text{for} \quad j = 1, \\ 3 & \text{for} \quad j = 2, 3, \cdots, n; \end{cases}$$
$$\lambda(e_{ij}) = \begin{cases} 1 & \text{for} \quad i + j \leq n + 1, \\ 2 & \text{for} \quad i + j = n + 2, \\ 3 & \text{for} \quad i + j \geq n + 3. \end{cases}$$

Thus, the weights of vertices u_i and v_j of $K_{n,n}$ (respectively) are:
$$wt(u_i) = n - 1 + 2i \quad \text{for} \quad i = 1, 2, \cdots, n,$$
$$wt(v_j) = n + 2j \quad \text{for} \quad j = 1, 2, \cdots, n.$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(K_{n,n}) = 3$ for $n \geq 3$. ∎

Theorem 4. *The total vertex irregularity strength of $K_{n,n+1}$, $tvs(K_{n,n+1}) = 3$ for $n \geq 3$.*

Proof. According to the Theorem 1, $tvs(K_{n,n+1}) \geq \left\lceil \frac{3n+1}{n+2} \right\rceil = \left\lceil 3 - \frac{5}{n+2} \right\rceil = 3$ for $n \geq 4$. For $n = 3$, we will proved that 2 is not irregularity strength of $K_{3,4}$. The smallest weight of vertices $K_{3,4}$ is 4, i.e. the sum of four numbers. As $K_{3,4}$ has 7 vertices, the largest weight of vertices $K_{3,4}$ is at least 10. If 10 is the sum of four numbers, then at least one edge has label at least 3. If 10 is the sum of five numbers, then all edges has label 2. This is not true because there is one edge has label 1. So $tvs(K_{3,4}) \geq 3$.

To show that $tvs(K_{n,n+1}) \leq 3$, we label the vertices and edges of $K_{n,n+1}$ as described in the following formula:

$$\lambda(u_i) = 2 \quad \text{for} \quad i = 1, 2, \cdots, n;$$
$$\lambda(v_j) = \begin{cases} 1 \text{ for } & j = 1, \\ 2 \text{ for } & j = 2, 3, \cdots, n+1; \end{cases}$$
$$\lambda(e_{ij}) = \begin{cases} 1 \text{ for } & i + j \leq n + 1, \\ 2 \text{ for } & i + j = n + 2, \\ 3 \text{ for } & i + j \geq n + 3. \end{cases}$$

Thus, the vertex-weights of $K_{n,n+1}$, i.e.
$$wt(u_i) = n + 2 + 2i \quad \text{for} \quad i = 1, 2, \cdots, n,$$
$$wt(v_j) = n - 1 + 2j \quad \text{for} \quad j = 1, 2, \cdots, n+1.$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(K_{n,n+1}) = 3$ for $n \geq 3$. ■

Theorem 5. *The total vertex irregularity strength of* $K_{n,n+2}$, $tvs(K_{n,n+2}) = 3$ *for* $n \geq 4$.

Proof. According to the Theorem 1, $tvs(K_{n,n+2}) \geq \left\lceil \frac{3n+2}{n+3} \right\rceil = \left\lceil 3 - \frac{7}{n+3} \right\rceil = 3$ for $n \geq 5$. For $n = 4$, we will proved that 2 is not irregularity strength of $K_{4,6}$. The smallest weight of vertices $K_{4,6}$ is 5, i.e. the sum of five numbers. As $K_{4,6}$ has 10 vertices, the largest weight of vertices $K_{4,6}$ is at least 14. If 14 is the sum of five numbers, then at least one edge has label at least 3. If 14 is the sum of seven numbers, then all edges has label 2. This is not true because there is one edge has label 1. So $tvs(K_{4,6}) \geq 3$.

