Magic coverings

A. Gutiérrez, A. Lladó*

Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya, Barcelona, Spain
allado@mat.upc.es

Abstract. A simple graph $G = (V, E)$ admits an *H-covering* if every edge in E belongs to a subgraph of G isomorphic to H. In this case we say that G is *H-magic* if there is a total labeling $f : V \cup E \to \{1, 2, \cdots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of G isomorphic to H, $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e))$ is constant. When $f(V) = \{1, \cdots, |V|\}$, we say that G is *H-supermagic*.
We study H–magic graphs for several classes of connected graphs. We also provide constructions of infinite families of H–magic graphs for an arbitrary given graph H.

1 Introduction

Let $G = (V, E)$ be a finite simple graph. An edge-covering of G is a family of different subgraphs $H_1, ..., H_k$ such that any edge of E belongs to at least one of the subgraphs H_i, $1 \leq i \leq k$. Then, it is said that G admits an $(H_1, \cdots, H_k)$–(edge)covering. If every H_i is isomorphic to a given graph H, then we say that G admits an *H–covering.*

Suppose that $G = (V, E)$ admits an H-covering. We say that a bijective function

$$f : V \cup E \to \{1, 2, \cdots, |V| + |E|\},$$

is an *H-magic* labeling of G if there is a positive integer $m(f)$, which we call *magic-sum*, such that for each subgraph $H' = (V', E')$ of G isomorphic to H we have,

$$f(H') \stackrel{def}{=} \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f).$$

In this case we say that the graph G is *H-magic.* When $f(V) = \{1, \cdots, |V|\}$, we say that G is *H-supermagic* and we denote its *supermagic-sum* by $s(f)$. In Figure 1 we show an example of a P_3-supermagic labelling of P_6.

* Supported by the Spanish Ministry of Science and Technology and the European Regional Development Fund (ERDF) under project-BFM-200200417 and the Catalan Research Council under grant 2001SGR-00079

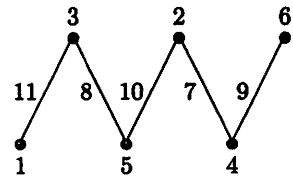

Fig. 1. P_3–supermagic labelling of P_6 .

When H is isomorphic just to one edge, the previous concept coincides with the well–known magic valuation introduced by A. Rosa [9] in 1966 and the supermagic case with the one treated with posteriority in [2], see also [1]. Notice that, in this case, an H-covering is also an H-decomposition. There are other closely related notions of magic labellings in the literature; see the survey of Gallian [4] and the references therein. Gallian suggests the denomination of *total edge–magic labellings* for what we just called magic labellings (when $H = K_2$.) For simplicity, in this paper we will use this shorter denomination.

In Section 2 we study star-magic coverings, particularly for complete bipartite graphs. We obtain general results which include the classical ones with respect to the basic magic property, which corresponds to the star with a single edge. Section 3 deals with path–magic coverings, another extension of the classical magic labellings. In particular we study the path–supermagic coverings of paths and cycles. We conclude in Section 4 by showing that, for any given graph H with some weak conditions, there are infinite families of H–magic graphs.

We use the following notations. For any two integers $n < m$ we denote by $[n, m]$ the set of all consecutive integers from n to m.

For any set $I \subset \mathbb{N}$ we write, $\sum I = \sum_{x \in I} x$. Note that, for any $k \in \mathbb{N}$,

$$\sum(I + k) = \sum I + k|I|.$$

Finally, given a graph $G = (V, E)$ and a total labelling f on it we denote by

$$f(G) = \sum f(V) + \sum f(E).$$

2 Star-magic coverings

In this Section we study star–magic coverings, which is an extension of the well-known problem of magic decompositions. We consider the basic families of complete and complete bipartite graphs with respect to the star–magic and star–supermagic properties.

It is clear that, for any pair of positive integers $n \geq h$, the star $K_{1,n}$ can be covered by a family of $\binom{n}{h}$ stars$K_{1,h}$. We start by easily proving that $K_{1,n}$ is $K_{1,h}$–supermagic for any $1 \leq h \leq n$.

