A Note on Linear Discrepancy and Bandwidth

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Abstract. Fishburn, Tanenbaum and Trenk [4] define the linear discrepancy $\mathrm{ld}(P)$ of a poset $P = (V, <_P)$ as the minimum integer $k \geq 0$ for which there exists a bijection $f : V \rightarrow \{1, 2, \ldots, |V|\}$ such that $u <_P v$ implies $f(u) < f(v)$ and $u||_P v$ implies $|f(u) - f(v)| \leq k$. In [5] they prove that the linear discrepancy of a poset equals the bandwidth of its cocomparability graph.

Here we provide partial solutions to some problems formulated in [4] about the linear discrepancy and the bandwidth of cocomparability graphs.

Keywords. Poset; cocomparability graph; linear discrepancy; bandwidth

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1 Introduction

In [4] Fishburn, Tanenbaum and Trenk introduce the notion of the linear discrepancy of a poset as a measure of its nonlinearity. At the end of [4] they formulate a number of problems about this parameter. In the present paper we will provide partial solutions to some of these problems. We start with notation and definitions.

Let $G = (V, E)$ be a finite and simple graph. The degree and neighborhood of a vertex $u \in V$ in the graph G are denoted by $d_G(u)$ and $N_G(u)$, respectively. The maximum degree of G is denoted by $\Delta(G)$.

Let $P = (V, <_P)$ be a finite poset, i.e. $<_P$ is an antisymmetric and transitive relation on the finite set V. For two different elements $x, y \in V$, we write $x \perp_P y$ ($x||_P y$), if either $x <_P y$ or $y <_P x$ (neither $x <_P y$ nor $y <_P x$) and call x and y comparable (incomparable).

An extension of $P = (V, <_P)$ is a poset $Q = (V, <_Q)$ such that $u <_P v$ implies $u <_Q v$ for $u, v \in V$. If Q is an extension of P and no two different elements of Q are incomparable, i.e. Q is a chain, then Q is called a linear extension of P.

The cocomparability graph G of a poset $P = (V, <_P)$ has vertex set V and two different elements $x, y \in V$ are adjacent in G if $x||_P y$. (*Note that the maximum degree $\Delta(G)$ of G equals the maximum number of elements that some element of V is incomparable to in P.*)

We will now define the *linear discrepancy* of a poset and the closely related *bandwidth* of a graph.

Let $P = (V, <_P)$ be a poset and let $Q = (V, <_Q)$ be a linear extension of P such that $u_1 <_Q u_2 <_Q \dots <_Q u_n$ for $n = |V|$. The *uncertainty* uncert(Q) of the linear extension Q of P is defined in [4] as 0, if P is a chain, or as $\max\{j - i \mid u_i||_P u_j,\ 1 \le i < j \le n\}$, if P is not a chain. The *linear discrepancy* $\mathrm{ld}(P)$ of P [4] is the minimum uncertainty of a linear extension of P. Alternatively, one can define the linear discrepancy of $P = (V, <_P)$ as the minimum integer $k \geq 0$ for which there exists a bijective mapping $f : V \to \{1, 2, \dots, |V|\}$ such that $u <_P v$ implies $f(u) < f(v)$ and $u||_P v$ implies $|f(u) - f(v)| \le k$.

Let $G = (V, E)$ be a graph. A bijective mapping $f : V \to \{1, 2, \dots, |V|\}$ such that $uv \in E$ implies $|f(u) - f(v)| \le k$, is called a *k-labeling* of G. The *bandwidth* $\mathrm{bw}(G)$ (cf. [3]) of G is the minimum k for which there exists a k-labeling of G.

The main result of [5] relates the linear discrepancy to the bandwidth.

Theorem 1 (Fishburn, Tanenbaum and Trenk [5])
If P is a poset and G is the cocomparability graph of P, then $\mathrm{ld}(P) = \mathrm{bw}(G)$.

We will now reformulate three of the problems posed at the end of [4].

(i) What is the maximum value of $\frac{\mathrm{uncert}(Q)}{\mathrm{ld}(P)}$ over all linear extensions Q of a poset P (cf. no. 5 in [4])?