To show that $tvs(K_{n,n+2}) \leq 3$, we label the vertices and edges of $K_{n,n+2}$ in the following way.
$$\lambda(u_i) = \begin{cases} 1 \text{ for } & i = 1, n, \\ 2 \text{ for } & i = 2, 3, \cdots, n-1; \end{cases}$$
$$\lambda(v_j) = \begin{cases} 1 \text{ for } & j = 1, 2, \cdots n+1 \\ 2 \text{ for } & j = n+2; \end{cases}$$
$$\lambda(e_{ij}) = \begin{cases} 1 \text{ for } & i + j \leq n + 1, \\ 2 \text{ for } & n + 2 \leq i + j \leq n + 4, \\ 3 \text{ for } & i + j \geq n + 5. \end{cases}$$

Thus, the vertex-weights of $K_{n,n+2}$, are as follows
$$wt(u_i) = \begin{cases} n + 3 + 2i & \text{for } \; i = 1, 2, \cdots n-1, \\ 3n + 2 & \text{for } \; i = n; \end{cases}$$
$$wt(v_j) = \begin{cases} n + j & \text{for } \; j = 1, 2, 3, 4, \\ n - 4 + 2i & \text{for } \; j = 5, 6, \cdots n+2. \end{cases}$$

It is easy to see that the labeling is vertex irregular. Thus $tvs(K_{n,n+2}) = 3$ for $n \geq 4$. ■

Theorem 6. *For every* $a > 1$ *and all* n, *the total vertex irregularity strength of* $K_{n,an}$, $tvs(K_{n,an}) = \left\lceil \frac{n(a+1)}{n+1} \right\rceil$.

Proof. We first show that $tvs(K_{n,an}) \geq \left\lceil \frac{n(a+1)}{n+1} \right\rceil$. There are an vertices of degree n. The weight of each vertex is the sum of $n+1$ numbers, the smallest weight must be $n+1$ and the largest weight at least $n + an = n(a+1)$. Then $tvs(K_{n,an}) \geq \left\lceil \frac{n(a+1)}{n+1} \right\rceil$.

To show that $tvs(K_{n,an}) \leq \left\lceil \frac{n(a+1)}{n+1} \right\rceil$, we label the vertices and edges of $K_{n,an}$ as described in the following formula:

$\lambda(u_i) = 1$ for $i = 1, 2, \cdots, n$;

$\lambda(v_j) = k$ with $k = 1, 2, \cdots, a$ for $n(k-1) + k \leq j \leq k(n+1)$.

$$\lambda(e_{ij}) = \begin{cases} k & \text{with } k = 1,2,\cdots,a, \\ & \text{for } n(k-1)+k \leq i+j \leq k(n+1), \text{ if } a \geq n, \\ a+1 & \text{for } a(n+1)+1 \leq i+j \leq n(a+1), \text{ if } a \leq n-1. \end{cases}$$

Thus, the weights of vertices u_i and v_j of $K_{n,an}$ are:

$$wt(u_i) = \begin{cases} 1 + \frac{1}{2}a(n-1)(a+1) + a + (a-1)i & \text{for } i \leq a, \\ 1 + \frac{1}{2}a(n-1)(a+1) + i + (a-1)i & \text{for } i \geq a+1, \end{cases}$$

$wt(v_j) = n + j$ for $j = 1, 2, \cdots, an$.

¿From this formula we immediately can see that $wt(u_i) \neq wt(u_j)$ if $i \neq j$ and $wt(u_i) \neq wt(v_j)$ for all i and j.

For $n = 1$ we have $tvs(K_{1,a}) = \lceil \frac{a+1}{2} \rceil$ (see [1]). For $n > 1$ we have $|E(K_{n,an})| \geq |V(K_{n,an})|$. So that $tvs(K_{n,an})$ can be determined by the largest label of edge labels. Now, consider three cases for a and n.

- **Case 1:** For $n = a$, then we obtain $a(n+1) = n(a+1)$. So the minimum value of the largest label over all edge labels $\lambda(e_{ij})$ is at most $a = \left\lceil \frac{n(a+1)}{n+1} \right\rceil$.
- **Case 2:** For $n < a$ write $n = a - r$, where $1 \leq r < a$. We obtain

$$a(n+1) = a(a-r+1) = a^2 + a - ar$$

and

$$n(a+1) = (a-r)(a+1) = a^2 + a - ar - r.$$

So that $a(n+1) > n(a+1)$. The minimum value of the largest label over all edge labels $\lambda(e_{ij})$ is at most k, with $k = 2, 3, \cdots, a$.
Furthermore there are 2 cases for r, i.e.

