Critical sets for a pair of mutually orthogonal cyclic latin squares of odd order greater than 9

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Abstract

To date investigations on critical sets for a set of mutually orthogonal latin squares (MOLS) have been carried out only for small orders $\leq$ 9. In this paper we deal with a pair of cyclic orthogonal latin squares of order n, $n \geq 11$, n odd. Through construction of a uniquely completable set we give an upper bound on the size of the minimal critical set. In particular for $n = 15$ a critical set achieving this bound is obtained.

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1 Introduction

In the last few years, a number of authors have worked on problems concerning *critical* and / or *uniquely completable* (UC) sets for latin squares (cf. Nelder [10], Smetanuik [12], Curran and Van Rees [3], Cooper, Donovan and Seberry [1], Cooper, McDonough and Mavron [2], Donovan, Cooper, Nott and Seberry [4], Donovan and Cooper [5], Fu, Fu and Rodger [7], and Donovan and Howse [6]). There is not a lot known about the critical sets for latin squares in general. However, a class of critical sets is known for a *back circulant* latin square which is a particular latin square having the initial row in the standard form and subsequent rows formed by translating the previous row one element to the left. A generalisation of the problem of finding critical sets of minimum size in a single latin square is that of finding the minimum number of cell entries needed to be prescribed in the members of a set of two or more mutually orthogonal latin squares (MOLS) so that the cell entries in the entire set of squares can thereby be uniquely determined. To date very few results in this direction are known. Keedwell [8] was the first to attempt this problem. He made a preliminary investigation on the size of the minimal critical set for a set consisting of a back circulant latin square of order upto 7 and its k cyclic orthogonal mates. Subsequently SahaRay, Adhikari and Seberry [11] carried out further investigation for order 7 and characterised a critical set, thereby providing an improved upper bound on the size of the minimal critical set. They also studied the case of order 9. They observed that the critical set obtained in the case of order 9, when generalised for order n, n odd, $n \geq 11$ can be easily shown to be UC in general, and provide an upper bound on the size of the minimal critical set. However the construction does not provide a critical set for any n. With a view to obtain a finer upper bound we further consider the same problem in this paper. In Section 3 through characterisation of a UC set we attain this goal. This UC set is also shown to be a critical set for order 15. Moreover, using computer programming, we obtain UC sets of further smaller size for orders 11 and 13 which is given in section 4, but we omit the details of the completion steps because of very complicated branching arising out of the *weakly completable* nature of the UC set.

Before discussing the main results some background information is needed which is given in the next section.

2 Preliminary Definitions and Notations

In this section, we draw the readers' attention to the definitions and known results on critical sets for latin squares of order n which will be used hereafter to derive the main results. A latin square L of order n is an $n \times n$ array with entries chosen from a set N of size n such that each element of N occurs precisely once in each row and in each column. In what follows N is assumed to be $\{1, 2, \cdots, n\}$. For convenience a latin square L of order n is sometimes represented by a set of orderd triplets $\{(i, j, k)|$ element k occurs in the position (i,j), $i,j,k \in N\}$.

A *partial latin square* P of order n is an $n \times n$ array with entries chosen from N such that each element of N occurs atmost once in each row and in each column of P. Then $|P|$ is said to be the size of the partial latin square and the set of positions $S_P = \{(i,j)|\ (i,j,k) \in P,\ \exists\ k \in N\}$ is said to determine the shape of P. Let P and P' be two partial latin squares of the same order, with the same size and shape. Then P and P' are said to be *mutually balanced* if the entries in each row (and column) of P are the same as those in the corresponding row (and column) of P'. They are said to be *disjoint* if no position in P' contains the same entry as the corresponding position in P. A *latin interchange* (also referred to as *latin trade*, cf. Keedwell [9]) I is a partial latin square for which there exists another partial latin square I' of the same order, size and shape with the property that I and I' are disjoint and mutually balanced.

A *Uniquely completable set* (UC set) U of triplets is such that it yields only one latin square L of order n which has element k in position (i,j), for each $(i,j,k) \in U$. A set C is said to be a *critical set* if

1. C is a UC set, and
2. no proper subset of C satisfies 1.

A *minimal critical set* is a critical set of the smallest possible size.

If C is a UC or critical set, a triple $(i,j,k) \in L \setminus C$ will be said to be *forced*, if either $\forall h \neq i$, $\exists\ z$ such that $(h,j;z)$ or $(h,z;k) \in C$, or $\forall\ h \neq j$, $\exists\ z$ such that $(z,h;k)$ or $(i,h;z) \in C$, or $\forall\ h \neq k, \exists\ z$ such that $(i,z;h)$ or $(z,j;h) \in C$.

A UC set of cell entries for a set of MOLS is called *strong* if the cell entries in the entire set of squares can be successively filled by a sequence of adjunctions of cell entries to individual squares of the set each of which is forced.

