# Exclusive Sum Labeling of Graphs

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**Abstract.** A graph $G(V, E)$ is called a *sum graph* if there is an injective labeling called *sum labeling* $L$ from $V$ to a set of distinct positive integers $S$ such that $xy \in E$ if and only if there is a vertex $w$ in $V$ such that $L(w) = L(x)+L(y) \in S$. In such a case $w$ is called a *working vertex*. Every graph can be made into a sum graph by adding some isolated vertices, if necessary. The smallest number of isolated vertices that need to be added to a graph $H$ to obtain a sum graph is called the *sum number* of $H$; it is denoted by $\sigma(H)$. A sum labeling which realizes $H \cup \overline{K_{\sigma(G)}}$ as a sum graph is called an *optimal sum labeling* of $H$.

Sum graph labeling offers a new method for defining graphs and for storing them digitally. Traditionally, a graph is defined as a set of vertices and a set of edges, specified by pairs of vertices which are the endpoints of an edge. To record a graph on a computer, the edges are usually stored either in the form of an adjacency matrix or as a linked list. Using sum graph labeling we only need to store the set of vertices, together with some additional isolates, if needed. While previously the edges in a graph were specified explicitly, using sum graphs, edges can be specified implicitly.
A sum labeling $L$ is called an exclusive sum labeling with respect to a subgraph $H$ of $G$ if $L$ is a sum labeling of $G$ where $H$ contains no working vertex. The *exclusive sum number* $\epsilon(H)$ of a graph $H$ is the smallest number $r$ such that there exists an exclusive sum labeling $L$ which realizes $H \cup \overline{K_r}$ as a sum graph. A labeling $L$ is an *optimal exclusive sum labeling* of a graph $H$ if $L$ is a sum labeling of $H \cup K_{\epsilon(H)}$ and $H$ contains no working vertex. While the exclusive sum number is never smaller than the corresponding sum number of a graph, labeling graphs exclusively has other desirable features which give greater scope for combining two or more labeled graphs.

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In this paper we introduce exclusive sum graph labeling and we construct optimal exclusive sum graph labeling for complete bipartite graphs, paths and cycles. The paper concludes with a summary of known results in exclusive sum labeling and exclusive sum numbers for several classes of graphs.

# 1 Introduction

A simple undirected graph $G$ is called a *sum graph* if there exists a labeling $L$ of the vertices of $G$ into distinct positive integers such that any two distinct vertices $u$ and $v$ of $G$ are adjacent if and only if there is a vertex $w$ whose label $L(w) = L(u) + L(v)$. However, for the sake of simplicity, we will from now on identify vertices with their labels under $L$ and we will write simply $u$ instead of $L(u)$.

Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain at least one isolated vertex. If $G$ is not a sum graph, it is always possible to add some finite number of isolated vertices to $G$ to obtain a sum graph. The *sum number* $\sigma(G)$ of a graph $G$ is the smallest number of isolated vertices that will achieve this result.

Since the introduction of the notion of a sum graph by Harary [5] in 1990, there have been quite a few papers published on this topic. One of the earliest interesting results was due to Ellingham [2] who proved the conjecture of Harary [5] that $\sigma(T) = 1$ for every $T \neq K_1$. A particularly interesting and intriguing result is due to Gould and Rödl [4] who proved that there exist graphs $G = (V, E)$ such that $\sigma(G) \in \Theta(|V|^2)$. Subsequently, Nagamochi *et al.* [13] proved that graphs with sum number on the order of $\Theta(|V|^2)$ should in fact be quite common. However, the methods used in [4] and [13] provide no means of constructing such graphs and a concentrated effort by several researches over more than a decade has failed to produce an example of any such graph or a class of graphs. In fact, the only known class of graphs $G$ that even achieves as much as $\sigma(G) \in \Theta(|E|)$ is the class of wheels $W_n$ with $n$ spokes; as shown in [11] and [8].

$$\begin{aligned} \sigma(W_n) &= n/2 + 2 \qquad \text{for even } n; \\ &= n \qquad \text{for odd } n. \end{aligned}$$

