VERTEX MAGIC TOTAL LABELLINGS OF COMPLETE MULTIPARTITE GRAPHS

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ABSTRACT. A Vertex Magic Total Labelling of a graph G is a one to one map λ from $E(G) \cup V(G)$ onto the set of integers $\{1, 2, ..., e+v\}$ such that for all $x \in V$ we have $\lambda(x)+\sum \lambda(xy) = h$ for some constant h, where the sum is taken over all vertices y adjacent to x In this paper we present several theorems on the existence of such labellings for multipartite graphs and give constructions for labellings for two infinite families of complete tripartite graphs, namely $K_{1,n,n}$ for odd n and $K_{2,n,n}$ for $n \equiv 3 \pmod 4$.

1. INTRODUCTION

Throughout we will denote the vertex set of a graph G by $V(G)$ and the edge set of G by $E(G)$. A Vertex Magic Total Labelling (VMTL) of a graph G is a one to one map λ from $V(G) \cup E(G)$ onto the set of integers $\{1, 2, ..., e+v\}$ such that for all $x \in V$ we have $\lambda(x) + \sum \lambda(xy) = h$, for some constant h, where the sum is taken over all vertices y adjacent to x and is called the weight of the vertex x, denoted $wt(x)$. The constant h is called the 'magic constant' of the graph and a graph which has a VMTL will be referred to as 'vertex magic'. These labellings were introduced in [1] where their basic properties were described and constructions provided for labellings of several families of graphs. For an overview of results on VMTL's we refer the reader to [2]. The problem of finding vertex magic labellings for the complete bipartite graphs was studied by Gray, MacDougall, Simpson and Wallis in [3]. A complete solution was given there in terms of the following theorem.

Theorem 1. *The complete bipartite graph $K_{m,n}$ (where $m \leq n$) has a VMTL if and only if $m \geq n-1$ (i.e. only for $K_{n,n}$ and $K_{n-1,n}$).*

For the proof we direct the reader to [2] or [3].

In what follows, we look for a natural generalisation of this theorem, namely in the context of complete multipartite graphs. We derive some conditions under which VMTL's cannot exist. On the other hand, we show

how to construct VMTL's for the complete tripartite graphs $K_{1,n,n}$ and $K_{2,n,n}$.

2. Multipartite Graphs That Cannot Be Labelled

Theorem 1 above provides both necessary and sufficient conditions for labelling $K_{m,n}$. We are able to prove a generalisation of the 'only if' part of the theorem, which applies to a large family of graphs. This is given in *Theorem 2* below. A further result applying to multipartite graphs is given in *Corollary 3.*

Before giving *Theorem 2* we remind the reader that the graph G formed from the product $G = H_1 \vee H_2$ is the graph with $V(G) = V(H_1) \cup V(H_2)$ and $E(G) = E(H_1) \cup E(H_2) \cup S$ where S is the set of edges connecting every vertex of H_1 to every vertex of H_2.

Theorem 2. *If H is any graph such that $G = H \vee \overline{K_n}$ then G can only be vertex magic if $|V(H)| \geq n - 1$.*

Proof. Let $|V(H)| = m$, $|E(H)| = s$, $V(H) = \{x_1, ..., x_m\}$ and $V(\overline{K_n}) = \{y_1, ..., y_n\}$ We have $m + n$ vertices and $s + nm$ edges in G for a total of $m+n+s+nm$ labels. The largest possible sum on the $\{y_1, ..., y_n\}$ vertices is obtained by using the largest $n + nm$ labels on those vertices and their incident edges. So

$$nh \leq \sum_{k=m+s+1}^{n+mn+m+s} k.$$

The smallest possible sum on the $\{x_1, ..., x_m\}$ vertices is obtained by using the smallest $m + s + mn$ labels on those vertices and their incident edges, however we must note that the s edges are counted twice in h and so we should assign the smallest possible labels to these. Consequently we have that

