Critical Sets in Edge-Magic Total Labelings*

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Abstract. Let λ be an edge-magic total (EMT) labeling of graph $G(V,E)$. Let $W \subset V(G) \cup E(G)$. Any restriction of λ to W is called a *partial EMT labeling* on G. A partial EMT labeling π is a *critical set* in λ if λ is the only edge-magic total labeling having π as its partial EMT labeling, and no proper restriction of π satisfies the first condition. In this paper, we study the property of critical sets in such a labeling. We determine critical sets in an EMT labeling for a given graph G.

Keywords : critical sets, edge-magic total labeling

1 Introduction

The notion of critical sets in Latin squares was first introduced and studied by J. Nelder in 1977 [7]. Now, this study has developed, see for instances [1–3]. In this paper we introduce a similar notion for edge-magic total labelings on a given graph.

An *edge-magic total (EMT) labeling* on graph $G(V,E)$ with n vertices and m edges is a bijection λ from $V(G) \cup E(G)$ onto the set of integers $1, 2, \cdots, m+n$ such that there exists a positive integer k satisfying

$$\lambda(x) + \lambda(xy) + \lambda(y) = k,$$

for each edge $xy \in E(G)$. We shall follow [9] to call $\lambda(x) + \lambda(xy) + \lambda(y)$ the *edge sum* of xy, and k the *magic constant* of graph G. A graph that admits an EMT labeling is said to be *edge-magic total* (*EMT*).

The notion of edge-magic total labeling was introduced by Kotzig and Rosa [4] under the name of magic valuation.

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For any EMT labeling λ on graph G with n vertices and m edges, its dual labeling λ' is defined by

$$\lambda'(v_i) = M - \lambda(v_i),$$

for any vertex v_i, and

$$\lambda'(x) = M - \lambda(x),$$

for any edge x, where $M = n + m + 1$.

It is easy to see that if λ is an EMT labeling with magic constant k then λ' is an EMT labeling with magic constant $k' = 3M - k$. The sum of vertex labels of λ' is $s' = nM - s$, where s is the sum of vertex labels of λ.

Either s or s' will be less than or equal to $\frac{1}{2}nM$. This means that, in order to see whether a given graph is EMT, it suffices to check either all cases with $s \le \frac{1}{2}nM$ or all cases with $s \ge \frac{1}{2}nM$ (equivalently, $k \le \frac{3}{2}M$ or $k \ge \frac{3}{2}M$).

Let λ be an EMT labeling on graph $G(V, E)$. Let $W \subset V(G) \cup E(G)$. Any restriction of λ to W is called a *partial EMT labeling* on G. A partial EMT labeling π on graph G is said to be *uniquely completable* if there is only one EMT labeling having π as its partial EMT labeling on G. A partial EMT labeling π is a *critical set* in an EMT labeling λ on G if

1. π is uniquely completable, and
2. no proper restriction of π satisfies the property 1.

Example 1. Let λ be the EMT labeling on path P_3 as in Fig. 1(a). Thus, we can write $\lambda = \{(1,1),(2,5),(3,3),(4,4),(5,2)\}$, by noting that the first integers in each tuples represent the position. It can be easily verified that $\{(1,1),(4,4)\}$ as Fig. 1(b) is a critical set in λ. However, the set $\{(3,3),(4,4)\}$ as in Fig. 1(c) is not a critical set, since other than λ, the labeling $\{(1,5),(2,1),(3,3),(4,4),(5,2)\}$ also contains $\{(3,3),(4,4)\}$ as its partial EMT labeling.

The labeling in Fig. 1(a) can be represented as $\{1,5,3,4,2\}$ if we know in which order we label the graph. Then, the critical set in Fig. 1(b) can be represented as $\{1,\_,\_,4,\_\}$, or for short we write $\{1,4\}$. Therefore, we may consider any critical set as a set. Precisely, it is a set of some labels in a given edge-magic total labeling of a graph.

This paper studies the property of critical sets in an edge-magic total labeling.