(ii) If G is a cocomparability graph, is it true that $\mathrm{bw}(G) \le \lfloor \frac{3\Delta(G)-1}{2} \rfloor$ (cf. no. 6 in [4])?

(iii) Characterize the posets with linear discrepancy equal to 2 (cf. no. 1 in [4]).

In the following section we prove a best-possible upper bound on the expression $\frac{\mathrm{uncert}(Q)}{\mathrm{ld}(P)}$ in Problem (i) and an upper bound on the bandwidth of a cocomparability graph G which implies an affirmative answer for Problem (ii) in the case $\Delta(G) \le 3$. Regarding the third problem, we first show how to solve the algorithmical version of this problem combining known results. Finally, we formulate a conjecture about a structural characterization of the posets with linear discrepancy at most 2 and prove a related result.

2 Results

Proposition 1 *Let $P = (V, <_P)$ be a poset which is not a chain and let $Q = (V, <_Q)$ be a linear extension of P. Let $G = (V, E)$ be the cocomparability graph of P. Then*

$$\text{uncert}(Q) \leq 2\Delta(G) - 1 \quad (1)$$

$$\frac{\text{uncert}(Q)}{\text{ld}(P)} \leq \frac{2\Delta(G) - 1}{\max\left\{\frac{1}{3}\left(|N_G(u) \cup N_G(v)| - 1\right) \mid uv \in E\right\}} \quad (2)$$

$$\frac{\text{uncert}(Q)}{\text{ld}(P)} \leq \frac{2\Delta(G) - 1}{\lceil\frac{\Delta(G)}{2}\rceil} \leq 4 - \frac{2}{\Delta(G)}. \quad (3)$$

Proof: Let $u_1 <_Q u_2 <_Q \ldots <_Q u_n$ for $n = |V|$ and let $u_i||_P u_j$ for some $1 \leq i < j \leq n$. Since Q is a linear extension of P, if $u_i \perp_P u_l$ and $u_l \perp_P u_j$ for some $i < l < j$, then $u_i <_P u_l <_P u_j$ which implies the contradiction $u_i <_P u_j$. Hence $u_i||_P u_l$ or $u_l||_P u_j$ for all $i < l < j$, and thus $\{u_i, u_{i+1}, \ldots, u_j\} \subseteq N_G(u_i) \cup N_G(u_j)$ which implies that

$$2\Delta(G) \geq d_G(u_i) + d_G(u_j) \geq |N_G(u_i) \cup N_G(u_j)| \geq j - i + 1.$$

Thus $j - i \leq 2\Delta(G) - 1$ which implies (1).

The inequalities (2) and (3) follow immediately from two known lower bounds on the bandwidth:

$$\text{bw}(G) \geq \max\left\{\frac{1}{3}(|N_G(u) \cup N_G(v)| - 1) \mid uv \in E\right\}$$

(cf. Lemma 2.3 in [8]) and $\text{bw}(G) \geq \lceil\frac{\Delta(G)}{2}\rceil$ (cf. [2] or Lemma 18 in [4]). Q.E.D.

We will illustrate that Proposition 1 is best-possible. For $l \geq 0$ let $P = (V, <_P)$ be the poset such that $V = \{x, y\} \cup \{u_1, u_2, \ldots, u_{2l+2}\}$, $u_i <_P u_j$ for $1 \leq i < j \leq 2l+2$, $x||_P y$, $x||_P u_i <_P y$ for $1 \leq i \leq l+1$ and $x <_P u_i||_P y$ for $l+2 \leq i \leq 2l+2$. See Figure 1 for a Hasse diagram of P.

If $G = (V, E)$ is the cocomparability graph of P and $l \equiv 0 \bmod (3)$, then it is easy to check that $\Delta(G) = l + 2$ and $\text{bw}(G) = \frac{2}{3}l + 1 = \frac{1}{3}(|N_G(x) \cup N_G(y)| - 1)$. If Q is the linear extension of P with $u_{2l+2} <_Q y$ and $x <_Q u_1$, then $\text{uncert}(Q) = 2l + 3 = 2\Delta(G) - 1$. Hence (1) and (2) are satisfied with equality for P and Q. Furthermore, (3) is satisfied with equality for $l = 0$.

