Some new results on the super edge-magic deficiency of graphs *

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Abstract. A (p, q) graph G is called edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(u) + f(v) + f(uv)$ is a constant for each edge $uv \in E(G)$. Also, G is said to be super edge-magic if $f(V(G)) = \{1, 2, \ldots, p\}$. Furthermore, the super edge-magic deficiency, $\mu_s(G)$, of a graph G is defined to be either the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n.

In this paper, the super edge-magic deficiency of certain forests and 2-regular graphs is computed which in turn leads to some conjectures on the super edge-magic deficiencies of graphs in these classes. Additionally, some edge-magic deficiency analogues to the super edge-magic deficiency results on forests are presented.

1 Introduction

All graphs that we consider in this paper are finite and simple, that is, without loops or multiple edges. Now, for most of the graph theory terminology utilized here, the authors refer the reader to Chartrand and Lesniak [1]; however, to make this paper reasonably self-contained, we mention that for a graph G, we denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. Moreover, for the sake of brevity, we will denote $[a, b] \cap \mathbb{Z}$ by simply writing $[a, b]$.

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The seminal paper in edge-magic labelings was published in 1970 by Kotzig and Rosa [12], who called these labelings: magic valuations; these were later *rediscovered* by Ringel and Lladó [13], who coined one of the now popular terms for them: edge-magic labelings. More recently, they have also been referred to as edge-magic total labelings by Wallis [14]. For a (p,q) graph G, a bijective function $f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p+q\}$ is an *edge-magic labeling* of G if $f(u)+f(v)+f(uv)$ is a constant k (called the *valence* of f) for any edge $uv \in E(G)$. If such a labeling exists, then G is said to be an *edge-magic graph.* In [2], Enomoto, Lladó, Nakamigawa and Ringel defined an edge-magic labeling f of a graph G to be a *super edge-magic labeling* of G if f has the additional property that $f(V(G)) = \{1, 2, \ldots, p\}$. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling. Lately, super edge-magic labelings and super edge-magic graphs are called by Wallis [14] strong edge-magic total labelings and strongly edge-magic graphs, respectively.

For every graph G, Kotzig and Rosa [12] proved that there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This motivated them to define the edge-magic deficiency of a graph. The *edge-magic deficiency,* $\mu(G)$, of a graph G is the smallest nonnegative integer n for which the graph $G \cup nK_1$ is edge-magic. In [5], the authors analogously defined the concept of *super edge-magic deficiency,* $\mu_s(G)$, of a graph G to be either the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. It follows immediately that $\mu(G) \leq \mu_s(G)$ for any graph G.

The authors refer the reader to the survey paper by Gallian [8] for some of the latest developments in these and other types of graph labelings.

To present the new results contained in this paper, the following lemma taken from [3] will prove to be useful.

Lemma 1. *A (p,q) graph G is super edge-magic if and only if there exists a bijective function $f: V(G) \rightarrow \{1, 2, \ldots, p\}$ such that the set*

$$S = \{f(u) + f(v) \,|uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min(S)$ and

$$\begin{aligned} S &= \{f(u) + f(v) \,|uv \in E(G)\} \\ &= \{k-(p+1), k-(p+2), \ldots, k-(p+q)\}. \end{aligned}$$

Due to Lemma 1, it is sufficient to exhibit the vertex labeling of a super edge-magic graph; however, we will provide the valences to increase the clarity of our results.

We will also utilize the following lemma from [3].

Lemma 2. *If a (p,q) graph G is super edge-magic with a super edge-magic labeling f, then*

$$\sum_{v \in V(G)} f(v) \deg v = qs + \binom{q}{2},$$

where s is defined as in the previous lemma.

A graph G is said to be an *even graph* if all of its vertices have even degree. Thus, with this definition in hand, we are able to state one more technical lemma found in [5].

Lemma 3. *If G is an even graph of size q, where $q/2$ is odd, then $\mu_s(G) = +\infty$.*

The above lemma shows that adding a finite number of isolated vertices to a graph that is *not* super edge-magic need not produce a super edge-magic graph. This was unexpected in light of the fact that Kotzig and Rosa [12] proved that every graph has finite edge-magic deficiency. Indeed, it is easy to construct a graph G such that $\mu_s(G) - \mu(G) = +\infty$ (see [5] for examples).

