Labellings of trees with maximum degree three - an improved bound

Ljiljana Brankovic[1], Alex Rosa[2], Jozef Širáň[3]

[1] School of Electrical Engineering and Computer Science University of Newcastle, Callgahan, NSW 2308, Australia,
Ljiljana.Brankovic@newcastle.edu.au
[2] Department of Mathematics and Statistics McMaster University Hamilton, Ontario, Canada L85 4K1
rosa@mcmaster.ca
[3] Department of Mathematics University of Auckland, Auckland, New Zealand
siran@math.auckland.ac.nz

Abstract. We improve the lower bound for the α-size of trees with maximum degree three.

1 Introduction

If $T = (V, E)$ is a tree with n vertices, a *(vertex-)labelling* of T is a bijection $\phi : V \to \{1, \ldots, n\}$. The *induced edge-labelling* $\bar{\phi} : E \to \{1, 2, \ldots, n-1\}$ is defined by $\bar{\phi}(e) = |\phi(u) - \phi(v)|$ for each edge $e = uv \in E$. The *size* of the labelling ϕ is $|\bar{\phi}(E)|$, i.e. the number of distinct induced edge labels. A labelling of T is *graceful* if its size is $n-1$. The famous Graceful Tree Conjecture (GTC) states that every tree has a graceful labelling. The conjecture is by now forty years old but remains wide open [Gallian, J., 2005]. In particular, GTC is still open even for trees of maximum degree three. The approximation approach outlined below was introduced in [Rosa, A., Širáň, J., 1995], and is pursued further in this article.

A labelling of T is *bipartite* if there exists a number k such that $\phi(u) \leq k < \phi(v)$ (or $\phi(v) \leq k < \phi(u)$) if and only if u and v are vertices of different colour (in the 2-colouring of the vertices of T). A labelling of T which is both graceful and bipartite is an *α-labelling.*

The *gracesize* $gs(T)$ of the tree T is the maximum size of a labelling of T. The *α-size* $\alpha(T)$ of the tree T is the maximum of $|\bar{\phi}(E)|$ where ϕ ranges over all bipartite labellings of T. Obviously, $\alpha(T) \leq gs(T)$.

Let $gs(n)$ and $\alpha(n)$ be the minimum of $gs(T)$ and $\alpha(T)$, respectively, taken over all trees with n vertices. While the graceful tree conjecture is equivalent to the statement that $gs(n) = n-1$ for all n, the situation for α-labellings is quite different. Indeed, it was shown in [Rosa, A., Širáň, J., 1995] that $5n/7 \leq \alpha(n) \leq (5n+4)/6$ for all $n \geq 4$. Since the trees on n vertices

JCMCC 55 (2005), pp. 159-169

that have been used in [Rosa, A., Širáň, J., 1995] to prove the upper bound on $\alpha(n)$ contain vertices of large degree, it is of interest to ask about upper and lower bounds on $\alpha(T)$ within the class of trees T on n vertices with a small maximum degree. Let $\alpha_k(n)$ be the smallest $\alpha(T)$ over all trees on n vertices with maximum degree k. For $k = 3$ the lower bound was improved in [Bonnington, C.P., Širáň, J., 1999] to $\alpha_3(n) \geq 5n/6$ for all $n \geq 12$. In this note we improve the lower bound on α_3 to $\alpha_3(n) \geq \lfloor \frac{6n}{7} \rfloor - 1$.

2 Preliminaries

We assume from now on that all trees that we consider have maximum degree three. To prove our results below we will make use of those trees with maximum degree three and with $n < 16$ vertices whose α-size equals $n - 2$ as described in [Bonnington, C.P., Širáň, J., 1999] (see Fig.1). These 7 trees in Fig.1 will be called *exceptional* trees. We note that the α-size of all other trees with maximum degree three and $n < 16$ vertices equals $n - 1$ [Bonnington, C.P., Širáň, J., 1999].