1. If $1 \leq r \leq n$, then $0 < \frac{r}{n+1} < 1$. So that $\left\lceil a - \frac{r}{n+1} \right\rceil = a$.
2. If $n < r < a$, then $1 \leq \frac{r}{n+1} < a - 1$. So that $2 \leq \left\lceil a - \frac{r}{n+1} \right\rceil \leq a - 1$.

So, if $n < a$ then the minimum value of the largest label over all edge labels is at most $2 \leq \left\lceil a - \frac{r}{n+1} \right\rceil = \left\lceil \frac{n(a+1)}{n+1} \right\rceil \leq a$.

- **Case 3:** For $n > a$ write $a = n - r$, where $1 \leq r < n$. We obtain

$$a(n+1) = (n-r)(n+1) = n^2 + n - nr - r$$

and

$$n(a+1) = n(n-r+1) = n^2 + n - nr.$$

So that $a(n+1) < n(a+1)$. Because of $0 < \frac{r}{n+1} < 1$, then the minimum value of the largest label over all edge labels $\lambda(e_{ij})$ is at most $a+1$, i.e.

$$a+1 = \left\lceil a + \frac{r}{n+1} \right\rceil = \left\lceil \frac{an+a+r}{n+1} \right\rceil$$
$$= \left\lceil \frac{an+n}{n+1} \right\rceil = \left\lceil \frac{n(a+1)}{n+1} \right\rceil.$$

So, the total vertex irregularity strength of $K_{n,an}$ is the minimum value of the largest label over all edge labels $\lambda(e_{ij})$, $tvs(K_{n,an}) \leq \left\lceil \frac{n(a+1)}{n+1} \right\rceil$. Therefore, for every $a > 1$ and all n, $tvs(K_{n,an}) = \left\lceil \frac{n(a+1)}{n+1} \right\rceil$ ■

For example, the labeling of $K_{4,12}$ is showed in following table.

$K_{4,12}$		v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	$wt(u_i)$
		1	1	1	1	1	2	2	2	2	2	3	3	
u_1	1	1	1	1	1	2	2	2	2	2	3	3	3	24
u_2	1	1	1	1	2	2	2	2	2	3	3	3	3	26
u_3	1	1	1	2	2	2	2	2	3	3	3	3	3	28
u_4	1	1	2	2	2	2	2	3	3	3	3	3	4	31
$wt(v_j)$		5	6	7	8	9	10	11	12	13	14	15	16	

In general, we give the lower bound on $tvs(K_{m,n})$ for $m \leq n$ as described in the following theorem.

Theorem 7. *For $m \leq n$, let $K_{m,n}$ be a complete bipartite graph different from $K_{2,2}$, then $tvs(K_{m,n}) \geq \max\left\{ \left\lceil \frac{m+n}{m+1} \right\rceil, \left\lceil \frac{2m+n-1}{n} \right\rceil \right\}$.*

Proof. Let $U = \{u_1, u_2, \cdots, u_m\}$ and $V = \{v_1, v_2, \cdots, v_n\}$ be the partite sets of $K_{m,n}$, where $m \leq n$.

- We know that the smallest label of vertices and edges of $K_{m,n}$ is 1. Then the smallest weights of vertices of $K_{m,n}$ is at least $m+1$, i.e. $wt(v_1) \geq m+1$ for $v_1 \in V$. Then the weights of vertices of $K_{m,n}$ are at least $m+1, m+2, m+3, \cdots$. If we can be make consecutive the weights of vertices in V, then the largest weights of vertices $v \in V$ is at least $m+n$, let $wt(v_n) \geq m+n$. So, for this case, at least one edge incident with v_n or v_n has label at least

$$\left\lceil \frac{m+n}{m+1} \right\rceil \quad (1)$$

- We have to obtain the weights of vertices of $K_{m,n}$ to be distinct and to give vertices and edges labels with the largest label as small as possible. In order to us get the minimum value of the largest label over all vertex and edge labels, we can make consecutive the weights of vertices. So the largest weights of vertices is at least $2m+n$. There are two cases for this weights:

