On super (a, d)-edge antimagic total labeling of disconnected graphs *

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Abstract. A graph G is called (a, d)-*edge antimagic total* ((a, d)-*EAT*) if there exist integers $a > 0$, $d \geq 0$ and a bijection $\lambda : V \cup E \to \{1, 2, ..., |V| + |E|\}$ such that $W = \{w(xy) : xy \in E\} = \{a, a + d, ..., a + (|E| - 1)d\}$, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$. An (a, d)-EAT labeling λ of graph G is *super* if $\lambda(V) = \{1, 2, \cdots, |V|\}$. In this paper we describe how to construct a super (a, d)-EAT labeling on some classes of disconnected graphs, namely $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$, for positive integer n.

Key words and phrases: (a, d)-*edge anti-magic total labeling, super* (a, d)-*EAT labeling, disconnected graphs*

1 Introduction

In this paper we consider undirected graphs without loops and multiple edges. $V(G)$ and $E(G)$ (in short, V and E) stand for the vertex-set and edge-set of graph G, respectively. Let $e = \{u, v\}$ (in short, $e = uv$) denote an edge connecting vertices u and v in G. Let P_n denote a path on n vertices. Other terminologies and notations for graph-theoretic ideas we follow the book of [4].

A graph G is called (a, d)-*edge antimagic total* ((a, d)-*EAT*) if there exist integers $a > 0$, $d \geq 0$ and a bijection $\lambda : V \cup E \to \{1, 2, ..., |V| + |E|\}$ such that the set of edge-weights is $W = \{w(xy) : xy \in E\} = \{a, a + d, ..., a + (|E| - 1)d\}$, where $w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$. We shall follow [7] to call

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$w(xy) = \lambda(x) + \lambda(y) + \lambda(xy)$ the *edge-weight* of xy, and W the *set of edge-weights* of the graph G. In particular, an (a, d)-*EAT* labeling λ of graph G is *super* if $\lambda(V) = \{1, 2, \cdots, |V|\}$.

For any super (a, d)-EAT labeling, the maximum edge-weight is no more than $|V| + (|V| - 1) + (|V| + |E|)$. Thus

$$a + (|E| - 1)d \leq 3|V| + |E| - 1 . \tag{1}$$

Similarly, the minimum possible edge-weight is at least $1 + 2 + |V| + 1$. Consequently

$$a \geq |V| + 4 . \tag{2}$$

So, from (1) and (2), we have

$$d \leq \frac{2|V| + |E| - 5}{|E| - 1} . \tag{3}$$

In general, for any (a, d)-EAT labeling, the maximum edge-weight is no more than $(|V| + |E| - 2) + (|V| + |E| - 1) + (|V| + |E|)$. Thus

$$a + (|E| - 1)d \leq 3|V| + 3|E| - 3 . \tag{4}$$

Similarly, the minimum possible edge-weight is at least $1 + 2 + 3$. Consequently

$$a \geq 6 . \tag{5}$$

So, from (4) and (5), we have

$$d \leq \frac{3|V| + 3|E| - 9}{|E| - 1} . \tag{6}$$

A number of classification studies on super (a, d)-EAT (resp. (a, d)-EAT) for connected graphs has been extensively investigated. For instances, in [2], Baca et al. showed that a wheel W_n has a super (a, d)-EAT labeling if and only if $d = 1$ and $n \equiv 1 (mod\ 4)$. A.A.G. Ngurah and E.T. Baskoro [5] proved that for every Petersen graph $P(n, m), n \geq 3, 1 \leq m \leq \frac{n}{2}$, has a super $(4n + 2, 1)$-EAT labeling.

Given any edge-magic total labeling λ on a graph G with p vertices and q edges. Then, its *dual* labeling λ' can be defined ([7]) by

$$\lambda'(x) = p + q + 1 - \lambda(x), \text{ for any vertex } x, \text{ and}$$

$$\lambda'(xy) = p + q + 1 - \lambda(xy), \text{ for any edge } xy.$$

By using the above duality property, we have the following theorem.