$$mh \geq \sum_{k=1}^{s} k + \sum_{k=1}^{m+mn+s} k.$$

This gives us the following two inequalities:

$$nh \leq \frac{(n + nm + m + s)(n + nm + m + s + 1) - (m + s)(m + s + 1)}{2},$$

$$mh \geq \frac{s(s + 1) + (m + mn + s)(m + mn + s + 1)}{2}.$$

Combining these we get

$$\frac{s(s+1)+(m+mn+s)(m+mn+s+1)}{m} \leq \frac{(n+nm+m+s)(n+nm+m+s+1) - (m+s)(m+s+1)}{n}.$$

Solving for n we get

$$n \leq \frac{m^2 - 2s + \sqrt{m^4 + 8m^3 + 4m^2(s+2) - 4s(s+2)}}{2m}.$$

Since the discriminant $< (m^2+4m+2s)^2$ we can substitute $(m^2+4m+2s)^2$ into the above expression to get the strict inequality

$$\begin{aligned} n &< \frac{m^2 - 2s + \sqrt{(m^2+4m+2s)^2}}{2m} \\ &= \frac{m^2 - 2s + m^2 + 4m + 2s}{2m} \\ &= m+2, \end{aligned}$$

i.e.

$$m \geq n-1$$

or

$$|V(H)| \geq n-1.$$

□

We note that since this theorem does not use the structure of H in any way, stronger results may hold for particular choices of H. We also note that, as a consequence of *Theorem 2*, $H \vee \overline{K_n}$ is not vertex magic for $n-1 < |V(H)|$.

Corollary 3. *The complete multipartite graph $K_{m_1,m_2,\ldots,m_r,n}$ can be vertex-magic only if*

$$\sum_{i=1}^{r} m_i \geq n-1.$$

Proof. Since a complete multipartite graph has the property of G in the preceding theorem the result follows immediately. □

In particular, for the tripartite graph $K_{k,m,n}$ to be vertex-magic we need $m+k \geq n-1$. No stronger result can be obtained by these kinds of arguments since there are known labellings for cases where $m+k = n-1$. By way of example a labelling for $K_{1,1,3}$ is shown in Figure 1. Computer searches for labellings of several other graphs close to the cut off point have also been successful. For example, $K_{1,2,4}$ as shown in Figure 2.

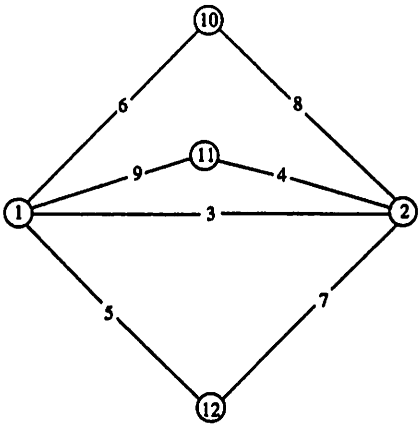

FIGURE 1. A vertex magic labelling for $K_{1,1,3}$ with magic constant $h = 24$.

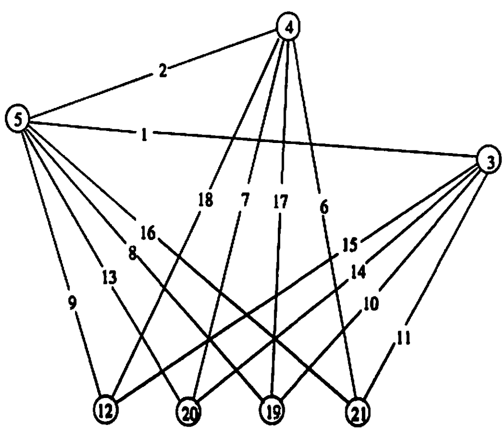

FIGURE 2. A vertex magic labelling for $K_{1,2,4}$ with magic constant $h = 54$.