  - **Case 1:** If $wt(u_m) = 2m+n$ for $u_m \in U$. At least one edge incident with u_m or u_m has label at least $\left\lceil \frac{2m+n}{n+1} \right\rceil$. But, because one of the edges which incident with u_m has got label 1, then at least one edge incident with u_m or u_m has label at least $\left\lceil \frac{2m+n-1}{n+1-1} \right\rceil = \left\lceil \frac{2m+n-1}{n} \right\rceil$.
  - **Case 2:** If $wt(v_i) = 2m+n$ for some $v_i \in V$, $i = 2, 3, \cdots, n$. At least one edge incident with v_i or v_i has label at least $\left\lceil \frac{2m+n}{m+1} \right\rceil$.

Now, we consider $\lceil \frac{2m-1}{n} + 1 \rceil$ and $\left\lceil \frac{2m+n}{m+1} \right\rceil$, and choose which minimum. Because of $m \le n$, then we can write $m = n - r$, where $0 \le r \le n-1$. We get

$$\left\lceil \frac{2m-1}{n} + 1 \right\rceil = \left\lceil \frac{2n-2r-1}{n} + 1 \right\rceil = \left\lceil \frac{3n-2r-1}{n} \right\rceil,$$

and

$$\left\lceil \frac{2m+n}{m+1} \right\rceil = \left\lceil \frac{3n-2r}{n+1-r} \right\rceil.$$

Of course $3n - 2r - 1 < 3n - 2r$, and

$$n+1-r = \begin{cases} n+1 & , r = 0, \\ n & , r = 1, \\ n-1 & , r = 2, \\ \vdots & \\ 2 & , r = n-1. \end{cases}$$

Then $n+1-r > n$ if $r = 0$ and $n+1-r \le n$ if $1 \le r \le n-1$.
So, $\frac{3n-2r}{n+1-r} > \frac{3n-2r-1}{n}$ if $1 \le r \le n-1$ or $m < n$. Thereby, if $m < n$ at least one edge incident with v in $K_{m,n}$ or v has label at least $\lceil \frac{2m+n-1}{n} \rceil$.

If $r = 0$ or $m = n$, we obtain $\frac{3n-2r}{n+1-r} = \frac{3n}{n+1}$ and $\frac{3n-2r-1}{n} = \frac{3n-1}{n}$. So that

$$\left\lceil \frac{3n}{n+1} \right\rceil = \left\lceil \frac{3n-1}{n} \right\rceil \text{ if } n \ne 2 \text{ and } \left\lceil \frac{3n}{n+1} \right\rceil < \left\lceil \frac{3n-1}{n} \right\rceil \text{ if } n = 2.$$

We can be chosen the minimum of the largest labels of $K_{m,n}$ is $\lceil \frac{2m-1}{n} + 1 \rceil$ if $n \ne 2$ and $\left\lceil \frac{2m+n}{m+1} \right\rceil$ if $n = 2$.

So, for this case, if $m \leq n$, with $(m,n) \neq (2,2)$, at least one edge incident with v in $K_{m,n}$ or v has label at least

$$\left\lceil \frac{2m+n-1}{n} \right\rceil \qquad (2)$$

Thereby, according to the equation 1 and 2, we obtain total vertex irregularity strength of $K_{m,n}$, $(m,n) \neq (2,2)$ is

$$tvs(K_{m,n}) \geq \max\left\{ \left\lceil \frac{m+n}{m+1} \right\rceil, \left\lceil \frac{2m+n-1}{n} \right\rceil \right\}. \qquad \blacksquare$$

References

1. Martin Bača, Stanislav Jendrol', Mirka Miller and Joseph Ryan, Total Irregular Labelings, Discrete Math. *to appear*.