2 Basic property and lower bound

In this section, we give some basic property of any critical set and we derive a lower bound of its size for a given EMT labeling of any particular graph.

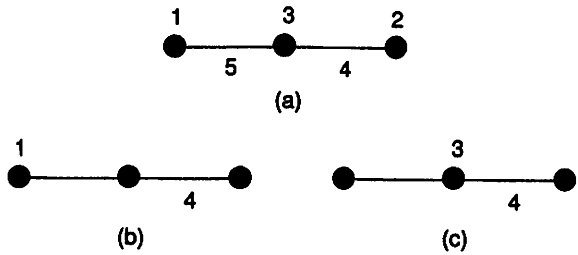

Fig. 1. The notion of critical set.

Theorem 1. *Let $Q = \{a_1, a_2, \cdots, a_q\}$ be a critical set of an EMT labeling λ on graph G with n vertices and m edges. Then $Q' = \{M - a_1, M - a_2, \cdots, M - a_q\}$ is a critical set of the dual labeling λ' on G where $M = n + m + 1$.*

Proof. Since Q is uniquely completable to λ then Q' is uniquely completable to λ' (by the duality property). Since no proper subset of Q can be uniquely completable, no proper subset of Q' can be also uniquely completable. Therefore, Q' is a critical set of λ' on G. □

Corollary 1 *Let s be a fixed positive integer. If an EMT labeling λ doesnot have any critical set of size s then neither does its dual labeling λ'.*

Proof. Suppose λ' has a critical set of size s. Then, by Theorem 1 the dual of λ', namely λ itself, also has a corresponding critical set of size s, a contradiction. □

The following theorem gives a lower bound and provide some information about which labels any critical set should consist of.

Theorem 2. *Let λ be any EMT labeling of graph G. If r is the number of leaves in G then the size of each critical set of λ is greater than r. Furthermore, if x is the label of any leaf and y is the label of the edge adjacent to it then each critical set in λ must contain either x or y, not both.*

Proof. Let $\{v_1, v_2, \dots, v_r\}$ be the set of all the leaves of G, and $\{e_1, e_2, \dots, e_r\}$ be the set of all edges incident to these leaves, accordingly. Let $\lambda(v_i) = x_i$, $\lambda(e_i) = y_i$ where $c_i = \{x_i, y_i\}$, $i = 1, 2, \dots, r$. Let Q_λ be a critical set of λ and for a contrary assume $|Q_\lambda| < r$. Then, there exists some i, say $i = k$, such that neither x_k nor y_k is in Q_λ. Therefore, completing Q_λ by swapping these two labels x_k and y_k will get a EMT labeling other than λ on

G, a contradiction. Therefore, $|Q_\lambda| \geq r$. The second statement immediately follows. □

Corollary 2 *The size of any critical set of an EMT labeling on tree T_n is at least r, where r is the number of its leaves.* □

3 Stars and its disjoint unions

In the following we show that for some particular tree, for instance stars, the size of some of its critical set is exactly the same as the number of its leaves.

Wallis et al. [9] showed that in every EMT labeling on a star S_n of n vertices, the center point of S_n always recieves a label either 1, n or $2n-1$.

Theorem 3. *Let λ be an EMT labeling on a star S_n on n vertices. Then, the size of any critical set of λ is either $n-1$ or n.*

Proof. Let Q be any critical set of λ. From Theorem 2, $|Q| \geq n-1$. Thus it suffices to show that $|Q| < n+1$. For a contrary assume that there is a critical set Q of λ with $|Q| = n+1$. Then, one of the following situations must hold: (i) Q contains label of the center and labels of some leaf together with an edge incident to it, (ii) Q contains labels of two leaves together with two edges incident to them. By Theorem 2, both cases are impposible. □

Theorem 4. *Let λ be a given EMT labeling on a star S_n on n vertices. Let c be the center point of S_n. Let Q_λ be a critical set in λ. Then, we have the following:*