A UC set which is not strong is called *weak*.

In this paper, we start with a back circulant latin square L_1 and deal with its cyclic orthogonal mate L_2 having its initial row $\{p_1, p_2, \ldots, p_n\}$ where $\{p_1, p_2, \ldots, p_n\}$ is any permutation of the symbols $\{1, 2, \ldots, n\}$ and try to identify a minimal critical set for the set $S = \{L_1, L_2\}$. However, to simplify notations, without loss of generality, we refer to p_j as j in our subsequent discussion. It is to be noted that all mathematical operations discussed in this paper are performed modulo n, however, we use symbol n instead of 0.

Definition 2.1 *We say a latin square L_t of order n is a cyclic latin square if its (i,j)th cell contains the entry $1+(i-1)t+(j-1)$, $t = 1, 2, \ldots, n-1$, $i,j \in N$.*

We now quote below the best known lower bound on the size of the critical sets for latin squares of order n, n odd, due to Cooper, Donovan and Seberry [1].

Lemma 2.2 *Let $n = 2m + 1$, for some positive integer m and*

$$\begin{aligned} C &= \{(i, j, i + j - 1) \mid i = 1, \ldots, (n-1)/2 \text{ and } j = 1, \ldots, (n-1)/2 - i + 1\} \\ &\bigcup \{(i, j, i + j - 1) \mid i = (n+1)/2 + 1, \ldots, n, \text{ and } j = (n+3)/2 - i, \ldots, n\} \end{aligned}$$

Then C is a critical set for a back circulant latin square of order n.

Now we define the special transversal pertinent to our discussion and introduce notations for any row or column of a $n \times n$ array.

Definition 2.3 *In an $n \times n$ array, a transversal is a collection of n cells $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ where $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ represent permutations of the numbers $\{1, 2, \ldots n\}$.*

Definition 2.4 *In an $n \times n$ array, for $i \in N$, the transversal $\{(1, i), (2, i-1), \ldots,$
$(n, i - n + 1)\}$ is termed as the ith reverse transversal denoted by T_i.*

We denote the *ith* row and the *jth* column of any $n \times n$ array by R_i and C_j respectively.
Remark 1 : It is evident that in the back circulant latin square L_1, the *i*th symbol occurs in the *i*th reverse transversal. Also note that L_2, has its first row in the standard form and subsequent rows are formed by translating the previous row 2 elements to the left.

3 Critical sets for a pair of Mutually Orthogonal Latin squares of odd order

In this section we deal with the set $S = \{L_1, L_2\}$ (vide Definition 2.1) of two cyclic MOLS of order n, n odd, $n \geq 11$ and identify a UC set, which is used to obtain an upper bound on the size of the minimal critical set for S. The following partial latin square of order 15, which will be shown to be a critical set for L_2, as a member of S can be generalised to construct the required UC set for any odd n, $n \geq 11$.

1		3		5		7		9		11				
3		5		7		9		11		13				
5		7		9		11		13						
7		9		11		13								
9		11		13										
		13												
													15	
											15		2	
									15		2		4	
							15		2		4		6	
					15		2		4		6		8	
					2		4		6		8		10	
					4		6		8		10		12	

*Fig.*1

Theorem 3.1 *Let L_1 be the back circulant latin square of order n, n odd, $n \geq 11$ and L_2 be its cyclic orthogonal mate translating two elements to the left. Then*

$$\begin{aligned} C &= \{(i\,,\ j,\ 2i+j-2):\ i=\frac{n-j}{2}-2, \frac{n-j}{2}-3,\ldots,1;\ \ j=1,3,5,\ldots,n-6\} \\ &\bigcup \{(i\,,\ j,\ 2i+j-2):\ i=\frac{n-j}{2}, \frac{n-j}{2}-1,\ \ j=3,\ 5,\ldots,n-4\} \\ &\bigcup \{(i,\ j,\ 2i+j-2):\ \ i=n,n-1,\ldots,n-\frac{j}{2}+1;\ \ \ j=6,8,\ldots,n-1\} \end{aligned}$$

is a UC set of size $\frac{n^2-1}{4}-6$ *for L_2 as a member of a set $S=\{L_1,\ L_2\}$ of two pairwise orthogonal cyclic latin squares.*

Proof : We start with a back circulant latin square L_1 and assume that it has been completed from a partial latin square of size $\frac{n^2-1}{4}$ given in Lemma 2.2. Not to obscure the essential steps for unique completion of L_2 as a member of $S=\{L_1, L_2\}$ we breakup C into two disjoint subsets consisting of odd and even columns. We define

$$\begin{aligned} C_{UT} &= \{(i\,,\ j,\ 2i+j-2):\ \ i=\frac{n-j}{2}-2, \frac{n-j}{2}-3,\ldots,1;\ \ \ j=1,3,5,\ldots,n-6\} \\ &\bigcup \{(i\,,\ j,\ 2i+j-2):\ i=\frac{n-j}{2}, \frac{n-j}{2}-1,\ \ j=3,\ 5,\ldots,n-4\} \end{aligned}$$

and

$$C_{LT} = \{(i,\ j,\ 2i+j-2):\quad i = n, n-1, \ldots, n-\frac{j}{2}+1;\quad j = 6, 8, \ldots, n-1\}$$

Obviously $C = C_{UT} \bigcup C_{LT}$.