With the possible exception of the graphs considered in [11], no graphs are known whose sum number exceeds $|V|$ in an asymptotic sense. Efforts to find graphs of large sum number have of course led to a consideration of graphs with many edges; for example, complete graphs $K_n$ and complete bipartite graphs $K_{m,n}$. However, for these graphs it turns out that $\sigma(G) \in \Theta(|V|)$. Bergstrand *et al.* [1] proved that $\sigma(K_n) = 2n-3$. Hartsfield and Smyth [7] showed that $\sigma(K_{m,n}) \in \Theta(|V|)$ for all values of $m$ and $n$, and Miller *et al.* [12] proved that $\sigma(H_{2,n}) = 4n - 5$, where $H_{2,n}$ is the cocktail party graph.

A simple undirected graph $G$ is called *integral sum graph* if there exists a labeling $L$ of the vertices of $G$ into distinct integers such that any two distinct vertices $u$ and $v$ of $G$ are adjacent if and only if there is a vertex $w$ whose label $L(w) = L(u) + L(v)$. Both sum and integral sum graph labelings use distinct integer labels; the only difference is that in sum graphs the labels are positive integers while in integral sum graphs the labels can be also negative integers or zero. The *integral sum number* $\zeta(H)$ is the least number $r$ of isolated vertices such that $H \cup \overline{K_r}$ is an integral sum graph.

Motivated by our desire to find graphs of sum number on the order of $\Theta(|V|^2)$, we introduce *exclusive sum graph labeling* and *exclusive sum number* of a graph. In sum graphs, if $L$ is a sum graph labeling then so is $kL$ where $k$ is any positive integer. In exclusive sum labeling, not only is this also true but furthermore, $k_1 L + k_2$, under some conditions on the integers $k_1$ and $k_2$, is also an exclusive sum labeling.

Since finding exclusive sum graph labelings seems to be easier, we expect that there is a bigger chance to find a graph of order $\Theta(|V|^2)$ with exclusive sum labeling than with (inclusive) sum labeling. In this paper we present exclusive sum labelings for all complete bipartite graphs, paths and cycles. Later in this paper (Section 5) we also summarize known results in exclusive sum labelings and exclusive sum numbers for particular families of graphs.

## 2 Exclusive sum labeling

In a sum graph $G$, a vertex $w$ is said to *label* an edge $uv \in E(G)$ if and only if $w = u + v$. Alternatively, we also say that $w$ *witnesses* the edge $uv$. The *multiplicity* of $w$, denoted by $\mu(w)$, is defined to be the number of edges which are labelled by $w$. If $\mu(w) > 0$, then $w$ is called a *working vertex.*

If $L$ is a sum labeling of $G = H \cup \overline{K_r}$ in such a way that $H$ contains no working vertex then $L$ is said to be an *exclusive sum labeling of H within G* (or just *exclusive sum labeling of H* if $G$ is understood); otherwise, $L$ is said to be an *inclusive sum labeling of G with respect to H* (or just *inclusive sum labeling of H* if $G$ is understood). We also say that $L$ labels the graph $H$

exclusively if $L$ is an exclusive sum labeling of $H$ and $L$ labels the graph $H$ inclusively otherwise. Examples of exclusive sum labeling are the optimum sum labeling of complete graph $K_n$ for $n \geq 4$ and of odd wheels; examples of inclusive sum labeling are the sum labeling of trees and of even wheels. Note that wheels are not the only class of graphs that are labeled exclusively or inclusively depending on order; the exclusiveness or otherwise of complete bipartite graphs $K_{m,n}$ also depends on $m$ and $n$.

The *exclusive sum number*, $\epsilon(H)$ of a graph $H$ is the smallest number $r$ of isolated vertices such that $G = H \cup \overline{K_r}$ is a sum graph and $H$ is labeled exclusively.

Obviously, every exclusive sum graph is a sum graph but not vice versa and so the exclusive sum number is always greater than or equal to the sum number, that is,

**Observation 1** *For any graph $G$, $\epsilon(G) \geq \sigma(G)$.*

It is easy to see that for every sum graph labeling (exclusive or not) we have

**Observation 2** *If $L$ is a sum graph labeling of a graph $G$ then so is $kL$, where $k$ is any positive integer.*

However, in the case of an exclusive sum labeling we can do better, as shown in [10].