Open Question: *Do all graphs satisfying Theorem 2 have VMTL's?*

3. Constructions of Labellings for Tripartite Graphs

The '*if*' part of *Theorem 1* was proved by constructing labellings for the permissable cases. An analogous result for multipartite graphs will necessarily be far more difficult, however there are certain families of tripartite graphs for which we have found labellings. It seems likely that no universal construction exists for those $K_{k,m,n}$ which are vertex magic. Here we present some constructions for two families of complete tripartite graphs.

Theorem 4. *The tripartite graph $G = K_{1,n,n}$ has a vertex magic labelling with magic constant $h = \frac{1}{2}(n^3 + 6n^2 + 9n + 2)$ when n is odd.*

Proof. Let us denote the vertices and edges of G by:

$$\begin{aligned} V(G) &= \{x_i, y_j, z \mid 1 \le i \le n,\ 1 \le j \le n\}, \\ E(G) &= \{x_i y_j, x_i z, y_j z \mid 1 \le i \le n, 1 \le j \le n\}. \end{aligned}$$

Throughout the proof i and j run from $1 \ldots n$.

We construct a labelling for G, in which the labels $1, .., n^2 + 4n + 1$ have been used exactly once each. Let A be an $n \times n$ magic square using the integers $\{2n + 1, ..., n^2 + 2n\}$. The magic constant for this square is given by

$$a = \frac{1}{2}(n^3 + 4n^2 + n).$$

We use $a_{i,j}$ to denote the i, jth entry of A and construct the labelling as follows:

$$\begin{aligned} \lambda(z) &= n^2 + 4n + 1, \\ \lambda(x_i) &= n^2 + 4n + 1 - i, \\ \lambda(y_j) &= \begin{cases} n^2 + 3n + 1 - j & \text{if } j \text{ is odd} \\ 2n + 1 - j & \text{if } j \text{ is even,} \end{cases} \\ \lambda(x_i y_j) &= a_{ij}, \\ \lambda(x_i z) &= i, \\ \lambda(y_j z) &= \begin{cases} n + j & \text{if } j \text{ is odd} \\ n^2 + 2n + j & \text{if } j \text{ is even.} \end{cases} \end{aligned}$$

The weights on the x_i vertices are given by

$$\begin{aligned} wt(x_i) &= \lambda(x_i) + \sum_{j=1}^{n} \lambda(x_i y_j) + \lambda(x_i z) \\ &= n^2 + 4n + 1 - i + a + i \\ &= n^2 + 4n + 1 + \frac{1}{2}(n^3 + 4n^2 + n) \\ &= \frac{1}{2}(n^3 + 6n^2 + 9n + 2) \\ &= h. \end{aligned}$$

The weights on the y_j vertices are given by

$$\begin{aligned} wt(y_j) &= \lambda(y_j)+\sum_{i=1}^{n}\lambda(x_iy_j)+\lambda(y_jz) \\ &= \begin{cases} n^2+3n+1-j+a+n+j & \text{if } j \text{ is odd} \\ 2n+1-j+a+n^2+2n+j & \text{if } j \text{ is even} \end{cases} \\ &= n^2+4n+1+\tfrac{1}{2}(n^3+4n^2+n) \\ &= \frac{1}{2}(n^3+6n^2+9n+2) \\ &= h. \end{aligned}$$

Finally the weight on the single vertex z is given by

$$\begin{aligned} wt(z) &= \lambda(z)+\sum_{i=1}^{n}\lambda(x_iz)+\sum_{j=1}^{n}\lambda(y_jz) \\ &= n^2+4n+1+\sum_{i=1}^{n}i+\sum_{j=1}^{\frac{1}{2}(n+1)}(n-1+2j)+\sum_{j=1}^{\frac{1}{2}(n-1)}(n^2+2n+2j) \\ &= n^2+4n+1+\frac{1}{2}n(n+1)(\frac{3}{4}n^2+n+\frac{1}{4})+(\frac{1}{2}n^3+\frac{3}{4}n^2-n-\frac{1}{4}) \\ &= \frac{1}{2}(n^3+6n^2+9n+2) \\ &= h. \end{aligned}$$

□

Unfortunately the same construction doesn't work for even n but perhaps something similar could work. It would be nice to find labellings for all graphs of the form $K_{m,n,n}$, however these constructions apparently become ever more complex as m increases. By example we present the following theorem.