The steps for completion of L_2 differ at some places depending on $n \equiv 1(mod\ 4)$ or $n \equiv 3(mod\ 4)$. So we discuss below the steps for unique completion for $n \equiv 1(mod\ 4)$ and whenever any deviation arrises for $n \equiv 3(mod\ 4)$ we mention that within bracket. We first make the following observations:

Fact 1 : In C_{UT},

x is present along the back transversals $T_{\frac{x+1}{2}},\ T_{\frac{x+3}{2}},\ \ldots, T_x$
for $x = 1,\ 3,\ \ldots,\ n-6$,

$n-4$ is present along $T_{\frac{n-1}{2}},\ T_{\frac{n+1}{2}}, \ldots, T_{n-4}$, and

$n-2$ is present along $T_{\frac{n+1}{2}},\ T_{\frac{n+3}{2}}, \ldots, T_{n-3}$.

Fact 2 : In C_{LT},

x is present along $T_{x+1},\ T_{x+2},\ \ldots,\ T_{\frac{n+x-1}{2}}$
for $x = 4, 6, \ldots, n-1$,

2 is present along $T_4,\ T_5,\ \ldots,\ T_{\frac{n+1}{2}}$,

n is present along $T_3,\ T_4, \ldots, T_{\frac{n-1}{2}}$.

Fact 3 : In L_1,

i occurs along the back transversal T_i, $i = 1, 2, \ldots, n$, and hence orthogonality of L_2 to L_1 demands that in L_2 all the elements along the back transversal T_i should be different. Moreover, being a latin square means that all the elements along row R_i and column C_j of L_2 should be different, $i, j \in \{1, 2, \ldots, n\}$.

Noting these three points we argue as follows towards completion of a partial latin square L_2' to L_2 , L_2' having entries prescribed by C. We assume below that $\mathrm{n} = 4\mathrm{t}+1,\ \mathrm{t} \geq 4\ (4\mathrm{t}+3,\ \mathrm{t} \geq 3)$. The cases of $n = 11$ and 13 are dealt with differently, as the general rule does not go through in these two cases.

Step 1 : Using Fact 1 and Fact 3 noted above, x is placed in L_2' uniquely in $C_{n-1}, C_{n-3}, \ldots, C_2$,
$C_n, C_{n-2}, \ldots, C_{x+2}$ in order, for $x = n-6,\ n-8,\ \ldots,\ 2t-1\ (2t+1)$.

Step 2 : Using Fact 2 and Fact 3 noted above, x is placed uniquely in $C_1,\ C_3,\ \ldots,\ C_n$,
$C_2,\ C_4,\ \ldots,\ C_x$ in order, for $x = 4,\ 6,\ \ldots, 2t-2$.

Step 3 : Now $n-2$ can be assigned in C_{n-1} in one of the three available cells viz.

(a) $(1,\ n-1)$,

(b) $(\frac{n-1}{2},\ n-1)$

and (c) $(\frac{n+1}{2},\ n-1)$.

We will argue below that the cases (a) and (b) are infeasible. To this end, we start with the case (a), i.e we assume that $n-2$ is placed in $(1,\ n-1)$, then $n-2$ can be placed in C_{n-3} in one of the two available cells viz.

(a1) $(\frac{n+1}{2},\ n-3)$
and (a2) $(\frac{n+3}{2},\ n-3)$,
both of which will turn out to be infeasible in the course of our reasoning. Now to establish this, we start with (a) along with (a1) i.e assignment of $n-2$ in C_{n-1} at $(1,\ n-1)$ and in C_{n-3} at $(\frac{n+1}{2},\ n-3)$. Then $n-2$ is placed uniquely in $C_{n-5},\ C_{n-7},\ \ldots,\ C_{2t+2}\ (C_{2t+4})$ in order, proceeding downwards in the process of filling, $C_{2t}\ (C_{2t+2})$ is the first column to arise, where two places for $n-2$ are available. Now using Fact 1 for $n-2$ and the T_i's along which $n-2$ has been placed so far in alternate columns starting C_{n-1}, it can be easily seen that there is no place left for $n-2$ in C_{n-2}. Hence (a) with (a1) cannot happen. Now to settle that (a) with (a2) also cannot happen we argue along the same lines. If (a) and (a2) occur together then $n-2$ can be placed uniquely in $C_{n-5},\ C_{n-7},\ldots,\ C_{2t+2}\ (C_{2t+4})$ leaving no place for $n-2$ in C_{n-2}.