**Theorem 1** *[10] If $L$ is an exclusive sum graph labeling of a graph $H$ in $G = H \cup \overline{K_r}$ then so is the labeling $L'(u) = k_1L(u) + k_2$ for $u \in H$ and $L'(u) = k_1L(u) + 2k_2$ for $u \in \overline{K_r}$, where $k_2$ is any integer which results only in positive distinct values in $L'$ and $k_1$ is any positive integer that does not divide $6k_2$.*

The following observation gives a lower bound for exclusive sum number.

**Observation 3** *Let $\Delta$ be the maximum degree of vertices in a graph $G$. Then $\epsilon(G) \geq \Delta(G)$.*

Note that if $\epsilon(G) = \Delta(G)$ then the graph $G$ is called a *$\Delta$-optimum exclusive sum graph* of $G$. This will be considered in Section 5.

## 3 Exclusive Sum Labeling of Complete Bipartite Graphs

Hartsfield and Smyth [8] showed that the sum number of a complete bipartite graph $K_{p,q}$ for $q \geq p \geq 2$ is equal to $\lceil \frac{3p+q-3}{2} \rceil$. In 2001 He *et al.* [9]

showed that this result is only realised for a limited range of $p$ and $q$. The correct sum number for $K_{m,n}$ was given independently by three sets of authors [19],[9], [14], all published in the same issue of *Discrete Mathematics* in 2001.

Figure 1 shows an example of a sum labeling for $K_{9,3}$, using the method proposed by Pyatkin [14].

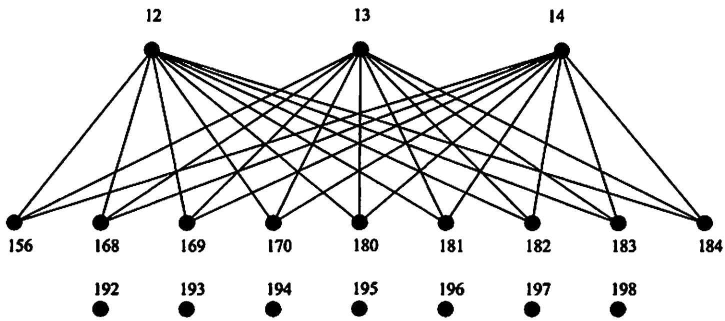


**Fig. 1.** Sum labeling for $K_{9,3}$.

However, in the case of exclusive sum number, the following lemma which is modified from the Hartsfield and Smyth's paper [8] is true in general and provides a lower bound on the exclusive sum number for the graph $K_{p,q}$.

**Lemma 1** *For* $p \geq 2$ *and* $q \geq 2, \epsilon(K_{p,q}) \geq p+q-1$.

**Proof**

Let $L$ be any exclusive sum labeling of a complete bipartite graph $K_{p,q}$, $q \geq 2$, $p \geq 2$. Let $P$ and $Q$ be the two partite sets, where $|P| = p \geq 2$, $|Q| = q \geq 2$. Suppose that the labels of $P = \{x_1, x_2, ..., x_p\}$ under $L$ are arranged into an ascending sequence, so that $x_j < x_{j+1}$, $1 \leq j \leq p-1$. Similarly, arrange the labels of $Q = \{y_1, y_2, ..., y_q\}$ into an ascending sequence. Observe that each of the following sums is distinct

$$x_1 + y_1 < x_2 + y_1 < ... < x_p + y_1 < x_p + y_2 < ... < x_p + y_q.$$

Since there are exactly $p+q-1$ distinct sums, it follows that at least $p+q-1$ isolated vertices are required to label the graph exclusively. □

Next choose $k$ such that $k > \max\{2p-2, p+q-2\}$ and suppose that $L$ labels the vertices of $P$ and $Q$ as follows.

$$P = \{1+4i|0 \le i \le p-1\}$$
$$Q = \{1+4j|k \le j \le k+q-1\}$$

Let $R$ be the set of isolated vertices which are labeled by

$$\{(1+4i)+(1+4k)|0 \le i \le p-2\} \cup \{(1+4(p-1))+(1+4j)|k \le j \le k+q-1\}.$$

It is clear that $|R| = p+q-1$. Note that the labels used for $P$ and $Q$ are 1 (mod 4) and the labels used for the isolated vertices $R$ are the sums of two numbers of 1 (mod 4), that is, 2 (mod 4). Therefore, $K_{p,q}$ contains no working vertex.