Theorem 1. *Let G be a graph with p vertices and q edges. If G has an (a, d)-EAT labeling then G has an $(3p + 3q + 3 - a - (q - 1)d, d)$-EAT labeling as its dual.*

More results concerning antimagic total labeling, see for instances [6, 1] and a nice survey paper by Gallian [3].

2 Main Results

The main goal of this paper is how to construct a super (a, d)-EAT (resp. (a, d)-EAT) labeling on disconnected graphs, especially $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$, for positive integer n.

Theorem 2. *Let G be a super (a, d)-EAT graph with p vertices and q edges. Let λ_1 be a super (a, d)-EAT labeling of G. Then, the labeling λ_1' defined:*

$$\lambda_1'(x) = p + 1 - \lambda_1(x),\ \forall x \in V, \text{ and}$$
$$\lambda_1'(xy) = 2p + q + 1 - \lambda_1(xy),\ \forall xy \in E$$

is a super $(4p + q + 3 - a - (q - 1)d, d)$-EAT labeling of G.

Proof. Let $xy \in E$. Then,

$$\begin{aligned}\lambda_1'(x) + \lambda_1'(xy) + \lambda_1'(y) &= (p + 1 - \lambda_1(x)) + (2p + q + 1 - \lambda_1(xy)) \\ &\quad + (p + 1 - \lambda_1(y)). \\ &= 4p + q + 3 - (\lambda_1(x) + \lambda_1(xy) + \lambda_1(y)).\end{aligned}$$

Thus, $W' = \{w(xy) : xy \in E\}$ under λ_1' constitutes an arithmatic progression starting from $4p + q + 3 - a - (q - 1)d$ with difference d. □

The labeling λ_1' is called a *dual super (a, d)-EAT labeling of* λ_1 on G.

2.1 $P_n \cup P_{n+1}$

In this section, we shall construct a super (a, d)-EAT (resp. (a, d)-EAT) labeling of graph $P_n \cup P_{n+1}$, namely a disjoint union of graphs P_n and P_{n+1}, for $n \geq 2$. We denote that

$$V(P_n \cup P_{n+1}) = \{u_{1,i} | 1 \leq i \leq n\} \cup \{u_{2,j} | 1 \leq j \leq n + 1\}, \text{ and}$$
$$E(P_n \cup P_{n+1}) = \{e_{1,i} | 1 \leq i \leq n - 1\} \cup \{e_{2,j} | 1 \leq j \leq n\}$$

where $e_{1,i} = u_{1,i}u_{1,i+1}$, for $1 \leq i \leq n - 1$, and $e_{2,j} = u_{2,j}u_{2,j+1}$, for $1 \leq j \leq n$.

By (3) and (6), we have: for every $n \geq 2$, there is no super (a, d)-EAT labeling of $P_n \cup P_{n+1}$ with $d \geq 4$; and there is no (a, d)-EAT labeling of $P_n \cup P_{n+1}$ with $d \geq 5$.

Theorem 3. *For every $n \geq 2$, the graph $P_n \cup P_{n+1}$ has a super $(4n + 4, 1)$-EAT labeling. This type of labeling is selfdual.*

Proof. Label the vertices and edges of $P_n \cup P_{n+1}$ in the following way

$$\begin{aligned}
\lambda(u_{1,i}) &= 2i, \text{ for } 1 \le i \le n. \\
\lambda(u_{2,j}) &= 2n+3-2j, \text{ for } 1 \le j \le n+1. \\
\lambda(e_{1,i}) &= \lambda(u_{1,i}u_{1,i+1}) = 4n+1-2i, \text{ for } 1 \le i \le n-1. \\
\lambda(e_{2,j}) &= \lambda(u_{2,j}u_{2,j+1}) = 2n+2j, \text{ for } 1 \le i \le n.
\end{aligned}$$