Now we eliminate the possibility (b) i.e placement of $n-2$ in $(\frac{n-1}{2},\ n-1)$. Suppose that (b) holds, i.e. $n-2$ is placed in $(\frac{n-1}{2},\ n-1)$. Then using Fact 1 as before, $n-2$ can be placed uniquely in $C_{n-3},\ C_{n-5},\ \ldots,\ C_{2t+2}\ (C_{2t+4})$ in order, and then in C_{n-2} at $(1, n-2)$. Now clearly $n-4$ and $2t-3\ (2t-1)$ can be placed sequentially in order, in C_{n-1} and then in $C_{n-3},\ \ldots,\ C_{2t+4}\ (C_{2t+6})$ in turn. Considering Fact 1 and T_i's along which $2t-3\ (2t-1)$ has been placed sequentially in alternate columns starting with C_{n-1}, we see that $2t-3\ (2t-1)$ can be placed uniquely in the rest of the columns in the order $C_{2t+2}\ (C_{2t+4}),\ C_{2t}\ (C_{2t+2}),\ \ldots,\ C_2,\ C_n,\ C_{n-2},\ \ldots,\ C_{2t-1}\ (C_{2t+1})$. Now going back to the placement of $n-2$ again, using Fact 1 and the T_i's along which $n-2$ has been placed so far, it follows that $n-2$ can be placed uniquely in $C_{2t}\ (C_{2t+2}),\ C_{2t-2}\ (C_{2t})\ \ldots,\ C_2,\ C_n$ in order. As a result, a contradiction arises since no place is left for $n-2$ in C_1. Hence possibilities (a) and (b) are eliminated and $n-2$ is fixed in C_{n-1} at $(\frac{n+1}{2},\ n-1)$.

Step 4 : Now $n-2$ is placed uniquely in $C_{n-3},\ C_{n-5},\ \ldots, C_{2t+2}\ (C_{2t+4})$ and then in C_{n-2} at $(1, n-2)$.

Step 5 : Now $n-4$ and $2t-3\ (2t-1)$ are fixed sequentially in order in C_{n-1} and then in $C_{n-3},\ldots, C_{2t+4}\ (C_{2t+6})$ in turn.

Step 6 : Now using the T_i's along which $2t-3\ (2t-1)$ is present in C_{UT} and also in the alternate columns starting with C_{n-1}, place $2t-3\ (2t-1)$ uniquely in $C_{2t+2}\ (C_{2t+4}),\ C_{2t}\ (C_{2t+2})$,
$\ldots, C_2,\ C_n,\ C_{n-2},\ldots,\ C_{2t-1}\ (C_{2t+1})$.

Step 7 : Now fix $n-2$ uniquely in $C_{2t}\ (C_{2t+2}),\ C_{2t-2}\ (C_{2t}),\ \ldots,\ C_2,\ C_n,\ C_1$.

Step 8: Now $n-4$ is fixed in $C_{2t+2}\ (C_{2t+4})$.

Step 9 : Now fix $2t-5\ (2t-3)$ in $C_{n-1},\ C_{n-3},\ \ldots, C_2,\ C_n,\ C_{n-2},\ \ldots,$ $C_{2t-3}\ (C_{2t-1})$.