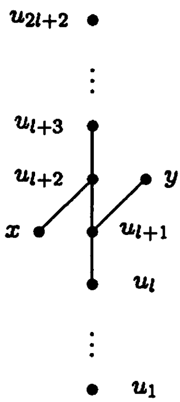

Figure 1: The poset P

Now we proceed to Problem (ii).

Since $K_{\Delta,\Delta}$ is the cocomparability graph of the disjoint union of two chains of length $\Delta - 1$ and satisfies $\mathrm{bw}(K_{\Delta,\Delta}) = \Delta + \lceil \frac{\Delta}{2} \rceil - 1 = \lfloor \frac{3\Delta - 1}{2} \rfloor$ (cf. Lemma 20 in [4]), the given bound would clearly be best-possible.

Let G be an arbitrary graph. It is easy to see that $\mathrm{bw}(G) = 1 = \lfloor \frac{3-1}{2} \rfloor$, if $\Delta(G) = 1$ and that $\mathrm{bw}(G) \leq 2 = \lfloor \frac{3\cdot 2-1}{2} \rfloor$, if $\Delta(G) = 2$. Hence the first non-trivial case of Problem (ii) is $\Delta(G) = 3$. In fact, there are planar graphs of maximum degree 3 that have arbitrarily large bandwidth (consider e.g. the so-called *walls* that even have arbitrarily large treewidth cf. [10]). Therefore, $\Delta(G) = 3$ is also the first case where the assumption that the graph is a cocomparability graph has to play some role.

In view of Theorem 1, Proposition 1 immediately implies $\mathrm{bw}(G) \leq 2\Delta(G) - 1$, if G is a cocomparability graph. We will now prove a small improvement of this bound which is sufficient to yield an affirmative answer to Problem (ii) for the case $\Delta(G) = 3$.

Theorem 2 *If $G = (V, E)$ is a cocomparability graph of maximum degree $\Delta(G) \geq 2$, then* $\mathrm{bw}(G) \leq 2\Delta(G) - 2$.

Proof: For contradiction we assume that G is a counterexample of minimum order $n = |V|$. Let G be the cocomparability graph of $P = (V, <_P)$ and let $\Delta = \Delta(G)$.

We choose a linear extension $Q = (V, <_Q)$ of P with $u_1 <_Q u_2 <_Q \ldots <_Q u_n$ such that $t = |\{u_i u_j \in E \mid 1 \leq i < j \leq n, j - i = 2\Delta - 1\}|$ is minimum. Note that, by Proposition 1, $u_i u_j \in E$ implies that $j - i \leq 2\Delta - 1$ for all $1 \leq i < j \leq n$. By the choice of G as a counterexample, we have $t \geq 1$.

Claim 1 If $u_iu_j \in E$ for some $1 \leq i < j \leq n$ with $j - i = 2\Delta - 1$, then $\{u_i, u_{i+1}, \ldots, u_j\} = N_G(u_i) \cup N_G(u_j)$ and $N_G(u_i) \cap N_G(u_j) = \emptyset$.

Proof of Claim 1: The fact that $\{u_i, u_{i+1}, \ldots, u_j\} \subseteq N_G(u_i) \cup N_G(u_j)$ follows exactly as in the proof of Proposition 1. If $(N_G(u_i) \cup N_G(u_j)) \setminus \{u_i, u_{i+1}, \ldots, u_j\} \neq \emptyset$, then

$$2\Delta \geq d_G(u_i) + d_G(u_j) \geq |N_G(u_i) \cup N_G(u_j)| \geq j - i + 1 + 1 = 2\Delta + 1$$

which is a contradiction. Hence $\{u_i, u_{i+1}, \ldots, u_j\} = N_G(u_i) \cup N_G(u_j)$.

If $N_G(u_i) \cap N_G(u_j) \neq \emptyset$, then

$$2\Delta - 1 \geq d_G(u_i) + d_G(u_j) - 1 \geq |N_G(u_i) \cup N_G(u_j)| \geq j - i + 1 = 2\Delta$$

which is a contradiction. Hence $N_G(u_i) \cap N_G(u_j) = \emptyset$ and the proof of the claim is complete.