The sum of any two numbers from $P$ or $Q$ cannot be in $R$ by the choice of $k$. Moreover, since numbers congruent to 3 (mod 4) and 0 (mod 4) do not occur in this labeling, we conclude that no extra edges are induced between the isolates or between the graph and the isolates. Therefore, we have shown that $L$ is an exclusive sum labeling of $K_{p,q}$ which realises the lower bound of $\epsilon(K_{p,q})$.

We have the following theorem:

**Theorem 2** *For* $p \ge 2$ *and* $q \ge 2, \epsilon(K_{p,q}) = p+q-1$.

Figure 2 shows an example of an exclusive sum labeling for $K_{3,9}$.

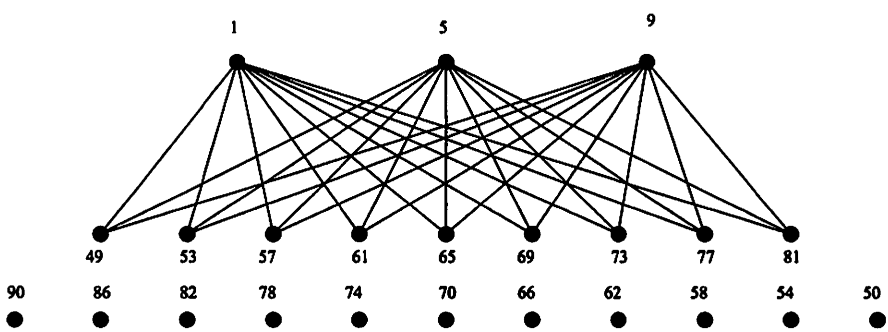


**Fig. 2.** Exclusive sum labeling for $K_{3,9}$.

In the next section we present the exclusive sum numbers of paths and cycles.

# 4 Paths and Cycles

## 4.1 Paths

Let $v_1, v_2, ..., v_n$ be the vertices of the path $P_n$. Label the vertices $v_i$ with $v_i = x + (\frac{i+1}{2} - 1)n$, for odd $i$ and $v_i = 2x - ((i/2) - 1)n$, for even $i$, where $x > n(n-2)$. Then $v_i + v_{i+1} = 3x$, for odd $i$ and $v_i + v_{i+1} = 3x + n$, for even $i$. Thus $P_n$ has an exclusive labelling with 2 isolated vertices, $v_{n-2} + v_{n-1}$ and $v_{n-1} + v_n$.

To prove that there is no additional edge between other vertices, we consider three cases as follows.

**(i)** There is no additional edge between $v_i$ and $v_j$ when $i, j$ are both even/odd.
If $i$ and $j$ are both even then we have $v_i + v_j = 4x - (\frac{i+j}{2} - 2)$. If $i$ and $j$ are both odd then we have $v_i + v_j = 2x + (\frac{i+j}{2} - 1)$. These values are not the same as the two isolated labels.
**(ii)** There is no additional edge between $v_i$ and $v_j$ when $i$ and $j$ have opposite parity.
Suppose that $i$ is even and $j$ is odd. Then $v_i + v_j = 3x - (\frac{i-j-1}{2} - 2)$. Again this value is not the same as the two isolated vertices labels.
**(iii)** There is no additional edge between $v_1$ and $v_n$.
For odd $n$, $v_1 + v_n = 2x + (\frac{n+1}{2} - 1)$ and for even $n$, $v_1 + v_n = 3x - (\frac{n}{2} - 1)$. Both values are not the same as the two isolated labels.

We have just proved

**Theorem 3** The exclusive sum number for paths, $\epsilon(P_n) = 2$, for $n \geq 3$.

## 4.2 Cycles

We start this subsection with the following lemma.