Proposition 1. *The star $K_{1,n}$ is $K_{1,h}$-supermagic for any $1 \leq h \leq n$.*

Proof. Denote by $V = \{v_1, \cdots, v_n, v_{n+1}\}$ the vertex set of $K_{1,n}$, where v_{n+1} is the vertex with maximum degree, and by $E = \{e_i = v_{n+1}v_i, 1 \leq i \leq n\}$ its set of edges.

Define a total labeling $f : V \cup E \rightarrow [1, 2n+1]$ as follows, set $f(v_{n+1}) = n+1$ and for each $1 \leq i \leq n$

$$f(v_i) = i, \qquad f(e_i) = 2(n+1) - i.$$

Clearly, $f(V) = [1, n+1]$ and $f(E) = [n+2, 2n+1]$.

For each subgraph H of $K_{1,n}$ isomorphic to $K_{1,h}$ we have,

$$f(H) = (n+1) + h(i + (2(n+1) - i)) = (n+1)(2h+1).$$

Therefore $K_{1,n}$ is $K_{1,h}$-supermagic for each $1 \leq h \leq n$. □

If a graph G is $K_{1,h}$-magic then we clearly have $h \leq \Delta(G)$. Next lemma says something stronger in this direction and gives as a corollary a sharp lower bound for h.

Lemma 1. *Let f be a $K_{1,h}$-magic labeling of a graph G. If the degree of a vertex $x \in V(G)$ verifies $d(x) > h$ then, for every vertex y adjacent to x, we have,*

$$f(y) + f(xy) = \frac{1}{h}(m(f) - f(x)).$$

Proof. Let $N(x) = \{y_1, \cdots, y_r\}$ be the neighborhood of x, with $r > h$, and denote by A the multiset

$$A = \{f(y_1) + f(xy_1), \cdots, f(y_r) + f(xy_r)\}.$$

For each pair of h-subsets X, Y of A we have,

$$\begin{aligned} m(f) &= f(x) + \sum X = f(x) + \sum Y \\ &= f(x) + (\sum X - \sum(X \setminus Y) + \sum(Y \setminus X)), \end{aligned}$$

which implies $\sum(Y \setminus X) = \sum(X \setminus Y)$.

By taking $Y \setminus X = \{f(y_i) + f(xy_i)\}$ and $X \setminus Y = \{f(y_j) + f(xy_j)\}$, $i \neq j$, we see that

$$f(y_1) + f(xy_1) = \cdots = f(y_r) + f(xy_r) = c,$$

for some constant c.

Therefore, $m(f) = f(x) + hc$, which gives the value of c. □

Corollary 1. *Let G be a $K_{1,h}$-magic graph with $h > 1$. Then, for every edge $e = xy$ of G,*

$$\min\{d(x), d(y)\} \leq h.$$

Proof. Suppose on the contrary that $\min\{d(x), d(y)\} > h$ and let f be a $K_{1,h}$-magic labeling of G. By Lemma 1 we have,

$$\begin{aligned} f(x) + f(xy) + f(y) &= \frac{1}{h}(m(f) - f(y)) + f(y) \\ &= f(x) + \frac{1}{h}(m(f) - f(x)), \end{aligned}$$

which implies $f(x) = f(y)$, a contradiction. □

As a direct consequence of Corollary 1 we obtain the next general result for regular graphs.

Corollary 2. *Let G be a d-regular graph. Then G is not $K_{1,h}$-magic for any $1 < h < d$.* □

It is well–known that the complete graph K_n is not magic for any order bigger than six, [3, 6, 8]. It is also known [7] that the complete bipartite graphs of any order are magic. As a direct consequence of Corollary 2 we obtain the following result.

Corollary 3.

(a) The complete graph K_n is not $K_{1,h}$-magic for any $1 < h < n - 1$.

(b) The complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-magic for any $1 < h < n$.

□

Using a classical result on the existence of magic squares it is easy to prove the extremal case for complete bipartite graphs.