Claim 2 If $u_iu_l, u_iu_k, u_ju_l \in E$ and $u_iu_j, u_ku_l \notin E$ for some $1 \leq i < j < k < l \leq n$ (cf. Figure 2), then $u_ju_k \in E$.

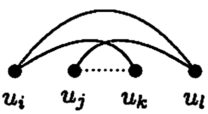

Figure 2: u_i, u_j, u_k **and** u_l

Proof of Claim 2: For contradiction we assume that $u_ju_k \notin E$. Since Q is a linear extension of P, we obtain $u_i <_P u_j <_P u_k <_P u_l$ which implies the contradiction $u_i <_P u_l$ and the proof of the claim is complete.

Now, let $u_iu_{i+2\Delta-1} \in E$ for some $1 \leq i \leq n - (2\Delta - 1)$.

Claim 3 If $u_iu_{i+2\Delta-1-\nu} \in E$ and $u_{i+2\Delta-1}u_{i+\nu} \in E$ for all $1 \leq \nu \leq j$ and some $0 \leq j \leq \Delta - 2$ (cf. Figure 3), then $u_iu_{i+2\Delta-1-j-1} \in E$ and $u_{i+2\Delta-1}u_{i+j+1} \in E$.

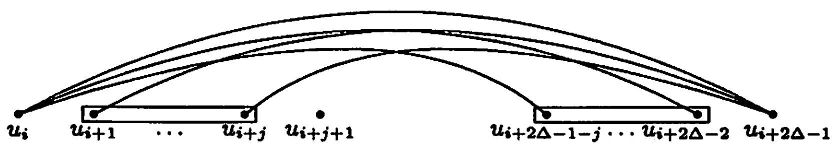

Figure 3: $u_i, \ldots, u_{i+2\Delta-1}$

Proof of Claim 3: For contradiction we may assume, by Claim 1 and symmetry, that $u_iu_{i+j+1} \in E$ and $u_{i+2\Delta-1}u_{i+j+1} \notin E$.

By Claim 2, we obtain that $u_\nu u_\mu \in E$ for $i \leq \nu \leq i+j$ and $i+2\Delta-1-j \leq \mu \leq i+2\Delta-1$ and $u_{i+\mu}u_{i+j+1} \in E$ for $0 \leq \mu \leq j$.

If $i+2\Delta \leq l \leq n$, then, by Claim 1, $u_{i+2\Delta-1} <_P u_l$. Since $u_{i+j+1} <_P u_{i+2\Delta-1}$, we obtain $u_{i+j+1} <_P u_l$. Hence $u_{i+j+1}u_l \notin E$ for $i+2\Delta-1 \leq l \leq n$.

If $u_\nu u_{\nu-l} \in E$ for some $i \leq \nu \leq i+j$ and some $l \geq 2\Delta-2$, then we obtain as in the proof of Proposition 1 that $\{u_{\nu-l}, u_{\nu-l+1}, \ldots, u_\nu\} \subseteq N_G(u_\nu) \cup N_G(u_{\nu-l})$. Since $\{u_{i+j+1}\} \cup \{u_{i+2\Delta-1-j}, u_{i+2\Delta-1-j+1}, \ldots, u_{i+2\Delta-1}\} \subseteq N_G(u_\nu) \cup N_G(u_{\nu-l})$, we obtain the contradiction

$$2\Delta \geq |N_G(u_\nu) \cup N_G(u_{\nu-l})| \geq l+1+1+j+1 \geq 2\Delta-2+3+j \geq 2\Delta+1.$$

Hence $u_\nu u_{\nu-l} \notin E$ for $i \leq \nu \leq i+j$ and $l \geq 2\Delta-2$.