**Lemma 2** There are at least three distinct edge labels in any exclusive sum labeling of $C_n$.

**Proof** It is obvious for $n = 3$. Now we assume that $n > 3$. Let $w$ be the largest vertex on $C_n$. Let $u$ and $v$ be adjacent to $w$, and $(u, w)$ and $(v, w)$ be their corresponding edges. Without loss of generality, we suppose that $u < v$. Since $n > 3$, it follows that there is a vertex $t$ and a corresponding edge $(t, u)$. It is clear that $t + u \neq u + w$ and $u + w \neq v + w$. Since $v > u$ and $w > t$, we see that $v + w \neq u + t$. Therefore, the sums $t + u, u + w$ and $v + w$ are all distinct. □

It is necessary to deal separately with odd and even cycles.

**Odd cycles** It is simple to find a required labelling for $C_3$, for example, label the cycle vertices with 1,3 and 8 (and isolates with 4,9 and 11). From now on, we assume $n \geq 4$.

Let $v_i$, for $1 \leq i \leq n$, be the vertices on the cycle $C_n$ for odd $n$. Suppose that we label the vertices as follows.

$$v_i = \begin{cases} 1 + (\frac{i-1}{2})d & \text{for odd } i, \quad i \leq d-1 \\ 2 + (\frac{i-3}{2})d + i & \text{for odd } i, \quad i \geq d+1 \\ v_n - (\frac{i}{2})d & \text{for even } i, \quad i \leq d \\ v_n - (\frac{i-2}{2})d - i & \text{for even } i, \quad i \geq d+2 \end{cases}$$

where $d = 2\lceil \frac{n}{4} \rceil$.
Now we sum each pair of adjacent vertices.

(i) For odd $i$, we consider three cases.

1. for $i \leq d-1$,
$$v_i + v_{i+1} = v_n - d + 1$$
2. for $i \geq d+1$,
$$v_i + v_{i+1} = v_n - d + 1$$
3. for $i = n$,
$$v_n + v_1 = v_n + 1$$

(ii) For even $i$, we consider the following three cases.

1. for $i \leq d-2$,
$$v_i + v_{i+1} = v_n + 1$$
2. for $i = d$,
$$v_i + v_{i+1} = v_n + 1$$
3. for $i \geq d-2$,
$$v_i + v_{i+1} = v_n + 3$$

We see that there are three distinct edge labels of the cycle $C_n$ for odd $n$, that is, $v_n - d + 1, v_n + 1$ and $v_n + 3$. Thus, in view of Lemma 2, the construction of an optimal exclusive labeling for odd cycles requires exactly three working vertices.

Since all labels on the cycle are odd and at most $v_n$ then cycle vertex labels cannot be the sum of two vertex labels on the cycle, or the sum of two working vertices. The cycle labels cannot be the sum of the working vertices ($v_n + 1$, resp., $v_n + 3$, and a vertex on the cycle). The remaining case concerns $v_n - d + 1$. Suppose $u, v \in C_n$ and $v_n - d + 1 + u = v \leq v_n$. Then $u \leq d-1$, so $u$ can only be 1 or 3. If $u = 1$ then $v = v_n - d + 2$ which is not a label of any vertex on the cycle. Similarly, if $u = 3$ then $v = v_n - d + 4$ which is not a label. So none of the labels on the cycle can be a working vertex.

All 3 working vertices are even and so cannot be the sum of vertex labels from the cycle and working vertex, that is, there are no edges between the cycle and the 3 working vertices. Furthermore, there are no edges between the working vertices since we cannot have, when $n \geq 3$,

$$v_n + 1 + v_n + 3 = v_n - d + 1$$
$$v_n + 1 + v_n - d + 1 = v_n + 3$$
$$v_n + 3 + v_n - d + 1 = v_n + 1$$

Finally, we need to show that the labels together with isolates induce edges of the cycle and no extra edges. Suppose that there is an extra edge. Then there are two cases.