Actually, more is true as given a nonnegative integer n, it is always possible to construct a graph G such that $\mu_s(G) - \mu(G) = n$. For example, Kotzig and Rosa [12] proved that all complete bipartite graphs are edge-magic, which implies that $\mu(K_{2,n+1}) = 0$. On the other hand, the authors showed in [5] that $\mu_s(K_{2,n+1}) = n$.

To conclude this introduction, we state a result by the authors (see [6]), and state and prove a corollary to it, which will later serve as the bases for some remarks and conjectures.

Theorem 1. *If G is a (super) edge-magic bipartite or tripartite graph and m is odd, then mG is (super) edge-magic.*

Corollary 1. *If G is a bipartite or tripartite graph and m is odd, then $\mu(mG) \leq m\mu(G)$ and $\mu_s(mG) \leq m\mu_s(G)$.*

Proof. Let G be a bipartite or tripartite graph and m be odd. First, notice that if $\mu_s(G) = +\infty$, then the inequality for μ_s is trivial. Hence, without loss of generality, we may assume that $\mu_s(G) < +\infty$. Moreover, as we commented above, it is an established fact that $\mu(G) < +\infty$ for any graph G. The remainder of the proof is identical for either μ or μ_s, and if G is bipartite or tripartite. Thus, without loss of generality, we proceed only with μ_s and assume that G is bipartite. Now, there exists a nonnegative integer n such that $G \cup nK_1$ is super edge-magic. Furthermore, $G \cup nK_1$ is certainly bipartite. Therefore, $m(G \cup nK_1) \cong mG \cup mnK_1$ is super edge-magic by the previous theorem, which implies that $\mu_s(mG) \leq m\mu_s(G)$.

2 Results on Forests

In [5], the authors give a constructive proof that the super edge-magic deficiency of forests is finite. That proof is too involved to easily glean from it an upper bound for $\mu_s(F)$ when F is a forest, but such a bound would certainly be a large quantity relative to the order of F. However, the authors believe that $\mu_s(F)$ is always a small number. The evidence for such a belief comes from the authors previous explorations on the super edge-magic properties of forests (see [4–7]) and are bolstered by the results included in this section.

Now, with the aid of the super edge-magic characterization of the forest $K_{1,1} \cup K_{1,n}$ found in [6], we are able to provide the following theorem.

Theorem 2. *For every positive integer n,*

$$\mu_s(P_2 \cup K_{1,n}) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First, notice that the forest $P_2 \cup K_{1,n} \cong K_{1,1} \cup K_{1,n}$ is shown to be super edge-magic in [6] if and only if n is even. Consequently, it is sufficient to verify that $\mu_s(P_2 \cup K_{1,n}) \leq 1$ when n is odd.

Now, assume that n is odd, and let $F \cong P_2 \cup K_{1,n} \cup K_1$ be the forest with

$$V(F) = \{u, w, x, y\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(F) = \{uv_i | 1 \leq i \leq n\} \cup \{xy\}.$$

Then the vertex labeling $f : V(F) \to \{1, 2, \ldots, n+4\}$ such that $f(x) = 1$; $f(y) = n + 4$; $f(u) = (n + 5)/2$;

$$f(v_i) = \begin{cases} i + 1, & \text{if } i \in [1, (n+1)/2]; \\ i + 2, & \text{if } i \in [(n+1)/2 + 1, n]; \end{cases}$$

and $f(w) = n + 3$ extends to a super edge-magic labeling of F with valence $(5n + 19)/2$.

Therefore, $\mu_s(P_2 \cup K_{1,n}) \leq 1$ when n is odd, which completes the proof.

The super edge-magic characterization of the forest $K_{1,2} \cup K_{1,n}$ found in [6] is extended in the following theorem.

Theorem 3. *For every positive integer n,*

$$\mu_s(P_3 \cup K_{1,n}) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. First, notice that the forest $P_3 \cup K_{1,n} \cong K_{1,2} \cup K_{1,n}$ is known to be super edge-magic if and only if $n \equiv 0 \pmod 3$; see [6]. Hence, in the remainder of the proof, assume that $n \not\equiv 0 \pmod 3$, and thus our goal is to show that in such a case, $\mu_s(P_3 \cup K_{1,n}) \leq 1$.

To do this, let $F \cong P_3 \cup K_{1,n} \cup K_1$ be the forest with

$$V(F) = \{u, w, x, y, z\} \cup \{v_i | 1 \leq i \leq n\}$$

and

$$E(F) = \{uv_i | 1 \leq i \leq n\} \cup \{xy, xz\},$$

and consider two cases.