Clearly, the set of edge-weights is

$$\begin{aligned}
W &= \{w(e_{1,i}) : 1 \le i \le n-1\} \cup \{w(e_{2,j}) : 1 \le j \le n\} \\
&= \{4n+3+2i : 1 \le i \le n-1\} \cup \{6n+4-2j : 1 \le j \le n\} \\
&= \{4n+5, 4n+7, ..., 6n+1\} \cup \{6n+2, 6n, ..., 4n+4\} \\
&= \{4n+4, 4n+5, ..., 6n+1, 6n+2\}.
\end{aligned}$$

By Theorem 2 it can be easily seen that this labeling is selfdual. □

Theorem 4. *For every $n \ge 2$, the graph $P_n \cup P_{n+1}$ has a super $(2n+6, 3)$-EAT labeling. This type of labeling is selfdual.*

Proof. Label the vertices and edges of $P_n \cup P_{n+1}$ in the following way

$$\begin{aligned}
\lambda(u_{1,i}) &= 2i, \text{ for } 1 \le i \le n. \\
\lambda(u_{2,j}) &= 2j-1, \text{ for } 1 \le j \le n+1. \\
\lambda(e_{1,i}) &= \lambda(u_{1,i}u_{1,i+1}) = 2n+2i+1, \text{ for } 1 \le i \le n-1. \\
\lambda(e_{2,j}) &= \lambda(u_{2,j}u_{2,j+1}) = 2n+2j, \text{ for } 1 \le i \le n.
\end{aligned}$$

Clearly, the set of edge-weights is

$$\begin{aligned}
W &= \{w(e_{1,i}) : 1 \le i \le n-1\} \cup \{w(e_{2,j}) : 1 \le j \le n\} \\
&= \{2n+3+6i : 1 \le i \le n-1\} \cup \{2n+6j : 1 \le j \le n\} \\
&= \{2n+9, 2n+15, ..., 8n-3\} \cup \{2n+6, 2n+12, ..., 8n\} \\
&= \{2n+6, 2n+9, ..., 8n-3, 8n\}.
\end{aligned}$$

By Theorem 2 this type of labeling is selfdual. This concludes the proof. □

Theorem 5. *For every odd n and $n \ge 3$, the graph $P_n \cup P_{n+1}$ has super $(4n+5, 1)$-EAT labeling and super $(3n+6, 2)$-EAT labeling.*

Proof. Define the vertex labeling λ_1 of $P_n \cup P_{n+1}$ in the following way:

$$\lambda_1(u_{1,i}) = \begin{cases} \frac{i+1}{2}, & \text{for} \quad i = 1, 3, ..., n \\ n+2+\frac{i}{2}, & \text{for} \quad i = 2, 4, ..., n-1 \end{cases}$$

$$\lambda_1(u_{2,j}) = \begin{cases} \frac{n+j}{2}+1, & \text{for} \quad j = 1,3,...,n \\ n+2, & \text{for} \quad j = n+1 \\ \frac{3(n+1)+j}{2}, & \text{for} \quad j = 2,4,...,n-1 \end{cases}$$

Define the edge labeling λ_2 in as follows:

$$\lambda_2(e_{1,i}) = \lambda_2(u_{1,i}u_{1,i+1}) = 4n+2-2i, \text{ for } 1 \le i \le n-1.$$

$$\lambda_2(e_{2,j}) = \lambda_2(u_{2,j}u_{2,j+1}) = \begin{cases} 4n+1-2j, & \text{for } 1 \le j \le n-1 \\ 2n+2, & \text{for} \quad j = n \end{cases}$$

Combining the vertex labeling λ_1 and the edge labeling λ_2 gives a super EAT labeling. Clearly, the set of edge-weights is $\{4n+5, 4n+6, ..., 6n+2, 6n+3\}$. This implies that the graph $P_n \cup P_{n+1}$ has a super $(4n+5,1)$-EAT labeling.