1. $Q_\lambda = \{1, 2, 4, \cdots, 2n-2\}$ *and* $Q_\lambda = \{1, n+1, n+2, \cdots, 2n-1\}$ *are the only critical sets of size n if $\lambda(c) = 1$;*
2. $Q_\lambda = \{1, 2, 3, \cdots, n\}$ *and* $Q_\lambda = \{n, n+1, n+2, \cdots, 2n-1\}$ *are the only critical sets of size n if $\lambda(c) = n$;*
3. $Q_\lambda = \{1, 2, \cdots, n-1, 2n-1\}$ *and* $Q_\lambda = \{2, 4, \cdots, 2n-2, 2n-1\}$ *are the only critical sets of size n if $\lambda(c) = 2n-1$.*

Proof. By Theorem 2, If the label of some leaf is not in Q_λ then the label of the edge adjacent to it must be included in Q_λ or vice versa. Thus, if $|Q_\lambda| = n$ then $\lambda(c) \in Q_\lambda$. Therefore, the only way to get $|Q_\lambda| = n$ if we have one of the following conditions.

1. If $\lambda(c) = 1$ then Q_λ is either the set of label 1 together with all even labels or the set of label 1 together with labels $n+1, n+2, \cdots$, and $2n-1$;

2. If $\lambda(c) = n$ then $Q_\lambda = \{1, 2, 3, \cdots, n\}$ or $Q_\lambda = \{n, n+1, n+2, \cdots, 2n-1\}$;
3. If $\lambda(c) = 2n-1$ then Q_λ is either the set of label $2n-1$ together with labels $1, 2, \cdots, n-1$, or the set of label $2n-1$ with all even labels. □

The following theorem enumerates all the critical sets of a given EMT labeling on a star S_n on n vertices.

Theorem 5. *For each EMT labeling λ of a star S_n on n vertices there are two critical sets of λ with size n and $2^{n-1} - 2$ critical sets of size $n-1$.*

Proof. This follows immediately from Theorems 2, 3 and 4. □

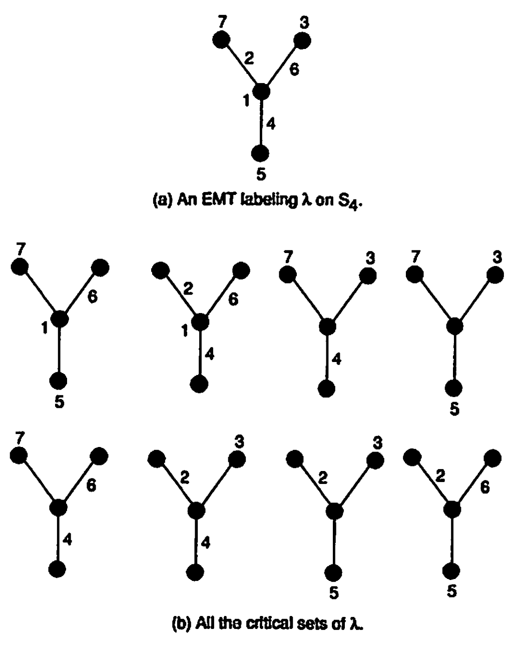

Fig. 2. List of all critical sets in a given EMT labeling on S_4.

In Figure 2(b), we list all critical sets of an EMT labeling on S_4 given in Fig. 2(a).

Now, let us denote by λ_1 and λ_2, respectively the EMT labelings of a star S_n with the vertex-set $\{c; v_1, v_2, \cdots, v_{n-1}\}$ and c the center vertex with the following properties.

$$\lambda_1(c)=1,\ \lambda_1(v_i)=i+1,\ \lambda_1(cv_i)=2n-i,\ \ \forall i.$$

$$\lambda_2(c)=n,\ \lambda_2(v_i)=i,\ \lambda_2(cv_i)=2n-i,\ \ \forall i.$$

Next, based on these two labelings we will determine a critical set in an EMT labeling of the unions tS_n of t disjoint copies of S_n, where t is odd. We know that tS_n is an EMT graph, see [8]. One way to label such a graph is the following. Take the EMT labeling λ_1 or λ_2 on S_n. Define a proper total colouring η in S_n by

$$\eta(c)=1,\quad \eta(v_i)=2,\quad \eta(cv_i)=3$$

for $i=1,2,\ldots,n-1$.