Case 1: Let n be odd, and define $f : V(F) \to \{1, 2, \ldots, n+5\}$ to be the vertex labeling such that $f(x) = 1$; $f(y) = n+4$; $f(z) = n+5$; $f(u) = (n+5)/2$;

$$f(v_i) = \begin{cases} i+1, & \text{if } i \in [1, (n+1)/2]; \\ i+3, & \text{if } i \in [(n+1)/2+1, n]; \end{cases}$$

and $f(w) = (n+7)/2$.
Then, by Lemma 1, f extends to a super edge-magic labeling of F with valence $(5n+23)/2$.

Case 2: Let n be even, and define $f : V(F) \to \{1, 2, \ldots, n+5\}$ to be the vertex labeling such that $f(x) = 1$; $f(y) = n+4$; $f(z) = n+5$; $f(u) = (n+6)/2$;

$$f(v_i) = \begin{cases} i+1, & \text{if } i \in [1, n/2]; \\ i+3, & \text{if } i \in [n/2+1, n]; \end{cases}$$

and $f(w) = (n+4)/2$.
Then, by Lemma 1, f extends to a super edge-magic labeling of F with valence $(5n+24)/2$.

Therefore, the proof is completed.

By labeling the vertices of degree 1 of the forest $F \cong P_1 \cup K_{1,n}$ with the first n positive integers, the isolated vertex of F with $n+1$ and the remaining vertex with $n+2$, we obtain a super edge-magic labeling of F with valence $3n+5$. Therefore, by the previous two results together with the fact presented in [6] that the forest $P_m \cup K_{1,n}$ is super edge-magic for every two integers $m \geq 4$ and $n \geq 1$, we immediately obtain the following theorem.

Theorem 4. *For every two positive integers m and n,*

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd} \\ & \quad \text{or } m = 3 \text{ and } n \not\equiv 0 \pmod 3; \\ 0, & \text{otherwise.} \end{cases}$$

Now, we study the super edge-magic properties of the forest $K_{1,m} \cup K_{1,n}$. For this purpose, let $F \cong K_{1,m} \cup K_{1,n} \cup K_1$ be the forest with

$$V(F) = \{v_i | 1 \leq i \leq m + n + 3\}$$

and

$$E(F) = \{v_2 v_i | 3 \leq i \leq m + 2\} \cup \{v_1 v_i | m + 4 \leq i \leq m + n + 3\}.$$

Then the vertex labeling $f : V(F) \to \{1, 2, \ldots, m + n + 3\}$ such that $f(v_i) = i$ $(1 \leq i \leq m + n + 3)$ induces a super edge-magic labeling of F with valence $2(m + n + 4)$. Thus, $\mu_s(K_{1,m} \cup K_{1,n}) \leq 1$ for every positive integer n.

On other hand, Ivančo and Lučkaničová proved in [10] that the forest $K_{1,m} \cup K_{1,n}$ is super edge-magic if and only if either m is a multiple of $n+1$ or n is a multiple of $m + 1$. Combining this with the above fact, we have the following theorem.

Theorem 5. *For every two positive integers m and n,*

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{either } m \text{ is a multiple of } n+1 \\ & \text{or } n \text{ is a multiple of } m+1; \\ 1, & \text{otherwise.} \end{cases}$$

We now inquire into the super edge-magic deficiency of the forest $P_m \cup P_n$. By Theorem 5, observe that $\mu_s(2P_n) = 1$ for $n = 2$ or 3. On the other hand, the authors proved in [7] that the forest $P_m \cup P_n$ is super edge-magic if and only if $(m, n) \notin \{(2, 2), (3, 3)\}$. Thus, we obtain the following theorem.

Theorem 6. *For every two positive integers m and n,*

$$\mu_s(P_m \cup P_n) = \begin{cases} 1, & \text{if } (m, n) \in \{(2, 2), (3, 3)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Now, we endeavor to find parallel edge-magic deficiency results to the ones on super edge-magic deficiency presented earlier in this section. This is interesting as Kotzig [11] proved that there are infinitely many forests that are not edge-magic. To do so, we start with the following corollary.