Step 10 : Now $n-4$ is placed uniquely in C_{2t} (C_{2t+2}), C_{2t-2} (C_{2t}), $\dots, C_2$, C_n, C_{n-2} and finally in C_1.
Step 11 : Now x is placed uniquely in $C_{n-1}, C_{n-3}, \dots, C_2, C_n, C_{n-2}, \dots$, C_{x+2} where x is odd, $x \neq 1$, $x = 2t-7$ $(2t-5)$, $2t-5$ $(2t-3)$, $\dots$, 3.
Step 12 : Now $C_{n-1}, C_{n-3}, \dots, C_6$ can be completed uniquely in order starting from R_1 and moving downwards.
Step 13 : Now we try to place 1 in C_4. Clearly, there are three places in C_4 available for 1, viz. $(\frac{n-1}{2}, 4)$, $(n-1, 4)$ and $(n, 4)$. If 1 is placed in $(n-1, 4)$ then no place is left for 1 in R_n. Similarly if 1 is placed at $(n, 4)$, 1 can not be placed in R_{n-1}. Hence 1 is placed in C_4 at $(\frac{n-1}{2}, 4)$ and henceforth in $C_2, C_n, C_{n-2}, \dots, C_3$.
Step 14 : Now using the fact that $2t$ is present in C_{LT} along $T_{\frac{n+1}{2}}, \dots$, $T_{\frac{n+2t-1}{2}}$, $2t$ is uniquely placed in $C_1, C_3, \dots, C_n, C_2, C_4$.
Step 15 : We now try to position 2. In C_2 only two places are available for 2 viz. $(1, 2)$ and $(n, 2)$ as the other empty cells in C_2 fall along the transversals $T_{t+2}, T_{t+3}, \dots, T_{\frac{n+1}{2}}$, $t \geq 3$ and 2 is already present in C_{LT} along $T_4, T_5, \dots, T_{\frac{n+1}{2}}$. If 2 is in $(n, 2)$ then in C_1 the only place available for 2 is $(\frac{n+3}{2}, 1)$ as 2 is already along $T_{\frac{n+1}{2}}$ and in the other rows of C_1. Now clearly no place for 2 is available in C_4 and hence 2 is fixed in $(1, 2)$ and consequently n is fixed in $(1, n)$. Hence in R_2, n is placed uniquely in $(2, n-2)$ and consequently 2 is placed in $(2, n)$. Thus in R_n, n is placed uniquely in $(n, 2)$ because of presence of n along T_n and T_3.
Step 16 : Now n is fixed in R_{n-1} at $(n-1, 4)$.
Step 17 : Now considering the possibilities of 2 in R_n we find that 2 can be at $(n, 1)$ or $(n, 4)$. To eliminate the placement of 2 in $(n, 1)$ we argue as follows. Suppose that 2 is placed in $(n, 1)$. Now we attempt to assign 2 in $R_{\frac{n+3}{2}}$. It is seen at this stage that in $R_{\frac{n+3}{2}}$ the empty cells are $(\frac{n+3}{2}, 1)$, $(\frac{n+3}{2}, 2t+1)$, $(\frac{n+3}{2}, 2t+3)$, $\dots$, $(\frac{n+3}{2}, n-2)$ which are along $C_1, T_1, T_3, T_5, \dots, T_{2t-1}$ respectively, whenever $n = 4t+1$. Thus we see that for $n = 4t+1$, 2 can be placed uniquely in $R_{\frac{n+3}{2}}$ at $(\frac{n+3}{2}, 2t+3)$ along T_3. Now because of the presence of 2 along T_n, 2 is placed in R_3 uniquely in $(3, n-4)$ and henceforth in $R_4, R_5, \dots, R_{t-1}$ uniquely, leading to a contradiction in placement of 2 in R_{t-1}. (Whenever $n = 4t+3$ the empty cells in $R_{\frac{n+3}{2}}$ are along $C_1, T_n, T_2, T_4, \dots, T_{2t}$ and thus it readily leads to a contradiction in the placement of 2 in $R_{\frac{n+3}{2}}$ as 2 is already in C_1 and along these transversals.) Hence in R_n, 2 should be placed at $(n, 4)$.

L_2' with the entries fixed so far can now be permuted to $\tilde{L}_2'$ which has the standard form of Cooper, Donovan and Seberry [1] and Smetanuik [12] where we now have a critical set in the back circulant latin square as given in Lemma 2.2. This allows us to uniquely complete $\tilde{L}_2'$ and reversing the permutations it gives back L_2 completed from L_2'.

Now we deal with the case $\mathbf{n} =$ **13.**

Place x uniquely in $C_{12}, C_{10}, \ldots, C_2, C_{13}, C_{11}, \ldots, C_{x+2}$ in order, for $x = 7$, 5. Then position 4 in C_1, $C_3, \ldots$, C_{13}, C_2, C_4 sequentially. In C_{12}, 11 can now occur in three places viz. (1,12), (6,12), (7,12). Now 11 in (1,12) leads to two possibilities for 11 in C_{10} viz. (7,10) and (8,10). Placement of 11 in (7,10) is followed by unique placement of 11 in C_8, C_{11}, C_6 leading to a contradiction of placement of 11 in C_{13}. Again placement of 11 in (8,10) is followed by placement of 11 in C_8 leading to a contradiction in placement of 11 in C_{11}. Thus choice of 11 in C_{12} at (1,12) is eliminated. Similarly choice of 11 in C_{12} at (6,12) is infeasible because it leads to the unique placement of 9 in C_{12} and C_{10}, then 3 in C_{12} and C_{10}, 11 in C_{10}, 3 and 11 simulteneously in order in C_8, C_6, C_4, C_2 sequentially, then 11 in C_{13} leaving no place for 11 in C_{11}. Thus 11 in C_{12} is fixed at (7,12). Now, in turn, 11 is fixed in C_{10} and C_8, 9 in C_{12} and C_{10}, 3 in C_{12}, C_{10}, C_8, 9 in C_8, 3 in C_6, 11 in C_{11}, C_6, C_4, C_2, C_{13}, C_1, 1 in C_{12}, C_{10}, C_8, C_6, 9 in C_6, C_4, C_2, C_{13}, C_{11} and finally in C_1. Then we follow step 11 onwards as discussed above for $n = 4t + 1$ to complete L_2 uniquely.