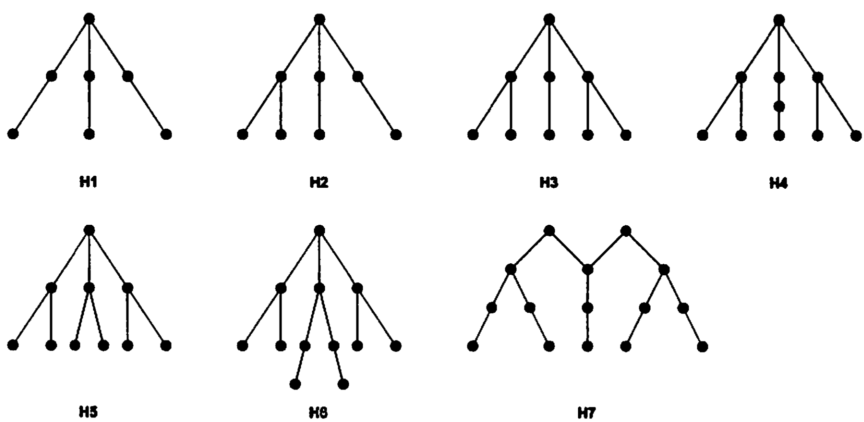

Fig. 1. The 7 exceptional trees.

Let T be a tree, and let T' be the (unique) homeomorphically irreducible tree homeomorphic to T. Thus in T' all vertices are either of degree three or of degree one (the latter are the *leaves*). Represent the tree T' as a rooted tree where the root is (for example, but not necessarily) a central vertex of T'. We say that the leaves have level 0; all vertices in T' having at least one child, and whose all children have level 0 will have level 1, and so on.

In general, all vertices in T' which have at least one child on level l and whose all children have level l or less will have level $l+1$. More formally, the level $l(v)$ of a vertex v is given by $l(v) = 1 + max_{c \in C_v} l(c)$ where C_v is the set of all children of the vertex v. We define the level of a vertex in T to be the same as the level of the corresponding vertex in T'; the levels of vertices of degree two in T (if any) are undefined.

Example 1. Let the tree T be as given in Fig.2. Then T' is as given in Fig.3a. For the sake of clarity, we labelled the vertices in T, and kept the same labels in T'. The vertices in T and T' with the same label are *corresponding* vertices. The labels in Fig.3b) represent the vertex levels.

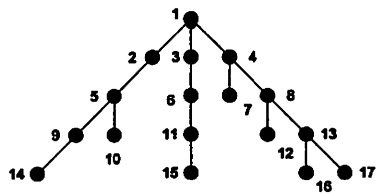

Fig. 2. A tree T with maximum degree three.

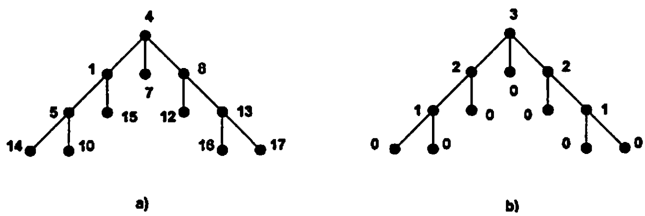

Fig. 3. a) T' with the labelling corresponding to T in Figure 2. **b)** T' with 'level' labeling.

We say that a tree T is *good* if T has at least $k \geq 7$ vertices, and deleting at most $\lfloor \frac{k-7}{7} \rfloor$ edges of T results in a forest whose each component has an α-labelling.

Before proceeding further, let us outline the strategy of our proof that $\alpha(n) \geq \lfloor \frac{6n}{7} \rfloor - 1$. We first show that each tree of maximum degree three and diameter at least five can be split (by deleting a suitable edge) into two trees, at least one of which is good. Then, using a direct argument we show that the α-size of a tree with maximum degree three, order n, and diameter at most four is at least $\lfloor \frac{6n}{7} \rfloor - 1$. Finally, we prove that $\alpha(n) \geq \lfloor \frac{6n}{7} \rfloor - 1$ by taking an arbitrary tree of maximum degree three, iteratively deleting edges until we are left with good trees and a tree of diameter at most four, and applying a simple result that allows to combine α-labellings under some special circumstances.