Theorem 5. *The tripartite graph $G = K_{2,n,n}$ has a vertex magic labelling with magic constant $h = \frac{n^3+10n^2+23n+12}{2}$ whenever $n \equiv 3 \pmod 4$.*

Proof. Let us denote the vertices and edges of G by:

$$\begin{aligned} V(G) &= \{x_i, y_j, z_k \mid 1 \le i \le n,\ 1 \le j \le n \text{ and } 1 \le k \le 2\}, \\ E(G) &= \{x_iy_j, x_iz_k, y_jz_k \mid 1 \le i \le n, 1 \le j \le n \text{ and } 1 \le k \le 2\}. \end{aligned}$$

Again i and j run from $1, \dots, n$ throughout.

We construct a labelling for G, for all cases where $n \geq 15$, in which the labels $1, \ldots, n^2 + 6n + 2$ have been used exactly once each, as follows:

$$\lambda(x_i) = \begin{cases} 3n + 3 - 2i & \text{for } i = 1, \ldots, \frac{n-11}{2} \\ 2n + 3 - 2i & \text{for } i = \frac{n-9}{2}, \ldots, \frac{n-3}{2} \\ n + 3 & \text{for } i = \frac{n-1}{2} \\ n + 2 & \text{for } i = \frac{n+1}{2} \\ n^2 + 2n + 4i & \text{for } i = \frac{n+3}{2}, \ldots, n-1 \\ n^2 + 6n - 2 & \text{for } i = n, \end{cases}$$

$$\lambda(y_j) = \begin{cases} 4n + 3 - 2j & \text{for } j = 1, \ldots, \frac{n-5}{2} \\ 5 & \text{for } j = \frac{n-3}{2} \\ 2 & \text{for } j = \frac{n-1}{2} \\ n^2 + 2n + 1 + 4j & \text{for } j = \frac{n+1}{2}, \ldots, n-1 \\ n^2 + 6n - 1 & \text{for } j = n, \end{cases}$$

$$\lambda(z_k) = \begin{cases} 1 & \text{for } k = 1 \\ 3n + 2 & \text{for } k = 2. \end{cases}$$

Let A be the $n \times n$ magic square formed from the numbers from $4n + 3$ up to $n^2 + 4n + 2$ which has magic number $a = \frac{n^3+8n^2+5n}{2}$. Let $a_{i,j}$ be the i,jth entry of A, then

$$\lambda(x_i y_j) = a_{i,j},$$

$$\lambda(x_i z_1) = \begin{cases} n^2 + 4n + 4i & \text{for } i = 1, \ldots, \frac{n-3}{2} \\ n^2 + 6n & \text{for } i = \frac{n-1}{2} \\ n^2 + 6n + 2 & \text{for } i = \frac{n+1}{2} \\ 3n + 3 - 2i & \text{for } i = \frac{n+3}{2}, \ldots, n-2 \\ 2n + 5 & \text{for } i = n - 1 \\ 2n + 4 & \text{for } i = n. \end{cases}$$

$$\lambda(x_i z_2) = \begin{cases} 2n + 3 - 2i & \text{for } i = 1, \ldots, \frac{n-11}{2} \\ 3n + 3 - 2i & \text{for } i = \frac{n-9}{2}, \ldots, \frac{n-3}{2} \\ 2n + 3 & \text{for } i = \frac{n-1}{2} \\ 2n + 2 & \text{for } i = \frac{n+1}{2} \\ 4n + 3 - 2i & \text{for } i = \frac{n+3}{2}, \ldots, n-2 \\ n + 5 & \text{for } i = n - 1 \\ n + 4 & \text{for } i = n, \end{cases}$$