In the Appendix, we present the details through a tree structure for $\mathbf{n} =$ **11** to allow easy comprehension for completion as the weakly completable nature of the UC set causes considerable minor details. ■

Now Lemma 2.2 and Theorem 3.1 provide an upper bound for the size of the minimal critical set for $S = \{L_1, L_2\}$.

Theorem 3.2 *Let L_1 be the back circulant latin square of order n, n odd, $n \geq 11$ and L_2 be its cyclic orthogonal mate translated two elements to the left. Then the size of the minimal critical set for $S = \{L_1, L_2\}$ is atmost* $\frac{n^2-1}{2} - 6$.

Proof: From Lemma 2.2 it follows that the back circulant latin square L_1 has a critical set of size $\frac{n^2-1}{4}$. It follows from Theorem 3.1 that the size of the critical set for L_2 as a member of S is atmost $\frac{n^2-1}{4} - 6$ as a critical set is embedded in a UC set. Hence for joint completion of L_1 and L_2 the size of the minimal critical set is atmost $\frac{n^2-1}{2} - 6$. ■

Now we show that the UC set C given in Theorem 3.1 is indeed a critical set for L_2 for $n =$ 15.

Theorem 3.3 *For $n =$* 15 *the set C given in Theorem 3.1 is a critical set for L_2 as a member of $S = \{L_1, L_2\}$.*

Proof: Now to prove that the UC set C in Fig. 1 is a critical set for L_2, we show that for each $(i,j,k) \in C$, there exist two latin interchanges L_{2t}, $t = 1,2$ in L_2 satisfying

$$C \bigcap L_{2t} = \{(i,j,k)\}, \quad t = 1,2.$$

It is easy to verify that for any $(i, j, k) \in C$, L_2 contains a partial latin square L_{21} of the form

$$L_{21} = \{(i,j,k),(i,j+\alpha,k+\alpha),(i+\alpha,j+\alpha,k),(i+\alpha,j+2\alpha,k+\alpha), (i+2\alpha,j+2\alpha,k),(i+2\alpha,j,k+\alpha)\},$$

which can be replaced by another partial latin square L_{22} of the form

$$L_{22} = \{(i,j,k+\alpha),(i,j+\alpha,k),(i+\alpha,j+\alpha,k+\alpha),(i+\alpha,j+2\alpha,k), (i+2\alpha,j+2\alpha,k+\alpha),(i+2\alpha,j,k)\},$$

yielding a different latin square $\tilde{L}_2$ orthogonal to L_1 where,

$$\alpha = 5 \quad \text{for } (i,j,k) \in C_{UT} \setminus \{(1, 11, 11), (2, 11, 13), (6, 3, 13)\} \bigcup \{(9, 14, 15), (10, 12, 15), (10, 14, 2)\}$$

and

$$\alpha = 10 \quad \text{for } (i,j,k) \in C_{LT}. \setminus \{(9, 14, 15), (10, 12, 15), (10, 14, 2)\} \bigcup \{(1, 11, 11), (2, 11, 13), (6, 3, 13)\}.$$

Orthogonality follows from the fact that (i, j, k) and $(i+\alpha, j+2\alpha, k+\alpha)$ fall along T_{i+j-1}, similarly $(i, j+\alpha, k+\alpha)$ and $(i+2\alpha, j+2\alpha, k)$ fall along $T_{i+j+\alpha-1}$ and $(i+\alpha, j+\alpha, k)$ and $(i+2\alpha, j, k+\alpha)$ fall along $T_{i+j+2\alpha-1}$ where k and $k+\alpha$ can be interchanged.
Thus if we remove any element from C then we can complete the subset to at least two latin squares orthogonal to L_1 each of which has one of the partial latin squares given above. So C with size 50 turns out to be a critical set for L_2. ■

Remark 2: Note that using orthogonality with L_1 the size of the critical set for L_2 can be reduced by 6 from 56, the size of the critical set of smallest size known so far, as given in Lemma 2.2 due to Cooper, Donovan and Seberry [1]. It is to be noted that there does not exist any cyclic latin square orthogonal to both L_1 and L_2.

4 UC set of further smaller size for $n = 11$ and 13

It is interesting to note that for $n = 11$ and 13 we obtain through computer verification UC sets of further smaller size, reduced three more elements in each case. As these sets are weakly completable, due to various branching at different stages we omit the deatils of the steps for completion, however we present below the computer output. Whether these sets are critical are yet to be verified.