Since u_{i+j+1} is incomparable to $u_i, u_{i+1}, \ldots, u_{i+j}$, the poset Q' with

$$\begin{aligned} &u_1 <_{Q'} u_2 <_{Q'} \ldots <_{Q'} u_{i-1} \\ &<_{Q'} u_{i+j+1} <_{Q'} u_i <_{Q'} u_{i+1} <_{Q'} \ldots <_{Q'} u_{i+j} \\ &<_{Q'} u_{i+j+2} <_{Q'} u_{i+j+3} <_{Q'} \ldots <_{Q'} u_n \end{aligned}$$

is a linear extension of P. (*Note that Q' arises from Q by 'shifting' u_{i+j+1} in front of u_i.*) If we rename the elements of V such that $v_1 <_{Q'} v_2 <_{Q'} \ldots <_{Q'} v_n$, then $|\{v_iv_j \in E \mid 1 \leq i < j \leq n, j-i = 2\Delta-1\}| \leq t-1$ which contradicts the choice of Q and the proof of the claim is complete.

By an inductive argument, Claim 3 and the fact that G has maximum degree Δ imply that the elements in $V' = \{u_i, u_{i+1}, \ldots, u_{i+2\Delta-1}\}$ induce a complete bipartite graph $K_{\Delta,\Delta}$ and that no edge of G joins a vertex in V' to a vertex in $V \setminus V'$. Let $G_1 = G[V']$ and $G_2 = G[V \setminus V']$. Since G_1 is $K_{\Delta,\Delta}$, $\text{bw}(G_1) \leq 2\Delta-2$. Since G_2 is a cocomparability graph with maximum degree at most Δ that has less vertices than G, $\text{bw}(G_2) \leq 2\Delta-2$. Since $\text{bw}(G) \leq \max\{\text{bw}(G_1), \text{bw}(G_2)\} \leq 2\Delta-2$, we obtain a contradiction to the choice of G and the proof is completed. Q.E.D.

Corollary 1 *If $G=(V,E)$ is a cocomparability graph of maximum degree $\Delta(G) \leq 3$, then* $\text{bw}(G) \leq \lfloor \frac{3\Delta(G)-1}{2} \rfloor$.

We will now turn our attention to Problem (iii).

It is well-known that graphs with bandwidth at most 2 can be recognized in linear time and that a 2-labeling of such graphs can also be found in linear time [6],[9],[1].

In view of Theorem 1, this implies that posets with linear discrepancy at most 2 can be recognized in linear time. Furthermore, we will describe now how the proof of Theorem 2 in [5] implies that a linear extension of uncertainty at most 2 of such posets can be found in polynomial time.

Given a 2-labeling of the cocomparability graph G of a poset P (*which can be found in linear time*), Fishburn, Tanenbaum and Trenk [5] reduce the problem of finding a linear extension of uncertainty at most 2 to a bandwidth-2 problem for an interval graph G'. Given P and the 2-labeling of G, the graph G' can be efficiently constructed and a special 2-labeling of G' is found using the linear time algorithm from [7]. Using the original 2-labeling of G and the special 2-labeling of G', it is then possible to obtain in polynomial time the desired linear extension of P using a so-called Switching Lemma (cf. Lemma 8 in [5]). This settles the algorithmical version of Problem (iii).

In [4] Fishburn, Tanenbaum and Trenk characterize the posets with linear discrepancy at most 1 in terms of three forbidden induced subposets (see Corollary 25 in [4]). We believe that this kind of characterization generalizes as follows to posets with linear discrepancy at most 2.

Conjecture 1 *A poset P satisfies $\mathrm{ld}(P) \leq 2$ if and only if it does not contain one of the following posets as an induced subposet:*

(i) $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$, *i.e. an antichain with 4 elements.*

(ii) The disjoint union of any poset on three elements and $\mathbf{1}+\mathbf{1}$.

(iii) The disjoint union of any poset on three elements and $\mathbf{2}$.

(iv) The disjoint union of any poset on five elements and $\mathbf{1}$.

(v) One of the two posets whose Hasse diagrams are given in Figure 4.

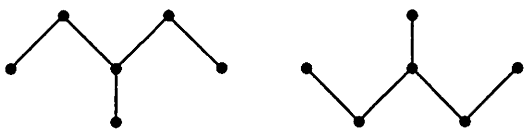

Figure 4

(vi) The poset whose Hasse diagram is given in Figure 5.