Theorem 1. *The complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for $n \geq 1$.*

Proof. It is well-known that there are magic squares of each order $m > 2$, that is, square matrices of order m with distinc entries in $[m^2]$ and all rows and columns adding up to $m(m^2+1)/2$.

Set U and W for the stable sets of $K_{n,n}$ and E for its set of edges. Define a labeling f from $U \cup W \cup E$, to $[n^2]$ as follows.

Take a magic square A of order $m = (n+1)$ with $a_{1,1} = (n+1)^2$ and for $1 \leq i,j \leq n$, define

$$f(u_i) = a_{1+i,1}, \qquad f(w_j) = a_{1,1+j}, \qquad f(u_i w_j) = a_{1+i,1+j}.$$

Therefore, f is a $K_{1,n}$-magic labeling of $K_{n,n}$. □

The extremal case for complete bipartite graphs with respect the star–supermagic property is not that easy, as it is shown in what follows. For that we will use the result contained in the next Lemma, dealing with 2-partions of sets of consecutive integers with cardinal bigger than two.

Lemma 2. *For an integer $m > 2$ let $\{X_1, X_2\}$ be a partition of $[1, m]$.*
If $\sum X_1 = \sum X_2$ then, $m \equiv 0, 3 \pmod 4$.

Proof. By the definition of X_1 and X_2 we have,

$$\sum X_1 + \sum X_2 = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$

and therefore

$$\sum X_1 = \sum X_2 = \frac{m(m+1)}{4}.$$

Hence, $m \equiv 0, 3 \pmod 4$. □

Corollary 4. *The complete bipartite graph $K_{n,n}$ is not $K_{1,n}$-supermagic for any integer $n > 1$.*

Proof. Let $U = \{u_1, \cdots, u_n\}$ and $W = \{w_1, \cdots, w_n\}$ the stable sets of $K_{n,n}$ and set $E = E(K_{n,n})$. Denote by H_i (resp. H_i') the star $K_{1,n}$ in $K_{n,n}$ with center u_i (resp. w_i), $1 \leq i \leq n$.

Suppose that f is a $K_{1,n}$-supermagic labeling of $K_{n,n}$ with supermagic-sum $s(f)$. Then, we have

$$\begin{aligned} ns(f) &= \sum_{i=1}^{n} f(H_i) = \sum f(U) + n \sum f(W) + \sum f(E) \\ &= \sum_{i=1}^{n} f(H_i') = n \sum f(U) + \sum f(W) + \sum f(E), \end{aligned}$$

which implies $\sum f(U) = \sum f(W)$.

Since f is a supermagic labeling, $\sum f(U) = n(2n+1)/2$ and $\sum f(E) = (n^2+2n+1)(n^2-2n)/2$. Moreover, $\{f(U), f(W)\}$ is a partition of $[1, 2n]$ and, by Lemma 2, n is even.

Substituting in the above equation, we have

$$s(f) = (n+1)(2n+1)/2 + n(n^2+2n+1)/2 = 2n(n+1) + (n^3+1)/2.$$

Therefore $s(f)$ is not an integer, a contradiction. □

We next study the same question for general complete bipartite graphs $K_{r,s}$ when $1 < r < s$. In [2] it is proved that the only bipartite complete graphs which are supermagic are the stars.

Corollary 1 ensures that if there exists a $K_{1,h}$-magic labeling of $K_{r,s}$ then $h \geq r$. Next Theorem says that in fact there is no integer $1 < h < s$ for which $K_{r,s}$ admits an $K_{1,h}$-supermagic labelling. It also states that $K_{r,s}$ is $K_{1,s}$-supermagic, which is an extension of the result given in [2].

Theorem 2. *For any pair of integers $1 < r < s$, the complete bipartite graph $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$.*

Proof. First we prove that $K_{r,s}$, $1 < r < s$, is not $K_{1,h}$-supermagic for any integer $1 < h \neq s$.

If $1 < h \leq r$, the result follows from Corollary 1.