1		3		5						
3		5		7		9				
5		7								
									11	
							11		2	
					11		2		4	
					2		4		6	
					4		6		8	

UC set for $n =$ 11, *Fig*.2

1		3		5		7		9				
3		5		7		9		11				
5		7		9								
7		9										
											13	
									13		2	
							13		2		4	
					13		2		4		6	
					2		4		6		8	
					4		6		8		10	

UC set for $n =$ 13, *Fig*.3

5 Concluding Remarks

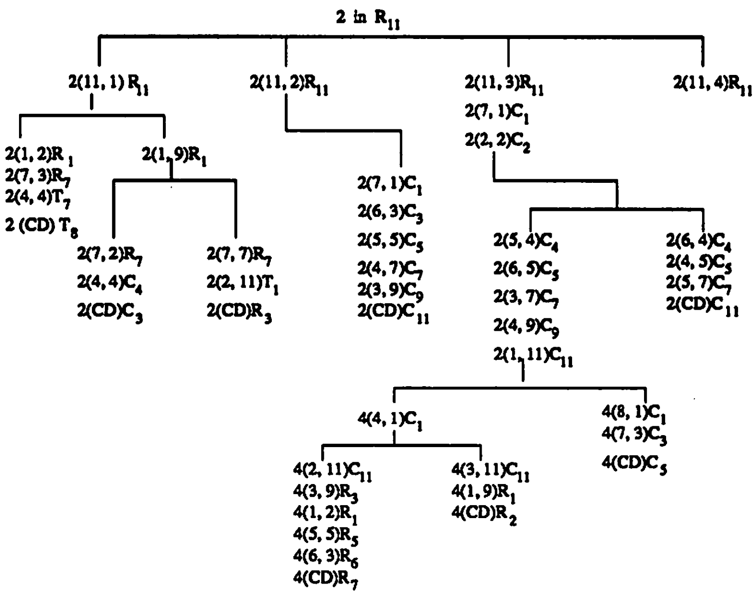
We strongly believe that as n grows , n prime, $n > 15$ there is a UC set of smaller size but to write down the steps for unique completion in general, becomes very much more complicated. Investigation is going on as to whether we can arrive at a reduction in the required size of the UC set depending on n.

Appendix:

We now present the steps for completion of L_2 for $n = 11$. We use the notation i $(j,\ k)$ Q where $Q = R_j$ or C_k or T_l, $i,\ j,\ k,\ l \in \{1, 2, \ldots, 11\}$ to denote that i is placed in R_j or C_k or along T_l in $(j,\ k)$ respectively. Let $i(CD)Q$ denote that a contradiction arises in the placement of i in Q.

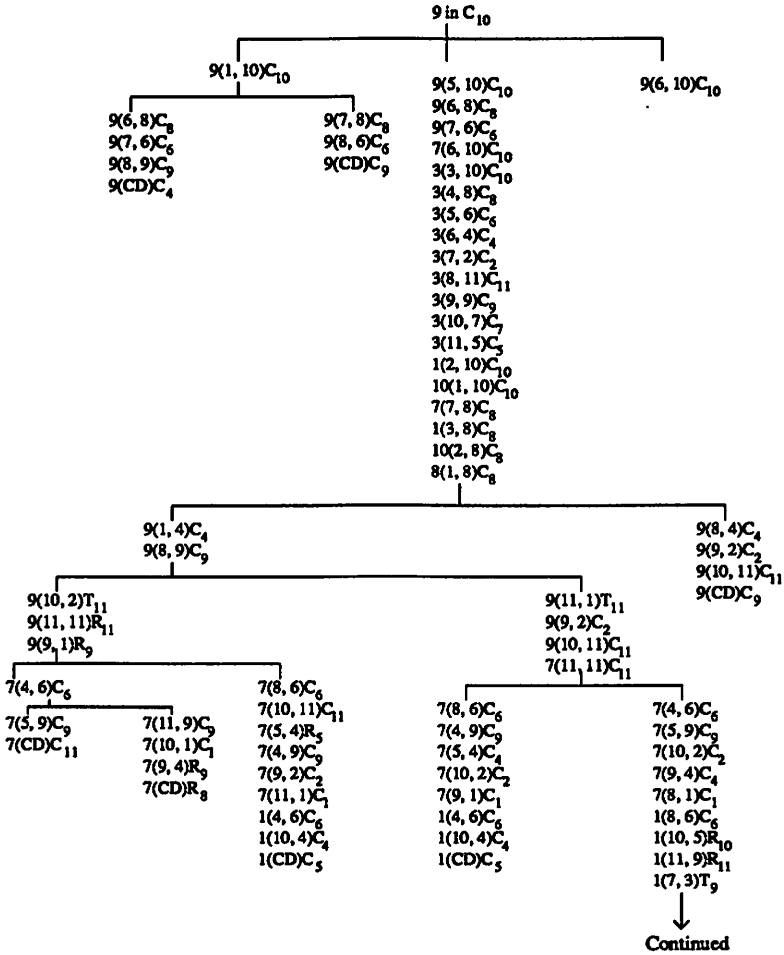
Step 1: 5 is placed in L_2' uniquely in C_{10}, $C_8, \ldots,$ C_2, C_{11}, C_9, C_7 in order.