Corollary 2. *For every two positive integers m and n,*

$$\mu(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{if } mn \text{ is even;} \\ 1, & \text{if } mn \text{ is odd.} \end{cases}$$

Proof. First, by Theorem 5, we have that

$$\mu(K_{1,m} \cup K_{1,n}) \leq \mu_s(K_{1,m} \cup K_{1,n}) \leq 1.$$

Second, notice that the authors proved in [6] that the forest $K_{1,m} \cup K_{1,n}$ is edge-magic if and only if mn is even.

If we let $n = 1$ and 2 in the above corollary, then we can compute the edge-magic deficiencies of the forests $K_{1,1} \cup K_{1,n} \cong P_2 \cup K_{1,n}$ and $K_{1,2} \cup K_{1,n} \cong P_3 \cup K_{1,n}$, respectively. By adding these facts to Theorem 4, we have the following corollary.

Corollary 3. *For every two positive integers m and n,*

$$\mu(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

Now, by the above corollary, $2P_3$ is edge-magic and $2P_2$ is not. Therefore, by Theorem 6, we have the following corollary.

Corollary 4. *For every two positive integers m and n,*

$$\mu(P_m \cup P_n) = \begin{cases} 1, & \text{if } (m, n) = (2, 2); \\ 0, & \text{otherwise.} \end{cases}$$

The following two conjectures are born out of the results in this section and those found in [4–7]. First, we state the weaker of the two.

Conjecture 1. If F is a forest with two components, then $\mu(F) \leq 1$.

In fact, we believe that more is true.

Conjecture 2. If F is a forest with two components, then $\mu_s(F) \leq 1$.

3 Results on 2-Regular Graphs

A recurrent object of interest for the authors are the super edge-magic properties of 2-regular graphs; for example, see [4, 6, 7]. Indeed, a long term goal has become the evincing of the super edge-magic deficiencies of all of these graphs. Therefore, the results in this section are intended to narrow the gap between what is known and our objective.

Now, consider the following result found in [4].

Theorem 7. *The 2-regular graph mC_n is super edge-magic if and only if m and n are odd.*

The above theorem allows us to compute the super edge-magic deficiency of the 2-regular graph $2C_n$.

Theorem 8. *For every integer $n \geq 3$,*

$$\mu_s(2C_n) = \begin{cases} 1, & \text{if } n \text{ is even;} \\ +\infty, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First, assume that n is odd. Then, by Lemma 3, we obtain that $\mu_s(2C_n) = +\infty$.

Next, assume that n is even. Then, by Theorem 7, the 2-regular graph $2C_n$ is not super edge-magic. Hence, $\mu_s(2C_n) \geq 1$.

To show that $\mu_s(2C_n) \leq 1$, let $G \cong 2C_n \cup K_1$ be the graph with

$$V(G) = \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\} \cup \{z\}$$

and

$$\begin{aligned} E(G) &= \{x_i y_i | 1 \leq i \leq n\} \\ &\cup \{x_i y_{i+1} | 1 \leq i \leq n/2 - 1\} \cup \{x_{n/2} y_1\} \\ &\cup \{x_i y_{i+1} | n/2 + 1 \leq i \leq n - 1\} \cup \{x_n y_{n/2+1}\}, \end{aligned}$$

and consider two cases.

Case 1: For $n = 4k$, where k is a positive integer, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 8k + 1\}$ such that

$$f(x_i) = \begin{cases} 4k + i + 1, & \text{if } i \in [1, k]; \\ 8k - i + 2, & \text{if } i \in [k + 1, 2k]; \\ i - k + 1, & \text{if } i \in [2k + 1, 3k - 1]; \\ i + 1, & \text{if } i \in [3k, 4k]; \end{cases}$$

$$f(y_i) = \begin{cases} i, & \text{if } i \in [1, k + 1]; \\ 4k - i + 2, & \text{if } i \in [k + 2, 2k]; \\ 8k - i + 2, & \text{if } i \in [2k + 1, 3k]; \\ 4k + i + 1, & \text{if } i \in [3k + 1, 4k]; \end{cases}$$

and $f(z) = 2k + 1$.