Now, construct the edge labeling λ_3 in the following way.

$$\lambda_3(e_{1,i}) = \lambda_3(u_{1,i}u_{1,i+1}) = 2n+1+i, \textit{ for } 1 \le i \le n-1.$$

$$\lambda_3(e_{2,j}) = \lambda_3(u_{2,j}u_{2,j+1}) = \begin{cases} 3n+1+j, & \text{for} \quad 1 \le j \le n-1 \\ 3n+1, & \text{for} \quad j = n \end{cases}$$

Label the vertices and edges of $P_n \cup P_{n+1}$ by λ_1 and λ_2. We can see that the resulting label is total and the set of edge-weights consists of the consecutive integers $\{3n+6, 3n+8, ..., 7n+2\}$. □

By applying the Duality property to Theorems 5 we have the following corollary.

Corollary 1. *For every odd n and $n \ge 3$, the graph $P_n \cup P_{n+1}$ has super $(4n+3,1)$-EAT labeling and super $(3n+4,2)$-EAT labeling.*

Theorem 6. *For every $n \ge 2$, the graph $P_n \cup P_{n+1}$ has $(6n+1,1)$-EAT labeling and $(4n+3,3)$-EAT labeling.*

Proof. Define the vertex labeling and two edge labelings of $P_n \cup P_{n+1}$ in the following way:

$$\lambda_1(u_{1,i}) = 2n-1+2i, \text{ for } 1 \le i \le n.$$
$$\lambda_1(u_{2,j}) = 2n-2+2j, \text{ for } 1 \le j \le n+1.$$

$$\lambda_2(e_{1,i}) = \lambda_2(u_{1,i}u_{1,i+1}) = 2n-2i, \text{ for } 1 \le i \le n-1.$$
$$\lambda_2(e_{2,j}) = \lambda_2(u_{2,j}u_{2,j+1}) = 2n-2j+1, \text{ for } 1 \le i \le n.$$

$$\lambda_3(e_{1,i}) = \lambda_3(u_{1,i}u_{1,i+1}) = 2i, \text{ for } 1 \leq i \leq n-1.$$
$$\lambda_3(e_{2,j}) = \lambda_3(u_{2,j}u_{2,j+1}) = 2j-1, \text{ for } 1 \leq i \leq n.$$

Combining the vertex labeling λ_1 and the edge labeling λ_2 gives a total labeling with $\{6n+1, 6n+2, ..., 8n-2, 8n-1\}$. Combining the vertex labeling λ_1 and the edge labeling λ_3 gives a $(4n+3, 3)$-EAT labeling of $P_n \cup P_{n+1}$. □

Theorem 7. *For every odd n and $n \geq 3$, the graph $P_n \cup P_{n+1}$ has $(6n, 1)$-EAT labeling and $(5n+1, 2)$-EAT labeling.*

Proof. Label the vertices of $P_n \cup P_{n+1}$ by λ_1 as follows:

$$\lambda_1(u_{1,i}) = \begin{cases} 4n+1-\frac{i+1}{2}, & \text{for } \; i = 1, 3, ..., n \\ 3n-1-\frac{i}{2}, & \text{for } \; i = 2, 4, ..., n-1 \end{cases}$$

$$\lambda_1(u_{2,j}) = \begin{cases} \frac{7n-j}{2}, & \text{for } \; j = 1, 3, ..., n \\ 3n-1, & \text{for } \; j = n+1 \\ \frac{5n-1-j}{2}, & \text{for } \; j = 2, 4, ..., n-1. \end{cases}$$

If we label the edges of $P_n \cup P_{n+1}$ by:

$$\lambda_2(e_{1,i}) = \lambda_2(u_{1,i}u_{1,i+1}) = 2i, \text{ for } 1 \leq i \leq n-1,$$

$$\lambda_2(e_{2,j}) = \lambda_2(u_{2,j}u_{2,j+1}) = \begin{cases} 2j+1, & \text{for } \; 1 \leq j \leq n-1 \\ 1, & \text{for } \; j = n, \end{cases}$$

then the resulting labeling is total and we have $\{6n, 6n+1, \cdots, 8n-3, 8n-2\}$ as the set of edge-weights. This implies that the graph $P_n \cup P_{n+1}$ has an $(6n, 1)$-EAT labeling. If we label the edges of $P_n \cup P_{n+1}$ by λ_3 as follows:

$$\lambda_3(e_{1,i}) = \lambda_3(u_{1,i}u_{1,i+1}) = 2n-i, \text{ for } 1 \leq i \leq n-1,$$

$$\lambda_3(e_{2,j}) = \lambda_3(u_{2,j}u_{2,j+1}) = \begin{cases} n-j, & \text{for } \; 1 \leq j \leq n-1 \\ n, & \text{for } \; j = n, \end{cases}$$

then, the resulting labeling is total and the set of edge-weights is $\{5n+1, 5n+3, ..., 9n-3\}$. □

2.2 $nP_2 \cup P_n$

In this section, we shall construct a super (a, d)-EAT labeling of graph $nP_2 \cup P_n$, for $n \geq 2$. The graph nP_2 is a disjoint union of n copies of graph P_2. The graph $nP_2 \cup P_n$ is a disjoint union of graphs nP_2 and P_n. We denote that

$$V(nP_2 \cup P_n) = \{u_{1,i}, u_{2,i} | 1 \le i \le n\} \cup \{u_{3,j} | 1 \le j \le n\}, \text{ and}$$
$$E(nP_2 \cup P_n) = \{e_{1,i} | 1 \le i \le n\} \cup \{e_{2,j} | 1 \le j \le n-1\},$$

where $e_{1,i} = u_{1,i}u_{2,i}$, for $1 \le i \le n$ and $e_{2,j} = u_{3,j}u_{3,j+1}$, for $1 \le j \le n-1$.

By (3) and (6), we have: for every $n \ge 2$, there is no super (a,d)-EAT labeling of $nP_2 \cup P_n$ with $d \ge 4$, and there is no (a,d)-EAT labeling of $nP_2 \cup P_n$ with $d \ge 8$.

Theorem 8. *For every $n \ge 2$, the graph $nP_2 \cup P_n$ has super (6n+2,1)-EAT labeling and super (5n+3,2)-EAT labeling. These types of labelings are selfdual.*

Proof. Label the vertices of $nP_2 \cup P_n$ by λ_1 as follows:

$\lambda_1(u_{1,i}) = i$, for $1 \le i \le n$
$\lambda_1(u_{2,i}) = 2n + i$, for $1 \le i \le n$
$\lambda_1(u_{3,j}) = n + j$, for $1 \le j \le n$
$\lambda_1(e_{1,i}) = \lambda(u_{1,i}u_{2,i}) = 4n + 1 - i$, for $1 \le i \le n$,

and the edges by λ_2:

$\lambda_2(e_{2,j}) = \lambda_2(u_{3,j}u_{3,j+1}) = 5n - j$, for $1 \le j \le n-1$.

Clearly, the resulting labeling is super EAT and the set of edge-weights is

$$\begin{aligned} W &= \{w(e_{1,i}) : 1 \le i \le n\} \cup \{w(e_{2,j}) : 1 \le j \le n-1\} \\ &= \{6n + i + 1 : 1 \le i \le n\} \cup \{7n + 1 + j : 1 \le j \le n-1\} \\ &= \{6n + 2, 6n + 3, ..., 7n + 1\} \cup \{7n + 2, 7n + 3, ..., 8n\} \\ &= \{6n + 2, 6n + 3, ..., 8n - 1, 8n\}. \end{aligned}$$

This implies that the graph $nP_2 \cup P_n$ has a super $(6n+2, 1)$-EAT labeling.

Next, if the vertices of $nP_2 \cup P_n$ are labelled by λ_1 and the edges are labelled by λ_3 as follows:

$\lambda_3(e_{1,i}) = \lambda_3(u_{1,i}u_{2,i}) = 3n + 2i - 1$, for $1 \le i \le n$
$\lambda_3(e_{2,j}) = \lambda_3(u_{3,j}u_{3,j+1}) = 3n + 2j$, for $1 \le j \le n-1$,

then the resulting labeling is still super EAT and the set of edge-weights is $\{5n + 3, 5n + 5, ..., 9n - 3, 9n - 1\}$. This implies that the graph $nP_2 \cup P_n$ has a super $(5n + 3, 2)$-EAT labeling.