3 An edge-deletion lemma

In this section we prove an auxiliary result about producing good trees out of arbitrary trees of maximum degree three and diameter at least five.

Lemma 1. *Let T be a tree of maximum degree three such that the diameter of T' is at least 5. Then it is always possible to delete an edge e in T such that $T - e = T_1 \cup T_{1R}$ where T_1 is a good tree.*

Proof. If T' has diameter at least 5, we can root T' at a peripheral vertex which will then become a vertex of level $d \geq 5$. We define a subtree of a vertex v of degree 3 in T to be a subtree that contains v and is obtained by deleting the edge joining v to its parent. (Thus a subtree of a vertex v cannot contain the root.) We refer to a subtree of a vertex on level i as a *level i subtree*. We proceed as follows. First we consider all level 1 subtrees. If one of them is a good tree, we are done, otherwise we consider all level 2 subtrees. In doing so, we make use of the fact that none of the level 1 subtrees is a good tree. Similarly, if no level 2 subtree is good, we consider level 3 subtrees, and if neither of these is good, we show that all level 4 subtrees are good.

A level 1 subtree (see Fig.4) is good if it contains at least 7 vertices, since it is a snake (a tree with exactly two leaves), and all snakes have an α-labelling [Rosa, A., 1967].

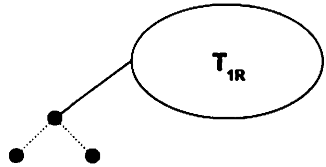

Fig. 4. A subtree of a vertex on level 1. Dashed line is used to represent not a single edge but a path with all intermediate vertices of degree 2.

If none of the level 1 subtrees is good, we proceed to the level 2 subtrees. There are two types of level 2 subtrees: Type 1 contains one level 1 subtree (Fig.5a) while Type 2 contains two level 1 subtrees (Fig.5b).

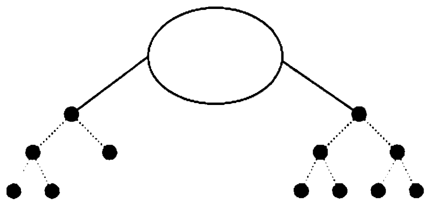

Fig. 5. a) Level 2 subtree of Type 1. **b)** Level 2 subtree of Type 2.

Type 1 level 2 subtree is a tree with exactly three endvertices (from now on we will call such a tree a *Y-tree*). A Y-tree always has an α-labelling, except when it is the tree H_1 in Fig.1 [Rosa, A., 1977]. Thus all type 1 level 2 subtrees are good trees, except for H_1, and trees containing 5 or 6 vertices.

It is useful to note at this point that if in any of these exceptions we delete the edge e_{12} incident to a level 2 vertex that is on the path that connects level 2 vertex to a level 1 vertex, we obtain either a Y-tree or a snake, and another snake each of which has an α-labelling. That is, the Y-tree is not H_1 as then the original level 2 subtree would contain at least 9 vertices, and thus would be a good tree.

A type 2 level 2 subtree is always a good tree. Indeed, it contains at least 7 vertices, and it has exactly 4 leaves. If it contains less than 16 vertices then it has an α-labelling since it cannot be H_2, the only exceptional tree with 4 leaves, as the two vertices of degree three are adjacent in H_2 and are nonadjacent in a type 2 level 2 subtree. On the other hand, if a type 2 level 2 subtree contains 16 or more vertices, we delete an edge incident to one of the vertices of degree 3. In this way we obtain one Y-tree (but not H_1 as each of the level 1 subtrees has at most 6 vertices, so the path connecting them contains at least 4 vertices), and a snake both of which have an α-labelling. Thus if no level 2 subtree is a good tree then all level 2 subtrees are H_1 or contain 5 or 6 vertices.