$$\lambda(y_jz_1) = \begin{cases} n+2-2j & \text{for } j = 1, \ldots, \frac{n-5}{2} \\ 3n+6 & \text{for } j = \frac{n-3}{2} \\ 3n+3 & \text{for } j = \frac{n-1}{2} \\ 5n+3-2j & \text{for } j = \frac{n+1}{2}, \ldots, \frac{3n-5}{4} \\ 2n+2-2j & \text{for } j = \frac{3n-1}{4}, \ldots, n-1 \\ 3 & \text{for } j = n, \end{cases}$$

$$\lambda(y_jz_2) = \begin{cases} n^2+4n+1+4j & \text{for } j = 1, \ldots, \frac{n-5}{2} \\ n^2+6n-5 & \text{for } j = \frac{n-3}{2} \\ n^2+6n+1 & \text{for } j = \frac{n-1}{2} \\ 2n+2-2j & \text{for } j = \frac{n+1}{2}, \ldots, \frac{3n-5}{4} \\ 5n+3-2j & \text{for } j = \frac{3n-1}{4}, \ldots, n-1 \\ 3n+4 & \text{for } j = n. \end{cases}$$

We now need to check that the weight on each of the vertices is equal to h. Then the weights on the vertices are given as follows:

$$\begin{aligned} wt(x_i) &= \lambda(x_i) + \lambda(x_iz_1) + \lambda(x_iz_2) + \sum_{j=1}^{n} \lambda(x_iy_j), \\ wt(y_j) &= \lambda(y_j) + \lambda(y_jz_1) + \lambda(y_jz_2) + \sum_{i=1}^{n} \lambda(x_iy_j), \\ wt(z_1) &= \lambda(z_1) + \sum_{i=1}^{n} \lambda(x_iz_1) + \sum_{j=1}^{n} \lambda(y_jz_1), \\ wt(z_2) &= \lambda(z_2) + \sum_{i=1}^{n} \lambda(x_iz_2) + \sum_{j=1}^{n} \lambda(y_jz_2). \end{aligned}$$

It is easily checked that $\lambda(x_i) + \lambda(x_iz_1) + \lambda(x_iz_2) = n^2 + 9n + 6$ for i from 1 to n. The $\lambda(x_iy_j)$ are the entries of the magic square A and so $\sum_{j=1}^{n} \lambda(x_iy_j) = a = \frac{n^3+8n^2+5n}{2}$.

Hence

$$\begin{aligned} wt(x_i) &= n^2 + 9n + 6 + \frac{n^3+8n^2+5n}{2} \\ &= \frac{n^3+10n^2+23n+12}{2} \\ &= h. \end{aligned}$$

A similar argument yields

$$wt(y_j) = h.$$

It remains to be shown that $wt(z_1) = wt(z_2) = h$. From above it can be checked that $\lambda(z_1) = 1$, $\sum_{i=1}^{n} \lambda(x_iz_1) = \frac{1}{2}n^3 + \frac{15}{4}n^2 + 6n + \frac{15}{4}$ and

$\sum_{j=1}^{n} \lambda(y_j z_1) = \frac{5}{4}n^2 + \frac{11}{2}n + \frac{5}{4}$. So

$$\begin{aligned} wt(z_1) &= 1 + \frac{1}{2}n^3 + \frac{15}{4}n^2 + 6n + \frac{15}{4} + \frac{5}{4}n^2 + \frac{11}{2}n + \frac{5}{4} \\ &= \frac{n^3 + 10n^2 + 23n + 12}{2} \\ &= h. \end{aligned}$$