Case 2: For $n = 4k + 2$, where k is a positive integer, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 8k + 5\}$ such that

$$f(x_i) = \begin{cases} 6k - i + 5, & \text{if } i \in [1, k - 1]; \\ 6k + i + 4, & \text{if } i \in [k, 2k + 1]; \\ i - 2k - 1, & \text{if } i \in [2k + 2, 3k + 2]; \\ 6k - i + 5, & \text{if } i \in [3k + 3, 4k + 2]; \end{cases}$$

$$f(y_i) = \begin{cases} 2k - i + 2, & \text{if } i \in [1, k]; \\ 2k + i + 2, & \text{if } i \in [k + 1, 3k + 3]; \\ 10k - i + 7, & \text{if } i \in [3k + 4, 4k + 2]; \end{cases}$$

and $f(z) = 2k + 2$.

Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $5n + 4$.

With the aid of Theorem 7, we are now able to provide the super edge-magic deficiency of the 2-regular graph $3C_n$.

Theorem 9. *For every integer $n \geq 3$,*

$$\mu_s(3C_n) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \equiv 0 \pmod 4; \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. It follows from Theorem 7 that $\mu_s(3C_n) = 0$ when n is odd; so we now assume that $n \equiv 0 \pmod 4$. Then, by Theorem 7, the 2-regular graph $3C_n$ is not super edge-magic, that is, $\mu_s(3C_n) \geq 1$.

To verify that $\mu_s(3C_n) \leq 1$, let $G \cong 3C_n \cup K_1$ be the graph with

$$V(G) = \{x_i | 1 \leq i \leq 3n/2\} \cup \{y_i | 1 \leq i \leq 3n/2\} \cup \{z\}$$

and

$$\begin{aligned} E(G) = & \{x_i y_i | 1 \leq i \leq 3n/2\} \\ & \cup \{x_i y_{i+1} | 1 \leq i \leq n/2 - 1\} \cup \{x_{n/2} y_1\} \\ & \cup \{x_i y_{i+1} | n/2 + 1 \leq i \leq n - 1\} \cup \{x_n y_{n/2+1}\} \\ & \cup \{x_i y_{i+1} | n + 1 \leq i \leq 3n/2 - 1\} \cup \{x_{3n/2} y_{n+1}\}, \end{aligned}$$

and consider two cases.

Case 1: For $n = 4$, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 13\}$ such that

$$\begin{aligned} (f(x_i))_{i=1}^6 &= (1, 3, 2, 7, 4, 5); \\ (f(y_i))_{i=1}^6 &= (8, 9, 6, 12, 11, 13); \text{ and} \\ f(z) &= 10. \end{aligned}$$

Case 2: For $n = 4k$, where k is an integer with $k \geq 2$, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 12k + 1\}$ such that $f(x_i) = i$, if $i \in [1, 6k]$;

$$f(y_i) = \begin{cases} 8k - 1, & \text{if } i = 1; \\ 6k + i - 1, & \text{if } i \in [2, k]; \\ 9k, & \text{if } i = k + 1; \\ 6k + i - 2, & \text{if } i \in [k + 2, 2k]; \\ 6k + i, & \text{if } i \in [2k + 1, 3k - 1]; \\ 6k + i + 3, & \text{if } i \in [3k, 4k - 1]; \\ 8k, & \text{if } i = 4k; \\ 6k + i + 2, & \text{if } i \in [4k + 1, 5k]; \\ 9k + 2, & \text{if } i = 5k + 1; \\ 6k + i + 1, & \text{if } i \in [5k + 2, 6k]; \end{cases}$$

and $f(z) = 9k + 10$.

Thus, by Lemma 1, f extends to a super edge-magic labeling of G with valence $15n/2+3$, and we conclude that $\mu_s(3C_n) = 1$ when $n \equiv 0 \pmod 4$.

Finally, the remaining case immediately follows from Lemma 3.

Now, in light of the above results, it seems natural to explore the super edge-magic deficiency of the 2-regular graph $4C_n$. However, the authors have only been able to provide a partial solution to this question, which is contained in the following theorem and the subsequent comment.

Theorem 10. *For every positive integer* $n \equiv 0 \pmod 4$, $\mu_s(4C_n) = 1$.