Next we consider level 3 subtrees. There are three types of these: Type 1 contains one level 2 and one level 0 subtree (Fig.6a), Type 2 contains one level 2 and one level 1 subtree (Fig.6b), and Type 3 contains two level 2 subtrees (Fig.6c).

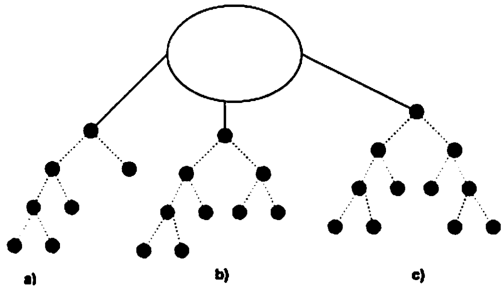

Fig. 6. **a)** Type 1 level 3 subtree. **b)** Type 2 level 3 subtree. **c)** Type 3 level 3 subtree.

First we note that all level 3 subtrees contain at least 7 vertices. If a Type 1 level 3 subtree contains less than 16 vertices then it is always good unless it is H_2. If it has at least 16 vertices then we delete the edge e_{12} incident to a level 2 vertex as shown in Fig.7. Thus we obtain a Y-tree (that is not H_1 as then our level 2 subtree would be good) and a snake both of which have an α-labelling, or two snakes.

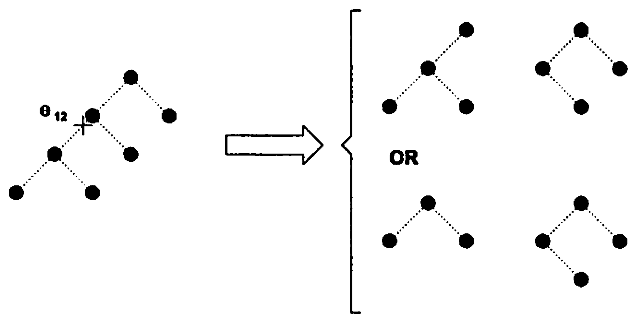

Fig. 7. The edge to be deleted in Type 1 level 3 subtree with 16 or more vertices.

If a Type 2 level 3 subtree has less than 16 vertices then it has an α-labelling: indeed it has 5 leaves and it cannot be H_3 or H_4 because there is no vertex of degree 3 adjacent to both other vertices of degree 3 as in H_3 and H_4. It is not H_7, the only other exceptional tree with 5 leaves, as then the level 2 subtree would have 9 vertices, and thus would be a good tree. If a Type 2 level 3 subtree has at least 16 vertices then we delete the edge e_{12}incident to a level 2 vertex, as shown in Fig.8. We thus obtain two Y-trees or one Y-tree and a snake. None of the Y-trees is H_1, by an argument as above. Therefore all Type 2 level 3 subtrees are good.

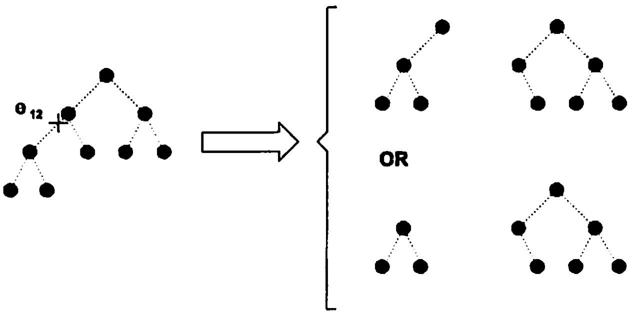

Fig. 8. The edge to be deleted in Type 2 level 3 subtree with 16 or more vertices.