By Theorem 2, it is easy to show that these labelings are selfdual. This concludes the proof. □

Theorem 9. *For every $n \ge 2$, the graph $nP_2 \cup P_n$ has an (7n,1)-EAT labeling and (6n+1,2)-EAT labeling.*

Proof. Define the vertex labeling λ_1 and two edge labelings λ_2, λ_3 of $nP_2 \cup P_n$ in the following way.

$$\begin{aligned}
\lambda_1(u_{1,i}) &= 5n - i, \text{ for } 1 \le i \le n \\
\lambda_1(u_{2,i}) &= 3n - i, \text{ for } 1 \le i \le n \\
\lambda_1(u_{3,j}) &= 4n - j, \text{ for } 1 \le j \le n,
\end{aligned}$$

$$\begin{aligned}
\lambda_2(e_{1,i}) &= \lambda_2(u_{1,i}u_{2,i}) = n + i - 1, \text{ for } 1 \le i \le n \\
\lambda_2(e_{2,j}) &= \lambda_2(u_{3,j}u_{3,j+1}) = j, \text{ for } 1 \le j \le n - 1,
\end{aligned}$$

$$\begin{aligned}
\lambda_3(e_{1,i}) &= \lambda_3(u_{1,i}u_{2,i}) = 2n - 2i + 1, \text{ for } 1 \le i \le n \\
\lambda_3(e_{2,j}) &= \lambda_3(u_{3,j}u_{3,j+1}) = 2n - 2j, \text{ for } 1 \le j \le n - 1.
\end{aligned}$$

Combining the vertex labeling λ_1 and the edge labeling λ_2 gives a total labeling with the set of edge-weights $\{7n, 7n + 1, ..., 9n - 3, 9n - 2\}$. Meanwhile, combining λ_1 and λ_3 gives a $(6n+1, 2)$-EAT labeling of $nP_2 \cup P_n$. □

2.3 $nP_2 \cup P_{n+2}$

In this section, we shall construct a super (a, d)-EAT labeling of graph $nP_2 \cup P_{n+2}$, for $n \ge 1$. The graph $nP_2 \cup P_{n+2}$ is a disjoint union of graph nP_2 and P_{n+2}. We denote that

$$\begin{aligned}
V(nP_2 \cup P_{n+2}) &= \{u_{1,i}, u_{2,i} | 1 \le i \le n\} \cup \{u_{3,j} | 1 \le j \le n + 2\}, \text{ and} \\
E(nP_2 \cup P_{n+2}) &= \{e_{1,i} | 1 \le i \le n\} \cup \{e_{2,j} | 1 \le j \le n + 1\}
\end{aligned}$$

where $e_{1,i} = u_{1,i}u_{2,i}$, for $1 \le i \le n$ and $e_{2,j} = u_{3,j}u_{3,j+1}$, for $1 \le j \le n + 1$.

By (3) and (6), we have the following facts: for every $n \ge 1$, there is no super (a, d)-EAT labeling of $nP_2 \cup P_n$ with $d \ge 5$, and there is no (a, d)-EAT labeling of $nP_2 \cup P_n$ with $d \ge 8$.

Theorem 10. *For every $n \ge 1$, the graph $nP_2 \cup P_{n+2}$ has super (6n+6,1)-EAT labeling and super (5n+6,2)-EAT labeling. These labelings are selfdual.*

Proof. Define the vertex label λ_1 and the two edge labelings λ_2 and λ_3 of $nP_2 \cup P_{n+2}$ in the following way.