Proof. Throughout this proof, assume that $n \equiv 0 \pmod 4$. Then notice by Theorem 7 that the 2-regular graph $4C_n$ is not super edge-magic. Thus, it suffices to show that $\mu_s(4C_n) \leq 1$. To do this, let $G \cong 4C_n \cup K_1$ be the graph with

$$V(G) = \{x_i | 1 \leq i \leq 2n\} \cup \{y_i | 1 \leq i \leq 2n\} \cup \{z\}$$

and

$$\begin{aligned} E(G) = & \{x_i y_i | 1 \leq i \leq n\} \\ & \cup \{x_i y_{i+1} \,|1 \leq i \leq n/2 - 1\} \cup \{x_{n/2} y_1\} \\ & \cup \{x_i y_{i+1} \,|n/2 + 1 \leq i \leq n - 1\} \cup \{x_n y_{n/2+1}\} \\ & \cup \{x_i y_{i+1} \,|n + 1 \leq i \leq 3n/2 - 1\} \cup \{x_{3n/2} y_{n+1}\} \\ & \cup \{x_i y_{i+1} \,|3n/2 + 1 \leq i \leq 2n - 1\} \cup \{x_{2n} y_{3n/2+1}\}, \end{aligned}$$

and consider five cases according to the possible values for the integer n.

Case 1: For $n = 4$, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 17\}$ such that

$$\begin{aligned} (f(x_i))_{i=1}^{8} &= (1, 3, 2, 6, 4, 8, 7, 9); \\ (f(y_i))_{i=1}^{8} &= (10, 11, 13, 15, 12, 14, 16, 17); \text{ and} \\ f(z) &= 5. \end{aligned}$$

Case 2: For $n = 8$, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 33\}$ such that

$$\begin{aligned} (f(x_i))_{i=1}^{16} &= (1, 2, 4, 5, 7, 10, 14, 15, \\ &\quad\; 3, 6, 8, 12, 11, 13, 17, 16); \\ (f(y_i))_{i=1}^{16} &= (18, 19, 20, 21, 22, 23, 28, 32, \\ &\quad\; 24, 25, 26, 27, 29, 30, 31, 33); \text{ and} \\ f(z) &= 9. \end{aligned}$$

Case 3: For $n = 12$, define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 49\}$ such that

$$\begin{aligned}
(f(x_i))_{i=1}^{24} &= (1,2,3,4,5,11,6,7,\\
&\quad 15,19,20,23,8,9,10,12,\\
&\quad 14,17,16,18,22,25,24,21);\\
(f(y_i))_{i=1}^{24} &= (26,27,28,29,30,31,32,33,\\
&\quad 34,41,45,48,35,36,37,38,\\
&\quad 39,40,42,43,44,47,49,46); \text{ and}\\
f(z) &= 13.
\end{aligned}$$

Case 4: For $n = 8k+8$, where k is a positive integer, assume that l is an integer with $1 \le l \le k$, and define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 32k+33\}$ such that

$$f(x_i) = \begin{cases} i, & \text{if } i \in [1, 4k+3];\\ i-1, & \text{if } i \in [4k+5, 6k+5]; \end{cases}$$

$$\begin{aligned}
f(x_{4k+4}) &= 8k+7; & f(x_{6k+6}) &= 10k+10;\\
f(x_{8k+7}) &= 15k+15; & f(x_{8k+8}) &= 15k+16;\\
f(x_{10k+11}) &= 8k+8; & f(x_{12k+12}) &= 10k+12;\\
f(x_{12k+13}) &= 10k+11; & f(x_{12k+14}) &= 10k+13;\\
f(x_{16k+15}) &= 10k+17; & f(x_{16k+16}) &= 10k+16;
\end{aligned}$$

$$f(x_i) = \begin{cases} 9k+6l+9, & \text{if } i = 6k+2l+5;\\ 9k+6l+10, & \text{if } i = 6k+2l+6;\\ 8k+2l+8, & \text{if } i = 10k+2l+10;\\ 8k+2l+9, & \text{if } i = 10k+2l+11;\\ 4k+6l+18, & \text{if } i = 10k+2l+15;\\ 4k+6l+19, & \text{if } i = 10k+2l+16;\\ 16k-6l+23, & \text{if } i = 14k+2l+13;\\ 16k-6l+22, & \text{if } i = 14k+2l+14; \end{cases}$$

$$f(x_{8k+i+8}) = 5k+i+5, \text{ if } i \in [1, 2k+2];$$

$$f(y_i) = 16k+i+17, \text{ if } i \in [1, 6k+6];$$

$$\begin{aligned}
f(y_{6k+7}) &= 26k+28; & f(y_{8k+8}) &= 32k+32;\\
f(y_{12k+13}) &= 26k+29; & f(y_{12k+14}) &= 26k+30;\\
f(y_{12k+15}) &= 26k+31; & f(y_{14k+16}) &= 32k+33;
\end{aligned}$$

$$f(y_i) = \begin{cases} 26k+6l+26, & \text{if } i = 6k+2l+6;\\ 26k+6l+27, & \text{if } i = 6k+2l+7;\\ 26k+6l+30, & \text{if } i = 12k+2l+14;\\ 26k+6l+31, & \text{if } i = 12k+2l+15;\\ 32k-6l+35, & \text{if } i = 12k+2l+17;\\ 32k-6l+34, & \text{if } i = 12k+2l+18; \end{cases}$$