If a Type 3 level 3 subtree has less than 16 vertices then it has an α-labelling: it is not H_5 or H_6 because there is no vertex of degree 3 adjacent to all other vertices of degree 3 as in H_5 and H_6. If a Type 3 level 3 subtree has 16 or 17 vertices, we delete the edge e_{12} incident to a level 2 vertex, thus creating one Y-tree (but not H_1) or a snake and one tree with 4 leaves which has an α-labelling as it has between 11 and 14 vertices and the only exceptional tree with 4 leaves (H_2) has 8 vertices. If a Type 3 level 3 subtree has at least 18 and at most 30 vertices, we delete an edge on the path connecting the two level 2 vertices in such a way that both trees with 4 leaves contain at least 9 and at most 15 vertices, and thus both have an α-labelling. Finally, if a Type 3 level 3 subtree contains more than 30 vertices, we delete the edge that is incident to the level 2 vertex in both level 2 subtrees. By doing so we create three trees where at least one is a snake, and each of the other two is either a snake or a Y-tree but not H_1.

Thus the only level 3 subtree which is not good is H_2.

If there are no good level 3 subtrees, we proceed to consider level 4 subtrees which can be one of the following four types. Type 1 level 4 subtree contains a level 3 subtree (that is, H_2) and a level 0 subtree (Fig.9a)). Type 2 level 4 subtree contains H_2 and a level 1 subtree (Fig.9b)), Type 3 level 4 subtree contains H_2 and a level 2 subtree (Fig.9c)) while Type 4 level 4 subtree contains two trees H_2 (Fig.9d)).

If a Type 1 level 4 subtree has less than 16 vertices then it has an α-labelling, as it is neither of H_3, H_4 or H_7. Similarly, if a Type 2 level 4 subtree contains less than 16 vertices, it has an α-labelling as it is neither of H_5 or H_6. On the other hand, if a Type 1 or Type 2 level 4 subtree has

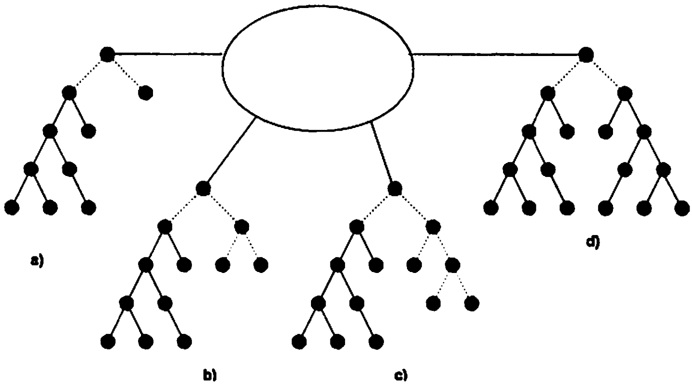

Fig. 9. a) Type 1 level 4 subtree. **b)** Type 2 level 4 subtree. **c)** Type 3 level 4 subtree. **d)** Type 4 level 4 subtree. Note again that solid lines represent edges and dashed lines represent paths where all intermediate vertices are of degree 2.

at least 16 vertices then we delete an edge connecting the level 2 and level 3 vertex in the level 3 subtree. We thus obtain a Y-tree and a snake for Type 1 and two Y-trees for Type 2 subtree. All of these new trees have an α-labelling and consequently all Type 1 and Type 2 level 4 subtrees are good.

A Type 3 level 4 subtree has at least 14 vertices; if it has less than 21 vertices, we delete an edge connecting the level 2 vertex to the level 3 vertex in the level 3 subtree. We thus obtain a Y-tree (not H_1) and a tree with 4 leaves that has between 8 and 14 vertices and is not an exceptional tree. If the tree has at least 21 vertices, we delete an edge connecting the level 2 and level 3 in the level 3 subtree and an edge e_{12} incident to the level 2 vertex in the other level 2 subtree. We obtain a Y-tree, a snake, and a Y-tree or a snake. All these trees have an α-labelling.