$$\begin{aligned}
\lambda_1(u_{1,i}) &= i, \text{ for } 1 \le i \le n \\
\lambda_1(u_{2,i}) &= 2n + i + 2, \text{ for } 1 \le i \le n \\
\lambda_1(u_{3,j}) &= n + j, \text{ for } 1 \le j \le n + 2
\end{aligned}$$

$$\begin{aligned}\lambda_2(e_{1,i}) &= \lambda_2(u_{1,i}u_{2,i}) = 4n - i + 3, \text{ for } 1 \le i \le n\\ \lambda_2(e_{2,j}) &= \lambda_2(u_{3,j}u_{3,j+1}) = 5n - j + 4, \text{ for } 1 \le j \le n+1,\end{aligned}$$

$$\begin{aligned}\lambda_3(e_{1,i}) &= \lambda_3(u_{1,i}u_{2,i}) = 3n + 2i + 2, \text{ for } 1 \le i \le n\\ \lambda_3(e_{2,j}) &= \lambda_3(u_{3,j}u_{3,j+1}) = 3n + 2j + 1, \text{ for } 1 \le j \le n+1.\end{aligned}$$

To obtain a super $(6n+6,1)$-EAT labeling combine the labelings λ_1 and λ_2. Whereas, combining the labelings λ_1 and λ_3 gives a super $(5n+6,2)$-EAT labeling. By Theorem 2 these labelings are selfdual. □

Theorem 11. *For every $n \ge 1$, the graph $nP_2 \cup P_{n+2}$ has (7n+6,1)-EAT labeling and (6n+6,2)-EAT labeling.*

Proof. Define the vertex labeling λ_1 and the two edge labelings λ_2 and λ_3 of $nP_2 \cup P_{n+2}$ in the following way.

$$\begin{aligned}\lambda_1(u_{1,i}) &= 5n + 4 - i, \text{ for } 1 \le i \le n\\ \lambda_1(u_{2,i}) &= 3n + 2 - i, \text{ for } 1 \le i \le n\\ \lambda_1(u_{3,j}) &= 4n + 4 - j, \text{ for } 1 \le j \le n+2\end{aligned}$$

$$\begin{aligned}\lambda_2(e_{1,i}) &= \lambda_2(u_{1,i}u_{2,i}) = n + 1 + i, \text{ for } 1 \le i \le n\\ \lambda_2(e_{2,j}) &= \lambda_2(u_{3,j}u_{3,j+1}) = j, \text{ for } 1 \le j \le n+1,\end{aligned}$$

$$\begin{aligned}\lambda_3(e_{1,i}) &= \lambda_3(u_{1,i}u_{2,i}) = 2n + 2 - 2i, \text{ for } 1 \le i \le n\\ \lambda_3(e_{2,j}) &= \lambda_3(u_{3,j}u_{3,j+1}) = 2n + 3 - 2j, \text{ for } 1 \le j \le n+1.\end{aligned}$$

To obtain a $(7n+6,1)$-EAT labeling combine the labelings λ_1 and λ_2. Whereas, combining the labelings λ_1 and λ_3 gives a $(6n+6,2)$-EAT labeling. By Theorem 2 these labelings are selfdual. □

To conclude this paper let us present the following open problems to work on.

Problem 1. Construct, if there exists,

- A super $(a,2)$-EAT labeling for $P_n \cup P_{n+1}$, n even.
- An $(a,2)$-EAT labeling for $P_n \cup P_{n+1}$, n even.
- An (a,d)-EAT labeling for $P_n \cup P_{n+1}$, $d = \{3,4\}$, and $n \ge 2$.

Problem 2. Construct, if there exists,

- A super $(a,3)$-EAT labeling for $nP_2 \cup P_n$, $n \ge 2$.
- An (a,d)-EAT labeling for $nP_2 \cup P_n$, $d = \{3,4,5,6,7\}$, and $n \ge 2$.

Problem 3. Construct, if there exists,

– A super (a,d)-EAT labeling for $nP_2 \cup P_{n+2}$, $d = \{3,4\}$, and $n \geq 1$.
– An (a,d)-EAT labeling for $nP_2 \cup P_{n+2}$, $d = \{3,4,5,6,7\}$, and $n \geq 1$.

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