Suppose that, for some $r < h < s$ there exists a $K_{1,h}$-supermagic labeling f of $K_{r,s}$ with supermagic sum $s(f)$.

Denote by $U = \{u_1, \cdots, u_r\}$ and $W = \{w_1, \cdots, w_s\}$ the stable sets of $K_{r,s}$, and for each $1 \leq i \leq r$ set $E_i = \{w_1u_i, \ldots, w_su_i\}$.

Lemma 1 ensures that, for any pair of subscripts $1 \leq i \leq r$ and $1 \leq j \leq s$,

$$f(w_j) + f(w_ju_i) = \frac{1}{h}(s(f) - f(u_i)) \tag{1}$$

Denote by $a_i = (s(f) - f(u_i))/h$ and set $A = \{a_1, \ldots, a_r\}$, where we may assume $a_1 < a_2 \cdots < a_r$. We have $f(U) = s(f) - h \cdot A$.

On the other hand, equality (2) implies

$$f(W) = a_1 - f(E_1) = a_2 - f(E_2) = \cdots = a_r - f(E_r).$$

Therefore $f(E) = \cup_{i=1}^{r} f(E_i)$ is the disjoint union

$$f(E) = \cup_{i=1}^{r}(f(E_1) + (a_i - a_1)).$$

Denote by $a = \min\{a_{i+1} - a_i, 1 \leq i < r\}$. Then, for some i,

$$f(E_1) \cap (f(E_1) + a) = ((f(E_1) + a_i) \cap (f(E_1) + a_{i+1})) - a_i = \emptyset.$$

In particular, the longest interval of integers contained in $f(E_1)$ has length at most a. Since $f(W) = a_1 - f(E_1)$ the same is true for $f(W)$. But the elements of $f(U) = s(f) - h \cdot A$ are at mutual distance at least $ha \geq 2a$. Since f is supermagic, we have $f(U) \cup f(W) = [1, r+s]$. If $a \geq 2$ then $f(W)$ contains an interval of length at least $2a - 1 > a$, a contradiction. If $a = 1$, since $s \geq r + 2$, $f(W)$ also contains an interval of length at least 2. This contradiction shows that we can not have a $K_{1,h}$–supermagic labelling of $K_{r,s}$ when $r < h < s$.

Assume now that $h = s$ and denote by H_i, $1 \leq i \leq r$, the copy of $K_{1,s}$ in $K_{r,s}$ containing the vertex $u_i \in U$.

We are going to give f, a $K_{1,s}$-magic labeling on $K_{r,s}$. In order to define it on $E(H_i)$ we will consider the following two translations of some interval $I = [1, s']$. For each $1 \leq i \leq r$ and $t = r + s$, we define,

$$I_i = I + t + s'(i-1) \;\text{ and }\; I_i' = I + t + s'(2r - i).$$

Suppose first that s is odd.

Define the following labeling on the stable sets of $V(K_{r,s})$,

$$f(U, W) = \begin{cases} f(u_i) = i, & 1 \leq i \leq r \\ f(w_i) = r + i, & 1 \leq i \leq s \end{cases}$$

Then, f is a bijection from $V(K_{r,s})$ to $[1, r+s]$.

Take $s' = \frac{s-1}{2}$ and define f on $E(H_i)$ by any bijection on $I_i \cup I_i' \cup \{t + rs - i + 1\}$.

Notice that, $\cup_{i=1}^{r} I_i = [t+1,\ t + s'r]$, $\cup_{i=1}^{r} I_i' = [t + rs' + 1,\ t + rs - r]$ and $\cup_{i=1}^{r}\{t + rs - i + 1\} = [t + rs - r + 1,\ t + rs]$. Therefore,

the family of $f(H_i)$ for $1 \leq i \leq r$ is a partition of the interval $[t+1,\ t+rs]$ and f is a bijection from $E(K_{r,s})$ to $[t+1,\ t+rs]$.