Denote the copies of S_n by $G_0,G_1,\ldots,G_{2r}$, where $2r+1=t$. Define matrix $A=[a_{\alpha\beta}]$ as follows:

$$\begin{pmatrix} 0 & 1 & \cdots & r-1 & r & r+1 & \cdots & 2r-1 & 2r \\ r+1 & r+2 & \cdots & 2r & 0 & 1 & \cdots & r-1 & r \\ 2r-1 & 2r-3 & \cdots & 1 & 2r & 2r-2 & \cdots & 2 & 0 \end{pmatrix}.$$

Then the graph element x of G_β, $\beta=0,1,\ldots,2r$, receives a label

$$\tau_j(x)=t(\lambda_j(x)-1)+1+a_{\eta(x)\beta}$$

for each $x\in V(tS_n)\cup E(tS_n)$ and $j=1,2$.

We can see that τ_1 and τ_2 are two different EMT labelings of tS_n.

Since all edge labels in τ_1 and τ_2 are exactly the same, namely $\tau_1(cv_i)=\tau_2(cv_i)$ for every $i=1,2\ldots n-1$, then by Theorem 2 the set of all edge labels plus label of one center vertex form a critical set in either τ_1 or τ_2 of tS_n. Therefore, we have the following theorem.

Theorem 6. *Let t be an odd integer. Let τ_1 be a super EMT labeling defined above on graph tS_n. Let $Q=\{1\}\cup\{\tau_1(xy)|xy\in E(tS_n)\}$. Then Q is a critical set of τ_1 with $|Q|=t(n-1)+1$.* □

4 Complete graphs

In [6], Kotzig and Rosa showed that no complete graph with more than 6 vertices is EMT. Furthermore, Wallis [9] enumerated all different EMT labelings on complete graphs with at most 6 vertices. Let k be the magic contant, S and s be the set of vertex labels and the sum of vertex labels, respectively. The complete list of all different EMT labelings for complete graphs is given below (see [9] and [8]). (Notice that in every case the solution for a given k is unique (if one exists).)

K_2 Trivially possible.
K_3 Sum values to be considered are $k = 9, 10, 11, 12$.
$k = 9$, $s = 6$, $S = \{1, 2, 3\}$.
$k = 10$, $s = 9$, $S = \{1, 3, 5\}$.
$k = 11$, $s = 12$, $S = \{2, 4, 6\}$.
$k = 12$, $s = 15$, $S = \{4, 5, 6\}$.
K_4 No solutions.
K_5 Sum values to be considered are $k = 18, 21, 24, 27, 30$.
$k = 18$, $s = 20$, $S = \{1, 2, 3, 5, 9\}$.
$k = 21$, $s = 30$, no solutions.
$k = 24$, $s = 40$, $S = \{1, 8, 9, 10, 12\}$.
$k = 24$, $s = 40$, $S = \{4, 6, 7, 8, 15\}$.
$k = 27$, $s = 50$, no solutions.
$k = 30$, $s = 60$, $S = \{7, 11, 13, 14, 15\}$.
K_6 Sum values to be considered are $k = 21, 25, 29, 33, 37, 41, 45$.
$k = 21$, $s = 21$, no solutions.
$k = 25$, $s = 36$, $S = \{1, 3, 4, 5, 9, 14\}$.
$k = 29$, $s = 51$, $S = \{2, 6, 7, 8, 10, 18\}$.
$k = 33$, $s = 66$, no solutions.
$k = 37$, $s = 81$, $S = \{4, 12, 14, 15, 16, 20\}$.
$k = 41$, $s = 96$, $S = \{8, 11, 17, 18, 19, 21\}$.
$k = 45$, $s = 111$, no solutions.