A Type 4 level 4 subtree has at least 17 vertices. If it has less than 21 vertices then we delete an edge connecting the level 2 to the level 3 vertex in the level 3 subtree. We obtain a Y-tree and a tree with 5 leaves and with the number of vertices between 11 and 14 which is thus not an exceptional tree. If a Type 4 level 4 subtree contains at least 21 vertices, we delete an edge connecting the level 2 and the level 3 vertex in both level 3 subtrees. We obtain two Y-trees and a snake all of which have an α-labelling.

Therefore all level 4 trees are good trees; note that we are assuming that none of the level 3, level 2 or level 1 trees are good. This completes the proof of Lemma 1. □

4 Proof of the main result

We start with a known result contained in [Rosa, A., Širáň, J., 1995].

Lemma 2. *Let e be an arbitrary edge of a tree T, and let $T - e = T_1 \cup T_2$. Then $\alpha(T) \geq \alpha(T_1) + \alpha(T_2)$.*

Proof. See [Rosa, A., Širáň, J., 1995]. □

One more auxiliary result is needed for the proof of our main theorem.

Lemma 3. *Let T be a tree with n vertices and with maximum degree three such that the diameter of the corresponding tree T' (as defined above) is at most 4. Then $\alpha(T) \geq \lfloor \frac{6n}{7} \rfloor - 1$.*

Proof. First we note that every tree T with the number of vertices $n \leq 6$ has $\alpha(T) = n - 1$. Let the root of T' be a central vertex of T'. If T is such that the root of T' has level 1 then $\alpha(T) = n - 1$ unless T' is H_1 (cf. [Rosa, A., 1977] in which case $\alpha(H_1) = n - 2 = \lfloor \frac{6n}{7} \rfloor - 1$ (as $n = 7$). If the root of T' has level 2, T is of diameter 3 or 4, and we have to consider three cases (see Fig.10).

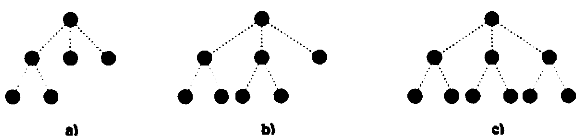

Fig. 10. The three cases for T where T' is of diameter 3 or 4.

If $n < 16$ then $\alpha(T) = n - 1$ unless T is one of the 7 exceptional trees in which case $\alpha(T) = n - 2$ [Bonnington, C.P., Širáň, J., 1999]. In any case, $\alpha(T) \geq \lfloor \frac{6n}{7} \rfloor - 1$. Therefore from now on we assume $n \geq 16$. It suffices to show that in all three cases we can delete at most three edges to obtain a forest whose all components have an α-labelling.

Case a). If the two adjacent vertices of degree 3 in T', say w and z, are joined by a path of length 3 in T, we delete the middle edge of this path. If w and z are joined in T by a path of length $\neq 3$, we delete an edge of the path incident with w or z. In either case, each of the resulting two components is either a snake or a Y-tree different from H_1. So the statement now follows from Lemma 2.

Case b). Let x, y, z be the 3 vertices of degree 3, with x adjacent to y and to z (but y and z nonadjacent) in T'. We delete an edge from the path

xy and an edge from the path xz in the following way. If x and y (or x and z, respectively) are joined in T by a path of length 3, we delete the middle edge of that path. Otherwise, we delete that edge of the path xy (or xz, respectively) which is incident with x. We get three components; each of them is either a snake or a Y-tree different from H_1. In all cases we get by applying Lemma 2 that the statement holds.

Case c). Let x, y, z, w be the four vertices of degree 3, with x adjacent to y, z and w in T' (y, z, w independent). If all three paths joining x to y (to z, and to w, respectively) in T are of length 3 (Fig.11a)), we delete the middle edge in each of these paths. The resulting four components are all Y-trees but none is H_1.