Step 2: Now 2 can be assigned in R_{11} in one of the four places viz. (11,1), (11,2), (11,3) and (11,4). We argue below through a tree structure of completion steps that the only feasible place for 2 in R_{11} is (11,4).



Hence 2 is fixed at (11,4) in R_{11} and subsequently in C_1, C_3, $\ldots,$ C_{11} and C_2 in order.

Step 3: Now we try to place 9 in C_{10}. Clearly there are three places in C_{10} available for 9, viz. (1,10), (5, 10), and (6,10). The following tree of completion steps eliminate the possibility of placement of 9 in (1,10) and (5,10).


9 in C_{10}
9(1, 10)C_{10}
9(5, 10)C_{10}
9(6, 10)C_{10}
9(6, 8)C_8
9(7, 6)C_6
9(8, 9)C_9
9(CD)C_4
9(7, 8)C_8
9(8, 6)C_6
9(CD)C_9
9(6, 8)C_8
9(7, 6)C_6
7(6, 10)C_{10}
3(3, 10)C_{10}
3(4, 8)C_8
3(5, 6)C_6
3(6, 4)C_4
3(7, 2)C_2
3(8, 11)C_{11}
3(9, 9)C_9
3(10, 7)C_7
3(11, 5)C_5
1(2, 10)C_{10}
10(1, 10)C_{10}
7(7, 8)C_8
1(3, 8)C_8
10(2, 8)C_8
8(1, 8)C_8
9(1, 4)C_4
9(8, 9)C_9
9(8, 4)C_4
9(9, 2)C_2
9(10, 11)C_{11}
9(CD)C_9
9(10, 2)T_{11}
9(11, 11)R_{11}
9(9, 1)R_9
9(11, 1)T_{11}
9(9, 2)C_2
9(10, 11)C_{11}
7(11, 11)C_{11}
7(4, 6)C_6
7(5, 9)C_9
7(CD)C_{11}
7(11, 9)C_9
7(10, 1)C_1
7(9, 4)R_9
7(CD)R_8
7(8, 6)C_6
7(10, 11)C_{11}
7(5, 4)R_5
7(4, 9)C_9
7(9, 2)C_2
7(11, 1)C_1
1(4, 6)C_6
1(10, 4)C_4
1(CD)C_5
7(8, 6)C_6
7(4, 9)C_9
7(5, 4)C_4
7(10, 2)C_2
7(9, 1)C_1
1(4, 6)C_6
1(10, 4)C_4
1(CD)C_5
7(4, 6)C_6
7(5, 9)C_9
7(10, 2)C_2
7(9, 4)C_4
7(8, 1)C_1
1(8, 6)C_6
1(10, 5)R_{10}
1(11, 9)R_{11}
1(7, 3)T_9
Continued

$1(6,11)C_{11}$
$1(9,7)R_9$
$1(5,2)R_5$
$1(4,4)R_4$
$4(4,1)C_1$
$4(8,3)C_3$
$4(2,2)C_2$
$4(5,4)C_4$
$4(7,5)C_5$
$4(3,11)C_{11}$
$4(1,9)C_9$
$4(6,7)C_7$
$6(9,3)C_3$
$6(1,6)C_6$
$6(5,1)C_1$
$6(3,2)C_2$
$6(4,11)C_{11}$
$6(2,9)C_9$
$6(7,7)C_7$
$6(8,5)C_5$
$6(CD)C_4$

$1(5,11)C_{11}$
$1(CD)R_9$

Hence 9 is fixed at (6,10) in C_{10} and subsequently in C_8 at (7,8).
Step 4: Now 3 is fixed uniquely in C_{10}, C_8, ..., C_2, C_{11}, C_9, ..., C_5 in order.
Step 5: Now 9 is placed uniquely in C_6, C_9, C_4 , C_2, C_{11}, C_1 in order.
Step 6: Now 7 is uniquely placed in C_{10} and C_8 in order.
Step 7: Position 1 uniquely in C_{10}, C_8 and C_6 in order.
Step 8: Now 7 is placed uniquely in C_6, C_4, C_2, C_{11}, C_9, C_1 in order.
Step 9: Complete C_{10}, C_8 and C_6 sequentially.
Step 10: Now 4 is positioned uniquely in C_1, C_3, ..., C_{11}, C_2, C_4 in order. So R_1 is fixed containing 11 at (1,11).
Step 11: 11 is placed uniquely in R_2 at (2,9).
Step 12: 1 is placed uniquely in R_{11} at (11,3).
Step 13: 11 is fixed at (11,2) in R_{11} and hence is placed at (10,4) in R_{10}.

Now as before, following the arguments given after step 17 in the proof of Theorem 3.1, L_2' can be uniquely completed to L_2.

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