In addition, for each $1 \leq i \leq r$ we have,

$$\sum f(V(H_i)) = i + rs + \binom{s+1}{2},$$

$$\sum f(E(H_i)) = 3t + rs + s'(2r-1) - i + 1 + 2\sum I =$$

$$= 2r + 3s + 2rs + (s')^2 + 1 - i.$$

Clearly, the addition $\sum f(V(H_i)) + \sum f(E(H_i))$ is independent of i and therefore, the supermagic property is satisfied.

Suppose now that s is even.

Define the following labeling on $V(K_{r,s})$. Take $f(u_i) = 2i - 1$, for $1 \leq i \leq r$, and $f(W)$ with any bijection on $[1, t] \setminus f(U)$. Therefore, f is a bijection from $V(K_{r,s})$ to $[1, t]$.

Define f on $E(H_i)$ by any bijection on $I_i \cup I'_i \cup \{s+rs-i+1,\ t+rs-i+1\}$ but now, $s' = \frac{s-2}{2}$. Then we have,

$$\cup_{i=1}^{r} I_i = [t+1,\ t+s'r], \quad \cup_{i=1}^{r} I'_i = [t+rs'+1,\ t+2rs'],$$

Therefore, $(\cup_{i=1}^{r} I_i) \cup (\cup_{i=1}^{r} I'_i) = [t+1,\ t+rs-r]$. As

$$\cup_{i=1}^{r}\{t+rs-i+\{1,1+r\}\} = [t+rs-r+1,\ t+rs],$$

we also have in this case that f is a bijection from $E(K_{r,s})$ to $[t+1, t+rs]$.

It is easily checked that for each $1 \le i \le r$,

$$\sum f(V(H_i)) = 2i + r + \binom{t+1}{2} - \binom{r+1}{2} - 1,$$

$$\sum f(E(H_i)) = 3t + rs + s + 2 - 2i + s'(s-1) + 2\sum I =$$

$$= 3r + 4s + 2rs + (s')^2 + ss' + 2 - 2i.$$

Therefore, $\sum f(V(H_i)) + \sum f(E(H_i))$ is also independent of i. This completes the proof. □

3 Path-magic coverings

In this Section we consider path–magic coverings, which is another extension of the already mentioned magic decompositions. The first result in this Section concerns the path–supermagic behavior of paths.

Theorem 3. *The path P_n is P_h-supermagic for any integer $2 \le h \le n$.*

Proof. Let $V = \{v_i,\ 1 \le i \le n\}$ and $E = \{e_i = v_iv_{i+1},\ 1 \le i < n\}$ be the vertex and edge set respectively of P_n.

For each $1 \le i \le n$ consider the decomposition $i = i_1 + i_2h$, with $1 \le i_1 \le h$ and $0 \le i_2 < n/h$ and write $\alpha(i) = (i_1, i_2)$. Let f_1 be the lexicographic ordering of the pairs $\alpha(i)$.

Similarly, for each $1 \le i < n$, consider $i = i_1 + i_2(h-1)$ with , $1 \le i_1 \le h-1$ and $0 \le i_2 < (n-1)/(h-1)$. Let f_2 be the lexicographic ordering of the pairs $\beta(i)$.

Consider the total labeling $f : V(P_n) \cup E(P_n) \to [1, 2n-1]$ defined as follows: $f(x_i) = f_1(\alpha(i))$ on the set of vertices and $f(e_i) = 2n - f_2(\beta(i))$ on the set of edges. It is clear that $f(V(P_n)) = [1, n]$ and $f(E(P_n)) = [n+1, 2n-1]$.

Let us show that f is a P_h-magic labeling. For this let $P_h^{(i)}$, $1 \le i \le n-h+1$, be the subpath of P_n with vertex set $\{x_i, x_{i+1}, \dots, x_{i+h-1}\}$ and edge set $\{e_i, e_{i+1}, \dots, e_{i+h-2}\}$.