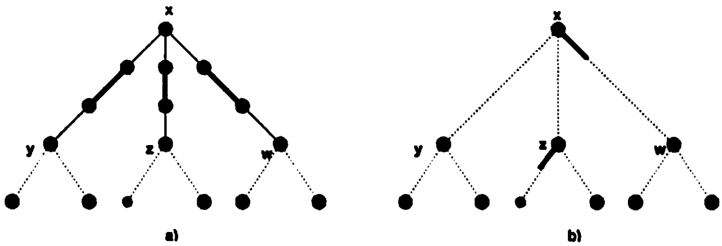

Fig. 11. **a)** Tree T in which all three paths (from x to y, z and w) are of length three. **b)** Tree T in which at least one of the paths xy, xz and xw is not of length three (we assume wlog that it is xw).

Otherwise (Fig.11b)), i.e. when at least one of these paths, say the path joining x to w in T is of length $\neq 3$, we delete the edge incident with x on this path, and also one of the edges adjacent to either y or z *not* on the path from x to y (or from x to z). Of the three resulting components, two are Y-trees (none of which is H_1), and one is a snake (possibly with no edges), or two are snakes and one is a Y-tree, different from H_1. As before, Lemma 2 is applied to complete the proof. □

We are now ready to proceed to our main result.

Theorem 1. *Let T be a tree with n vertices and with maximum degree three. Then* $\alpha(T) \geq \lfloor \frac{6n}{7} \rfloor - 1$.

Proof. If the tree T' that corresponds to the tree T has diameter at least 5, we delete an edge e in T such that $T - e = T_1 \cup T_{1R}$ where T_1 is a good tree; this is always possible by Lemma 1. If the tree T'_{1R} corresponding to T_{1R} is still of diameter at least 5, we then delete an edge e_1 in T_{1R} such that

$T_{1R} - e_1 = T_2 \cup T_{2R}$ where T_2 is a good tree. We continue this process m times, until the remaining tree T_{mR} is a tree whose corresponding tree T'_{mR} has diameter at most 4. Using Lemma 1 and Lemma 2 we first determine $\alpha(T_i), 1 \le i \le m$, to be

$$\alpha(T_i) \ge \sum_{j=1}^{\lfloor \frac{n_i-7}{7} \rfloor + 1} (n_{ij} - 1) = \lceil \frac{6n_i}{7} \rceil$$

where n_i is the number of vertices of T_i, and n_{ij} is the number of vertices of the component T_{ij} of the forest F obtained by deleting at most $\lfloor \frac{n_i-7}{7} \rfloor$ edges of T_i as described above.

From this inequality and Lemma 3 we get

$$\alpha(T) \ge \alpha(T_1) + \cdots + \alpha(T_m) + \alpha(T_{mR}) \ge \sum_{i=1}^{m} \lceil \frac{6|T_i|}{7} \rceil + \lfloor \frac{6|T_{mR}|}{7} \rfloor - 1 \ge \lfloor \frac{6n}{7} \rfloor - 1$$

and the proof is complete. □

5 Conclusion

The same technique as above could possibly be used to improve the lower bound for $\alpha(T)$, the α-size of a tree with maximum degree three, to $\alpha(T) \ge \lfloor \frac{7n}{8} \rfloor - 1$, however, the arguments would by necessity be much more complicated. The possibility suggested in [Bonnington, C.P., Širáň, J., 1999] that for trees T with maximum degree three $\alpha(T) \ge n - c$ may actually hold, for some integer constant $c \ge 1$.

References

Bonnington, C.P., Širáň, J.: Bipartite labeling of trees with maximum degree three. J. Graph Theory **31** (1999) 37–56

Gallian, J.: A dynamic survey of graph labelling, DS6, Electron. J. Combin. (2005) 148pp

Rosa, A.: On certain valuations of the vertices of a graph. Theory of Graphs, Proc. Internat. Sympos. Rome 1966. Dunod, Paris (1967) 349–355

Rosa, A.: Labelling snakes. Ars Combinat. **3** (1977) 67–74

Rosa, A., Širáň, J.: Bipartite labelings of trees and the gracesize. J. Graph Theory **19** (1995) 201–215