Likewise $\lambda(z_2) = 3n + 2$, $\sum_{i=1}^{n} \lambda(x_i z_2) = 2n^2 + 5n$ and $\sum_{j=1}^{n} \lambda(y_j z_2) = \frac{1}{2}n^3 + 3n^2 + \frac{7}{2}n + 4$. So

$$\begin{aligned} wt(z_2) &= 3n + 2 + 2n^2 + 5n + \frac{1}{2}n^3 + 3n^2 + \frac{7}{2}n + 4 \\ &= \frac{n^3 + 10n^2 + 23n + 12}{2} \\ &= h. \end{aligned}$$

Thus for $n \geq 15$, $n \equiv 3(4)$, $K_{2,n,n}$ has a VMTL. The pattern does not extend down to $n = 3, 7$ and 11 but similar constructions have also been found for these cases. In the interests of brevity we will omit the labellings for $K_{2,7,7}$ and $K_{2,11,11}$, however we illustrate in Figure 4 with a labelling for $K_{2,3,3}$. □

In constructing these labellings we have found it convenient to make use of a *magic triangle*. In such a triangle the entries in the boxes represent the labels of the graph as indicated and the diagonal and horizontal columns all sum to the magic number for the graph. Note the use of the magic square at the top of the triangle. The magic triangle for $K_{2,3,3}$ is shown in Figure 3 along with it's corresponding graph labelling in Figure 4.

Again it would have been nice to find labellings for $n \equiv 0, 1$ and 2 (mod 4) however finding constructions of this type is very time consuming and encourages finding a different approach.

Further progress on an analogue of *Theorem 1* for all multipartite graphs is more likely to come from a proof that does not rely as heavily on constructions as did the proof of *Theorem 1*.

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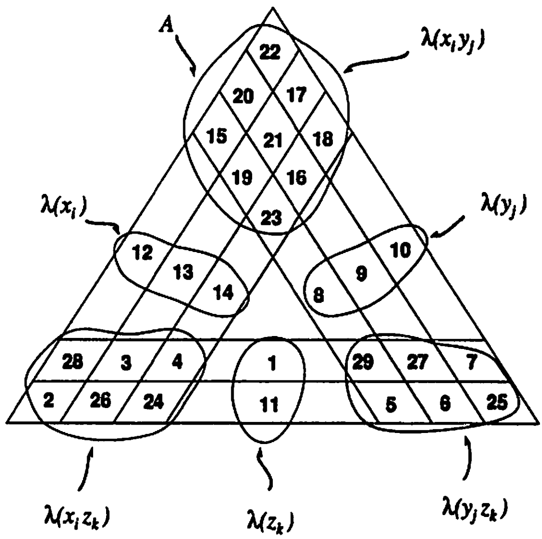

FIGURE 3. A magic triangle for the graph $K_{2,3,3}$

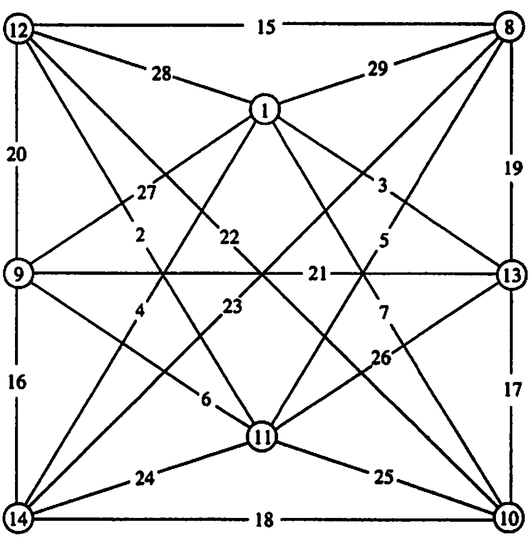

FIGURE 4. A vertex magic labelling for $K_{2,3,3}$ with magic constant $h = 99$.

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