For $1 \le i \le n-h$ we have,

$$f(P_h^{(i+1)}) - f(P_h^{(i)}) = f(x_{i+h}) - f(x_i) + f(e_{i+h-1}) - f(e_i).$$

If $\alpha(i) = (i_1, i_2)$ then $\alpha(i+h) = (i_1, i_2+1)$, so that $f(x_{i+h}) - f(x_i) = 1$. On the other hand, if $\beta(i) = (i_1, i_2)$, then $\beta(i+h-1) = (i_1, i_2+1)$ which implies $f(e_{i+h-1}) - f(e_i) = -1$.

Hence, for each $P_h^{(i)}$ subpath in P_n, we have that $f(P_h^{(i)}) = f(P_h^1)$ and f is P_h-supermagic. □

It has already been mentioned that the complete graph K_n is not magic for any integer $n > 6$. Corollary 3 shows that K_n is not P_3–magic. Next easy Lemma implies that complete graphs are not path-magic for any path of length larger than 2.

Lemma 3. *Let G be a P_h-magic graph, $h > 2$. Then G is C_h-free.*

Proof. Suppose that C_h is a cycle in G with edges $e_1, e_2, \cdots e_h$ and f is an P_h-magic labeling of G. Then, by taking the P_h subgraphs of C_h with edges $e_1, e_2, \cdots, e_{h-1}$ and $e_2, e_3, \cdots, e_h$ we have

$$\sum_{x \in V(C_h)} f(x) + \sum_{i=1}^{h-1} f(e_i) = \sum_{x \in V(C_h)} f(x) + \sum_{i=2}^{h} f(e_i)$$

which implies $f(e_1) = f(e_h)$ and f is not injective. □

Corollary 5. *The complete graph K_n is not P_h-magic for any $2 < h \le n$.* □

All cycles are magic, see [5]. It is also known that only the odd cycles are supermagic [2]. By adding an additional divisibility condition we get path–supermagic labelings of cycles as shown in the next Theorem.

Theorem 4. *The cycle C_n is P_h-supermagic for any $2 \le h < n$ such that* $\gcd(n, h(h-1)) = 1$.

Proof. Let n and h be two positive integers such that $2 \le h < n$ and $\gcd(n, h(h-1)) = 1$. We prove that C_n is P_h-supermagic.

Let $V = \{x_i,\ 0 \le i < n\}$ and $E = \{e_i = x_{i-1}x_i,\ 0 \le i < n\}$ be the vertex and edge set respectively of the cycle C_n, where the subscripts are taken modulo n.

We define a total labeling f from $V \cup E$ to $[1, 2n]$ as follows.

Let $f(x_{ih}) = 1+i$ and $f(e_{i(h-1)}) = 2n-i$, $0 \le i < n$. Since $\gcd(n, h) = 1$ and $\gcd(n, h-1) = 1$, f is well defined and clearly we have, $f(V) = [1, n]$ and $f(E) = [n+1, 2n]$.

Let us show that f is a P_h-magic labeling.

For each $0 \le i < n$, let $P_h^{(i)}$ be the subpath of C_n with vertex set $V(P_h^{(i)}) = \{x_i, x_{i+1}, \ldots, x_{i+h-1}\}$ and edge set $E(P_h^{(i)}) = \{e_{i+1}, e_{i+2}, \ldots, e_{i+h-1}\}$.

Then, we have

$$f(P_h^{(ih+1)}) - f(P_h^{(ih)}) = f(x_{(i+1)h}) - f(x_{ih}) + f(e_{(i+1)h}) - f(e_{ih+1}).$$

Notice that, $ih + 1 = i(h-1) + (i+1)$. Consider j defined by the equality $j(h-1) = i(h-1) + (i+1)$ in $\mathbb{Z}_n$. Then,

$$f(e_{(i+1)h}) - f(e_{ih+1}) = f(e_{(j+1)(h-1)}) - f(e_{j(h-1)}).$$

If $i \ne n-1$ then $f(x_{(i+1)h}) - f(x_{ih}) = 1$ and $f(e_{(j+1)(h-1)}) - f(e_{j(h-1)}) = -1$. If $i = n-1$ then $j = n-1$ as well and the terms in the right hand side of the equality are $f(x_0) - f(x_{n-h}) = 1-n$ and $f(e_0) - f(e_{n-(h-1)}) = -(1-n)$. Therefore, $f(P_h^{(ih)}) = f(P_h^{(0)})$ for $0 \le i < n$. This completes the proof. □

4 Constructing H-magic graphs

The object of this Section is to obtain infinite families of H-magic graphs for a given graph H.