$f(y_{8k+i+8}) = 22k + i + 23$, if $i \in [1, 4k+4]$; and $f(z) = 8k + 9$.

Case 5: For $n = 8k+12$, where k is a positive integer, assume that l is an integer with $1 \le l \le k$, and define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 32k+49\}$ such that

$$f(x_i) = \begin{cases} i, & \text{if } i \in [1, 4k+5]; \\ i-1, & \text{if } i \in [4k+7, 6k+8]; \end{cases}$$

$$\begin{aligned} f(x_{4k+6}) &= 8k+11; & f(x_{6k+9}) &= 10k+15; \\ f(x_{8k+10}) &= 16k+19; & f(x_{8k+11}) &= 16k+20; \\ f(x_{8k+12}) &= 16k+23; & f(x_{10k+16}) &= 8k+12; \\ f(x_{12k+18}) &= 10k+17; & f(x_{12k+19}) &= 10k+16; \\ f(x_{12k+20}) &= 10k+18; & f(x_{14k+21}) &= 16k+22; \\ f(x_{14k+22}) &= 16k+25; & f(x_{14k+23}) &= 16k+24; \\ f(x_{14k+24}) &= 16k+21; \end{aligned}$$

$$f(x_i) = \begin{cases} 10k+6l+13, & \text{if } i = 6k+2l+8; \\ 10k+6l+14, & \text{if } i = 6k+2l+9; \\ 10k+6l+17, & \text{if } i = 12k+2l+9; \\ 10k+6l+18, & \text{if } i = 12k+2l+10; \\ 16k-6l+22, & \text{if } i = 14k+2l+23; \\ 16k-6l+21, & \text{if } i = 14k+2l+24; \end{cases}$$

$$f(x_{8k+i+12}) = \begin{cases} 6k+i+7, & \text{if } i \in [1, 2k+3]; \\ 6k+i+9, & \text{if } i \in [2k+5, 4k+5]; \end{cases}$$

$$f(y_i) = 16k + i + 25, \text{ if } i \in [1, 6k+9];$$

$$\begin{aligned} f(y_{6k+10}) &= 26k+41; & f(y_{8k+11}) &= 32k+45; \\ f(y_{8k+12}) &= 32k+48; & f(y_{12k+19}) &= 26k+42; \\ f(y_{12k+20}) &= 26k+43; & f(y_{12k+21}) &= 26k+44; \\ f(y_{14k+22}) &= 32k+47; & f(y_{14k+23}) &= 32k+49; \\ f(y_{14k+24}) &= 32k+46; \end{aligned}$$

$$f(y_i) = \begin{cases} 26k+6l+39, & \text{if } i = 6k+2l+9; \\ 26k+6l+40, & \text{if } i = 6k+2l+10; \\ 26k+6l+43, & \text{if } i = 12k+2l+20; \\ 26k+6l+44, & \text{if } i = 12k+2l+21; \\ 32k-6l+48, & \text{if } i = 14k+2l+23; \\ 32k-6l+47, & \text{if } i = 14k+2l+24; \end{cases}$$

$f(y_{8k+i+12}) = 22k + i + 34$, if $i \in [1, 4k+6]$; and $f(z) = 8k + 13$.

Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $10n + 4$, which leads to conclude that $\mu_s(4C_n) = 1$.

When $n \not\equiv 0 \pmod 4$, our knowledge is limited to the value of $\mu_s(4C_6)$ which we determine to be 1 as follows: first, by Theorem 7, the 2-regular graph $4C_6$ is not super edge-magic; however, the graph $4C_6 \cup K_1$ is super edge-magic by labeling the vertices in its cycles with $14-1-15-2-16-5-14$, $17-3-19-4-21-10-17$, $18-6-20-8-22-11-18$ and $23-9-25-13-24-12-23$, and its isolated vertex 7 to obtain a valence of 64.