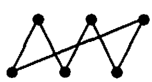

Figure 5

It is easy to check that all posets enumerated in Conjecture 1 have linear discrepancy 3. Furthermore, it is a simple task to determine all posets among the listed posets that are minimal with respect to this property.

By Theorem 2, posets whose cocomparability graphs have maximum degree at most 3 have linear discrepancy at most 4. We will now prove that forbidding induced subposets as in (iii) and (vi) of Conjecture 1 implies that these posets have linear discrepancy at most 3.

Theorem 3 *Let $P = (V, <_P)$ be a poset such that the cocomparability graph $G = (V, E)$ of P has maximum degree $\Delta(G) \leq 3$.*

If P does not contain an induced subposet as in (iii) or (vi) of Conjecture 1, then $\mathrm{ld}(P) \leq 3$.

Proof: By Theorem 2, $\mathrm{ld}(P) \leq 4$. We choose a linear extension $Q = (V, <_Q)$ of P with $u_1 <_Q u_2 <_Q \ldots <_Q u_n$ for $n = |V|$ such that

a) $|\{u_iu_j \in E \mid 1 \leq i < j \leq n, j - i \geq 5\}| = 0$ and subject to Condition a)

b) $t = |\{u_iu_j \in E \mid 1 \leq i < j \leq n, j - i = 4\}|$ is minimum.

For contradiction, we assume that $t \geq 1$.

Let $u_iu_{i+4} \in E$ for some $1 \leq i \leq n - 4$. As in the proof of Proposition 1, we have $\{u_i, u_{i+1}, \ldots, u_{i+4}\} \subseteq N_G(u_i) \cup N_G(u_{i+4})$. Since $\Delta(G) \leq 3$, we can assume, by symmetry, that $|N_G(u_i) \cap \{u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\}| = 3$.

Case 1.1 $u_{i+1} \notin N_G(u_i)$.

This implies that $u_i <_P u_{i+1}$ and that $u||_P v$ for all $u \in \{u_i, u_{i+1}\}$ and $v \in \{u_{i+2}, u_{i+3}, u_{i+4}\}$. Hence the set $\{u_i, u_{i+1}, \ldots, u_{i+4}\}$ induces a subposet of P as in (iii) of Conjecture 1 which is a contradiction.

Case 1.2 $u_{i+2} \notin N_G(u_i)$.

This implies $u_i <_P u_{i+2}$, $u_i||_P u_{i+1}$, $u_i||_P u_{i+3}$, $u_i||_P u_{i+4}$, $u_{i+2}||_P u_{i+3}$ and $u_{i+2}||_P u_{i+4}$.

If $u_{i+1}||_P u_{i+4}$, then $\Delta(G) \leq 3$ implies $u_{i+3} <_P u_{i+4}$ and $u_{i+1}||_P u_{i+3}$. Hence the set $\{u_i, u_{i+1}, \ldots, u_{i+4}\}$ induces a subposet of P as in (iii) of Conjecture 1 which is a contradiction. Therefore $u_{i+1} <_P u_{i+4}$.

First, we assume that $u_{i+1} <_P u_{i+3}$. If $n \geq i + 5$ and $u_{i+1}||_P u_{i+5}$, then $u_{i+3}||_P u_{i+5}$ and $u_{i+4}||_P u_{i+5}$. Since $u_i <_P u_{i+5}$, we obtain that the set $\{u_i, u_{i+1}, u_{i+3}, u_{i+4}, u_{i+5}\}$ induces a subposet of P as in (iii) of Conjecture 1 which is a contradiction. Therefore, either $n \leq i + 4$ or $u_{i+1} <_P u_j$ for all $i + 5 \leq j \leq n$. Since $u_j <_P u_i$ for all $1 \leq j \leq i - 1$, the poset Q' with

$$u_1 <_{Q'} \ldots <_{Q'} u_{i-1} <_{Q'} u_{i+1} <_{Q'} u_i <_{Q'} u_{i+2} <_{Q'} \ldots <_{Q'} u_n$$

is a linear extension of P. If we rename the elements of V such that $v_1 <_{Q'} v_2 <_{Q'} \ldots <_{Q'} v_n$, then $|\{v_iv_j \in E \mid 1 \leq i < j \leq n, j - i \geq 5\}| = 0$ and $|\{v_iv_j \in E \mid 1 \leq i < j \leq n, j - i = 4\}| \leq t - 1$ which is a contradiction.