We first need some preliminary results.

Let $¶ = \{X_1, \cdots, X_k\}$ be a partition of a set X of integers. When all sets have the same cardinality we say that ¶ is a *k-equipartition* of X. We identify the partition ¶ with a $\{1, \cdots, k\}$-coloring of X such that for every $1 \le i \le k$, the set of elements in X colored with the color i determine X_i.

We denote the set of subsets sums of the parts of ¶ by

$$\sum ¶ = \{\sum X_1, \cdots, \sum X_k\}.$$

Lemma 4. *Let h and k be two positive integers and let $n = hk$. For each integer $0 \le t \le \lfloor h/2 \rfloor$ there is a k-equipartition ¶ of $[1, n]$ such that $\sum ¶$ is an arithmetic progression of difference $d = h - 2t$.*

Proof. For each integer $0 \le t \le \lfloor h/2 \rfloor$, let $¶_t = \{X_1^{(t)}, \ldots, X_k^{(t)}\}$ be the k-equipartition of $[1, hk]$ given by the following coloring,

$$(1, 2, \cdots, k)^{h-t}(k, k-1, \cdots, 1)^t.$$

That is, for each positive integer $i \le k$,

$$X_i^{(t)} = (\cup_{j=1}^{h-t}\{(j-1)k+i\}) \cup (\cup_{j=h-t}^{h-1}\{(j+1)k-i+1\}).$$

Then, we have

$$\sum X_i^{(t)} = \sum X_1^{(t)} + (i-1)(h-t) + (1-i)t = \sum X_1^{(t)} + (i-1)(h-2t).$$

Therefore,

$$\sum ¶_t = \{\sum X_1^{(t)}, \sum X_1^{(t)} + (h-2t), \cdots, \sum X_1^{(t)} + (k-1)(h-2t)\}$$

forms an arithmetic progression of difference $h-2t$. □

Lemma 5. *Let h and k be two positive integers and let $n = hk$. In the two following cases there exists a k–equipartition ¶ of a set X such that $\sum ¶$ is a set of consecutive integers.*

(i) h or k are not both even and $X = [1, hk]$

(ii) $h = 2$ and k is even and $X = [1, 2k+1] \setminus \{k/2+1\}$.

Proof. (i) If h is odd the result follows from Lemma 4 by taking $t = (h-1)/2$.

Suppose that h is even and $k \ge 3$ is odd. Consider the partition $¶ = \{X_1, \cdots, X_k\}$ given by the coloring

$$(1,2,\cdots,k)^{\frac{h}{2}}(k,k-1,\cdots,1)^{\frac{h}{2}-1}(\tau(1),\tau(2),\cdots,\tau(k)),$$

where τ denotes the permutation of $\{1,\cdots,k\}$ given by

$$\tau(i) = \begin{cases} 1+k-2i, & 1 \le i \le (k-1)/2 \\ 1+2(k-i), & (k+1)/2 \le i \le k \end{cases}$$

Then, we clearly have

$$\begin{aligned} \sum X_i - \sum X_1 &= (h/2)(i-1) + (h/2-1)(1-i) + (\tau(i)-\tau(1)) \\ &= (h/2)(i-1) - (h/2-1)(i-1) + (\tau(i)-\tau(1)) \\ &= i + \tau(i) - k \\ &= \begin{cases} 1-i, & 1 \le i \le (k-1)/2 \\ 1+k-i, & (k+1)/2 \le i \le k \end{cases}. \end{aligned}$$