Theorem 7. *The only critical sets of an EMT labeling on K_3 are any partial EMT labeling of size 2 other than $\{1, 3\}$, $\{1, 6\}$, or $\{4, 6\}$.*

Proof. Let $\overline{6} = \{1, 2, 3, 4, 5, 6\}$. Since any $x \in \overline{6}$ is contained in two $S's$ then any singleton subset of $\overline{6}$ cannot be a critical set for λ of K_3. Clearly, a critical set cannot contain all vertex labels nor all edge labels. Thus, $2 \leq |Q| \leq 4$. Let $Q = \{a, b\}$ be any set of size 2 and $Q \neq \{1, 3\}, \{1, 6\}, \{4, 6\}$. Then, it can be verified that Q is uniquely completable and no singleton subset of Q has the same property. Therefore Q is a critical set. Next, we will show that no critical set of size more than 2 is possible. Suppose we have a critical set Q of size 3 and let $Q = \{a, b, c\}$. This yields that any 2-subset of Q cannot be uniquely completable. Therefore, the three 2-subset of Q must be $\{1, 3\}, \{1, 6\}$ and $\{4, 6\}$. This is impossible. As a consequence, there is no critical set of size 4 either. □

Theorem 8. *The number of critical sets of sizes* $1, 2,$ *and* 3 *are* $3, 36,$ *and* 7 *for the first labeling (as well as for the third labeling by the duality); while the critical sets of sizes* $1, 2,$ *and* 3 *are* $2, 34,$ *and* 19 *respectively for the* *second labeling (as well as for the fourth labeling by the duality) on K_5. A critical set of a greater size does not exist.*

Proof. We will enumerate all critical sets of all EMT labelings on K_5. Since the third labeling is the dual of the first one, and the fourth labeling is the dual of the second, so to enumerate all critical sets we suffice to enumerate critical sets for the first and second labelings. The critical sets of the third labeling are all the dual critical sets of the first labeling. Similarly, all critical sets of the fourth labeling are the dual of ones from the second labeling (by Theorem 1).

From now on, let Q be a critical set from one of the first two labelings on K_5 above (with $S = \{1,2,3,5,9\}$ or $S = \{1,8,9,10,12\}$).

Case 1. $|Q| = 1$.
Since any single edge label is contained in more than one EMT labelings on K_5 then no singleton critical set containing one edge label. If Q contains a vertex label x and $x \in \{2,3,5,10,12\}$ then Q is uniquely completable. Therefore Q is a critical set. In this case, all critical sets of size 1 for the first labeling are $\{2\},\{3\}$ and $\{5\}$; while for the second labeling we have $\{10\}$ and $\{12\}$.

Case 2. $|Q| = 2$.
If Q is a set of two vertex labels then the first labeling has no such a critical set and for the second labeling Q must be either $\{1,8\}$ or $\{8,9\}$.

If Q is a set of two edge labels then it can be verified that the first and second labelings have: $\{4,15\}$, $\{6,14\}$, $\{6,15\}$, $\{7,11\}$, $\{7,13\}$, $\{7,14\}$, $\{8,11\}$,$\{8,15\}$ as such critical sets. Additionally, the first labeling has further 17 such critical sets: $\{4,10\}$, $\{6,8\}$, $\{6,12\}$, $\{7,8\}$, $\{7,10\}$, $\{7,12\}$, $\{8,10\}$, $\{8,11\}$, $\{8,13\}$, $\{8,14\}$,$\{8,15\}$, $\{10,13\}$, $\{10,15\}$, $\{11,12\}$, $\{11,14\}$, $\{12,13\}$, $\{12,15\}$. Additionally, the second labeling also has further 9 such critical sets: $\{2,7\}$, $\{2,15\}$, $\{3,4\}$, $\{3,6\}$, $\{3,7\}$, $\{3,15\}$, $\{4,5\}$, $\{5,7\}$, $\{5,15\}$.