**(i)** Let $v_n + 1 = u + v$ then

$$u = v_n - (\frac{i-1}{2})d = v_{i-1} \Leftrightarrow v = 1 + (\frac{i-1}{2})d = v_i, \qquad i \leq d - 1$$

or

$$u = v_n - (\frac{i-3}{2})d - i - 1 = v_{i-1} \Leftrightarrow v = 2 + (\frac{i-3}{2})d + i = v_i, \qquad i \geq d + 1$$

**(ii)** Let $v_n + 3 = u + v$ then

$$u = v_n - (\frac{i-3}{2})d - i + 1 = v_{i-1} \Leftrightarrow v = 2 + (\frac{i-3}{2})d + i = v_i, \qquad i = d + 1$$

or

$$u = v_n - (\frac{i-1}{2})d = v_{i-1} \Leftrightarrow v = 3 + (\frac{i-1}{2})d = v_i$$

where, for the second case, $v_i$ cannot be a cycle vertex when $n \geq 3$.
**(iii)** Let $v_n - d + 1 = u + v$ then

$$u = 1 + (\frac{i-2}{2})d = v_{i-1} \Leftrightarrow v = v_n - (\frac{i-2}{2})d = v_i, \qquad i - 1 \leq d - 1$$

$$u = v_n - (\frac{i-3}{2})d - i - 1 = v_{i-1} \Leftrightarrow v = 2 + (\frac{i-3}{2})d + i, \qquad i \geq d$$

Thus, our labelling for odd cycles is an exclusive sum labeling which utilizes 3 isolates.

**Even cycles.** Let $v_i \in V(C_n)$, $1 \leq i \leq n$, $n$ even. Suppose that we label the vertices as follows.

$$v_i = \begin{cases} 4i - 3 & \text{if} \quad i \text{ is odd} \\ 4n - 4i + 5 & \text{if} \quad i \text{ is even} \end{cases}$$

Then the sum of each pair of adjacent vertices is: for $1 \leq i \leq n - 1$,

$$v_i + v_{i+1} = \begin{cases} 4n - 2 & \text{if} \quad i \text{ is odd} \\ 4n + 6 & \text{if} \quad i \text{ is even} \end{cases}$$

and

$$v_n + v_1 = 6.$$

Similarly to the odd case, and in view of Lemma 2, an optimum exclusive sum labeling of even cycles requires three isolates.

Thus, we have proved the following result.

**Theorem 4** $\epsilon(C_n) = 3$, for $n \geq 3$.

In Table 1 we summarize our knowledge of $\Delta$-optimum exclusive sum labeling of particular classes of graphs and their exclusive sum numbers.

| Graph $G$ | $\sigma(G)$ | $\epsilon(G)$ |
|---|---|---|
| Stars $S_n, n \geq 2$ | 1 [2] | $n$ [16] |
| Double stars $S_{m,n}, m, n \geq 2$ | 1 [2] | $\max\{m, n\}$ [17] |
| Caterpilar $S$ | 1 [2] | $\Delta(S)$ [17] |
| Trees $T_n, n \geq 3$ | 1 [2] | ? |
| Paths $P_n$ | 1 [2] | 2 |
| Cycles $C_4$ | 3 [5] | 3 |
| $C_n, n \geq 4$ | 2 [5] | 3 |
| Wheels | | |
| $W_n, n \geq 5, n$ odd | $n$ [11] | $n$ [16] |
| $n \geq 4, n$ even | $\frac{n}{2}+2$ [11] | $n$ [16] |
| Fans $f_n$, | | |
| $n = 3$ | 3 [18] | $n$ [16] |
| $n \geq 4, n$ even | 3 [18] | $n$ [16] |
| $n \geq 5, n$ odd | 4 [18] | $n$ [16] |
| Friendship graphs $F_n$ | 2 [3] | $2n$ [16] |
| Complete graphs $K_n, n \geq 3$ | $2n-3$ [1] | $2n-3$ [1] |
| Cocktail party graphs | | |
| $H_{2,n}$ | $4n-5$ [12] | $4n-5$ [12] |
| $H_{m,n}$ | ? | ? |
| Complete bipartite graphs, | | |
| $K_{n,n}$ | $\lceil\frac{(4n-3)}{2}\rceil$ [7] | $2n-1$ |
| $K_{m,n}$ | $\lceil k(n-1)/2 + m/(k-1)\rceil$ where $k = \lceil\frac{\sqrt{(1+(8m+n-1)(n-1))}}{2}\rceil$ [14] | $m+n-1$ |

**Table 1.** Sum numbers and exclusive sum numbers of various classes of graphs.

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