Thus $u_{i+1}||_P u_{i+3}$. Hence, $\Delta(G) \leq 3$ implies $u_{i+3} <_P u_j$ for $i+4 \leq j \leq n$. If $n \leq i+4$ or $n \geq i+5$ and $u_{i+1} <_P u_{i+5}$, then we obtain the same contradiction as above (*again switching u_i and u_{i+1}*). Hence $n \geq i+5$ and $u_{i+1}||_P u_{i+5}$. This implies $u_{i+4}||_P u_{i+5}$, $u_{i+1} <_P u_{i+2}$, $u_{i+2}||_P u_{i+5}$ and $u_i <_P u_{i+5}$. Therefore, the set $\{u_i, u_{i+1}, \ldots, u_{i+5}\}$ induces a subposet of P as in (vi) of Conjecture 1 which is a contradiction.

Case 1.3 $u_{i+3} \notin N_G(u_i)$.

This implies $u_i <_P u_{i+3}$, $u_i||_P u_{i+1}$, $u_i||_P u_{i+2}$, $u_i||_P u_{i+4}$ and $u_{i+3}||_P u_{i+4}$.

Since $u_j <_P u_i$ for $1 \leq j \leq i-1$, we obtain $u_j <_P u_{i+3}$ for all $1 \leq j \leq i-1$.

If $n \geq i+8$ and $u_{i+4}||_P u_{i+8}$, then $\{u_i, u_{i+3}\} \cup \{u_{i+4}, \ldots, u_{i+8}\} \subseteq N_G(u_{i+4}) \cup N_G(u_{i+8})$ which is a contradiction to $\Delta(G) \leq 3$. Hence either $n \leq i+7$ or $u_{i+4} <_P u_{i+8}$.

If $n \leq i+6$ or $n \geq i+7$ and $u_{i+4} <_P u_{i+7}$, then the poset Q' with

$$u_1 <_{Q'} \ldots <_{Q'} u_{i+2} <_{Q'} u_{i+4} <_{Q'} u_{i+3} <_{Q'} u_{i+5} <_{Q'} \ldots <_{Q'} u_n$$

is a linear extension of P. If we rename the elements of V such that $v_1 <_{Q'} v_2 <_{Q'} \ldots <_{Q'} v_n$, then $|\{v_iv_j \in E \mid 1 \leq i < j \leq n, j-i \geq 5\}| = 0$ and $|\{v_iv_j \in E \mid 1 \leq i < j \leq n, j-i = 4\}| \leq t-1$ which is a contradiction.

Hence $n \geq i+7$ and $u_{i+4}||_P u_{i+7}$. This implies that $u_{i+1} <_P u_{i+4}$, $u_{i+2} <_P u_{i+4}$, $u_{i+4} <_P u_{i+5}$, $u_{i+4} <_P u_{i+6}$, $u_{i+5}||_P u_{i+7}$, $u_{i+6}||_P u_{i+7}$ and $u_{i+7} <_P u_j$ for $i+8 \leq j \leq n$. The poset Q'' with

$$\begin{aligned} u_1 &<_{Q''} \ldots <_{Q''} u_{i+2} \\ &<_{Q''} u_{i+4} <_{Q''} u_{i+3} <_{Q''} u_{i+5} <_{Q''} u_{i+7} <_{Q''} u_{i+6} \\ &<_{Q''} u_{i+8} <_{Q''} \ldots <_{Q''} u_n \end{aligned}$$

is a linear extension of P. If we rename the elements of V such that $v_1 <_{Q''} v_2 <_{Q''} \ldots <_{Q''} v_n$, then $|\{v_iv_j \in E \mid 1 \leq i < j \leq n, j-i \geq 5\}| = 0$ and $|\{v_iv_j \in E \mid 1 \leq i < j \leq n, j-i = 4\}| \leq t-1$ which is a contradiction.

Since all three cases led to a contradiction, the proof is completed. Q.E.D.

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