Now, Lemma 3, all the results in this section and the authors' computation of $\mu_s(C_n)$ in [5] lead us to the following conjecture.

Conjecture 3. For every two integers $m \geq 1$ and $n \geq 3$, $\mu_s(mC_n) = 1$, if $mn \equiv 0 \pmod 4$.

Finally, to put forth a similar conjecture for the edge-magic deficiency of multiple copies of a cycle, we make some observations. First, if we combine Theorem 1 with Goldbold and Slater's result [9] or Kotzig and Rosa's result [12] that the cycle C_n is edge-magic, we have that the 2-regular graph mC_n is edge-magic when m is odd. Also, the authors have shown in [6] that mC_n is edge-magic when $m \equiv 2 \pmod 4$ and $n = 4$ or 6, or $n \equiv 1$, 5 or 7 (mod 12). Furthermore, Kotzig and Rosa [12] showed that $2C_3$ is not edge-magic, and mentioned that its edge-magic deficiency is 1. Indeed, an edge-magic labeling of $2C_3 \cup K_1$ is obtained by labeling its cycles with $2-6-13-2$ and $3-7-9-3$, and letting the valence be 20. Therefore, all these facts lead us to the following conjecture.

Conjecture 4. For every two integers $m \geq 1$ and $n \geq 3$,

$$\mu(mC_n) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n = 3; \\ 0, & \text{otherwise.} \end{cases}$$

4 Some Open Problems

We conclude this paper with some remarks on bounds for the edge-magic and super edge-magic deficiencies of graphs, and some open problems.

Notice that all the graphs that we study in this paper are either bipartite or tripartite, which means that the bounds of Corollary 1 hold for all of them. Unfortunately, these bounds are not sharp (for example, we know from that $\mu_s(C_{4n}) = 1$, which implies that $\mu_s(3C_{4n}) \leq 3$; however, we know by Theorem 9 that actually $\mu_s(3C_{4n}) = 1$). Nevertheless, these bounds are better than those found in [14, p. 62–64] (which are in terms of Sidon sequences), where Wallis commented that no good bounds were known for edge-magic deficiencies and the only known classes of graphs for which the exact values are known are the various classes of edge-magic graphs (which of course have edge-magic deficiency 0). Moreover, Kotzig and Rosa [12] easily found an upper bound for the edge-magic deficiency of a graph G of order p, namely, $\mu(G) \leq F_{p+2} - 2 - \binom{p+1}{2}$, where F_p is the p-th term of the Fibonacci sequence. This motivates us to propose the following problem.

Problem 1. Find a good upper bound for the (super) edge-magic deficiency of bipartite and tripartite graphs.

Finally, as mentioned in the introduction, the authors proved in [5] that $\mu_s(K_{2,n+1}) = n$ for every nonnegative integer n. This leads us to ask in the last problem whether a similar result is possible for edge-magic deficiency.

Problem 2. Given a nonnegative integer n, construct a graph G such that $\mu(G) = n$.

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References

1. G. Chartrand and L. Lesniak, *Graphs and Digraphs*, second edition. Wadsworth & Brooks/Cole Advanced Books and Software, Monterey (1986).
2. H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105–109.
3. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231** (1-3) (2001), 153–168.
4. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On super edge-magic graphs, *Ars Combin.* **64** (2002), 81–95.
5. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Ars Combin.*, to appear.
6. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On edge-magic labelings of certain disjoint unions of graphs, *Austral. J. Combin.*, to appear.
7. R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, A magical approach to some labeling conjectures, *Austral. J. Combin.*, submitted.
8. J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* 5 (2004), #DS6.
9. R. D. Goldbold and P. Slater, All cycles are edge-magic, *Bull. Inst. Combin. Appl.* **22** (1998), 93–97.

10. J. Ivančo and I. Lučkaničová, On edge-magic disconnected graphs, *SUT J. Math.* **38** (2002), 175–184.
11. A. Kotzig, On a class of graphs without magic valuations, *Publications du Centre de Recheres Mathemátiques Université de Montréal.* **136**, (1971).
12. A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.
13. G. Ringel and A. Lladó, Another tree conjecture, *Bull. Inst. Combin. Appl.* **18** (1996), 83–85.
14. W. D. Wallis, *Magic Graphs*. Birkhäuser, Boston (2001).