If Q is a critical set consisting of one vertex label a and one edge label b then $a \in \{1,8,9\}$. If $a = 1$ then ($b \in \{8,10,11,12,13\}$ for the first labeling) or ($b \in \{2,3,5,11,13\}$ for the second one). If $a = 9$ then ($b \in \{4,6,8,10,12,14\}$ for the first) or ($b \in \{2,3,4,5,6,14\}$ for the second). If $a = 8$ then $b \in \{4,6,7,15\}$ (only for the second labeling).

Case 3. $|Q| = 3$.
Next, we will enumerate all critical sets of size 3 for the first and second labelings. Let $Q = \{a,b,c\}$. Assume Q contains one vertex label, say a. Then $a \notin \{2,3,5,10,12\}$. Therefore the only possible values of a's are 1 and 9 (for the first labeling), 1, 8 and 9 (for the second labeling). This

means that there is no Q of size 3 consisting of three vertex labels, since $\{1, 8\}$ is a critical set. So, at least one of b and c are an edge label, say c. In the first labeling, if $a = 1$ then $c \in I = \{4, 6, 7, 14, 15\}$. If b is a vertex label then b must be 9. If b is an edge label then $b \in I$. Therefore no matter label b will be chosen, Q cannot be a critical set. If $a = 9$ then $c \in \{7, 11, 13, 15\}$. By a similar argument Q cannot be a critical set in this case. So, no critical set of size 3 contains at least one vertex label for the first labeling. By similar argument, we can also show that there is no such a critical set of the second labeling.

Now, in this case it remains to consider the case of Q containing three edge labels: a, b and c. Clearly, Q cannot contain any critical set of size 2 containing edge labels only. To count all such critical sets Q for the first labeling, start by letting $a = 4$ and $b = 6$. Then the possible values of c's are either $8, 10, 11, 12, 13, 14$, or 15. However, only $c = 8$ yields a critical set. Do the same process by changing b and a (respectively) with higher edge labels. It can be verified that the whole process will yield 7 critical sets: $\{4, 6, 8\}$, $\{4, 8, 13\}$, $\{4, 8, 14\}$, $\{4, 12, 14\}$, $\{6, 10, 11\}$, $\{12, 14, 15\}$, and $\{13, 14, 15\}$ for the first labeling. For the second labeling, by using the same process we have 19 such critical sets: $\{2, 3, 5\}$, $\{2, 3, 11\}$, $\{2, 4, 11\}$, $\{2, 4, 13\}$, $\{2, 4, 14\}$, $\{2, 5, 6\}$, $\{2, 5, 13\}$, $\{2, 6, 11\}$, $\{2, 6, 13\}$, $\{2, 11, 13\}$, $\{3, 5, 14\}$, $\{3, 11, 14\}$, $\{4, 11, 14\}$, $\{5, 6, 11\}$, $\{5, 6, 13\}$, $\{5, 13, 14\}$, $\{11, 13, 14\}$, $\{11, 14, 15\}$, and $\{13, 14, 15\}$.

Case 4. $|Q| \geq 4$.
As a consequence of Case 3, there is no critical set of size 4 or greater containing at least one vertex label. Therefore, the only critical sets Q of size ≥ 4 consists of edge labels only. Let now consider Q of size 4 for the first labeling, and let $Q = \{a, b, c, d\}$. For a start let $a = 4$, $b = 6$, $c = 7$ and $d \in \{8, 10, 11, 12, 13, 14, 15\}$. By considering all critical sets of lower sizes we can show no such Q with $a = 4$, $b = 6$, $c = 7$. Iterate values a, b and c over all possible edge labels. Since Q cannot contain any critical set of size ≤ 3 then it can be verified that no such Q is possible for the first labeling. Since there is no critical set Q of size 4 then neither is critical set of size ≥ 5. By a similar process, we can show that the second labeling has no critical sets of size ≥ 4. □

5 Open Problems

To conclude this paper, let us present the following open problem to work on.

Problem 1.

1. *Characterize all critical sets in an EMT labeling for other class of graphs ?*
2. *Construct an algorithm to verify whether a given partial EMT labeling is a critical set or not ?*
3. *Study the characteristics of critical sets in other type of labeling on graphs?*

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