Hence, $\sum ¶$ is the set of consecutive integers,

$$\sum X_1 + ([-(k-3)/2,\ 0] \cup [1,\ (k+1)/2]).$$

(ii) For $1 \leq i \leq k$ consider the sets

$$X_i = \begin{cases} \{i,\ 2(k-i)+3\} & 1 \leq i \leq k/2 \\ \{i+1,\ 2(k-i)+k+2\} & k/2 < i \leq k \end{cases},$$

Then $¶ = \{X_1, \ldots, X_k\}$ is a partition of $[1, 2k+1] \setminus \{k/2+1\}$ and as

$$\sum X_i = \begin{cases} 2k+3-i & 1 \leq i \leq k/2 \\ 3k+3-i & k/2 < i \leq k \end{cases},$$

we have that $\sum ¶$ is the set of consecutive integers

$$[\frac{3k}{2}+3,\ 2k+2] \cup [2k+3,\ \frac{5k}{2}+2].$$

This completes the proof. □

Lemmas 4 and 5 allow one to obtain infinite families of H-magic graphs for a given graph H under some weak conditions.

Lemma 4 has a simple application for the construction of an infinite family of H-magic non connected graphs.

Lemma 6. *Let H be any graph with $|V(H)| + |E(H)|$ even. Then the disjoint union $G = kH$ of k copies of H is H-magic.*

Proof. Let $h = |V(H)| + |E(H)|$, so that $|V(G)| + |E(G)| = kh$. By Lemma 4 with $t = h/2$, there is a k-equipartition $¶ = \{X_1, \cdots, X_k\}$ of $[1, kh]$ with $\sum X_1 = \sum X_2 = \cdots = \sum X_k$. By assigning to the i–th copy of H the labels of X_i we get an H-magic labelling of G. □

As an application of Lemma 5 we obtain the following result which provides infinite families of connected H–magic graphs. It is based on the following graph operation. Let G and H be two graphs and $e \in E(H)$ a distinguished edge in H. We denote by $G*eH$ the graph obtained from G by gluing a copy of H to each edge of G by the distinguished edge $e \in E(H)$. Figure 6 shows an example of $C_4 * eC_3$, where e is a any edge of C_3.

Theorem 5. *Let H be a 2–connected graph and G an H-free supermagic graph. Let k be the size of G and $h = |V(H)| + |E(H)|$. Assume that h and k are not both even. Then, for each edge $e \in E(H)$, the graph $G * eH$ is H-magic.*

Proof. Note that $G * eH$ admits an H-decomposition corresponding to the set of edges in G. Moreover, the only subgraphs of $G * eH$ isomorphic to H are precisely the copies of this graph glued to edges of G. Indeed, any subset $X \subset V(G*eH)$ of cardinality $|V(H)|$ different from the vertex set of a copy of H is either contained in $V(G)$ or intersects at least two copies of H. Let F be the subgraph of $G * eH$ induced by X. Then F does not contain a subgraph isomorphic to H, in the first case because G is H–free and in the second one because F is not 2–connected (any vertex $x \in V(G) \cap V(F)$ is a cutting vertex of F.)

Let f be a supermagic labeling of G. Denote by $e_i = x_i y_i$ the edge in G with label $f(e_i) = |V(G)| + i$ and by H_i the copy of H glued through it.

Define the total labeling f' of $G * eH$ as follows. The restriction of f' to $V(G)$ coincides with f. We proceed to define f' on each of the sets $Y_i = (V(H_i) \setminus \{x_i, y_i\}) \cup E(H_i)$.

By Lemma 5, there is a k-equipartition $\P = \{X_1, \cdots, X_k\}$ of the interval $[1, (h-2)k] + |V(G)|$ such that $\sum \P$ is a set of consecutive integers such that $\sum X_i = \sum X_1 + (i-1)$ for each $1 \leq i \leq k$.

We define f' by any bijection from Y_i to X_i for each $1 \leq i \leq k$. Then,

$$\begin{aligned} f'(H_i) &= f(x_i) + f(y_i) + \sum X_i = f(x_i) + f(y_i) + \sum X_1 + i - 1 \\ &= f(x_i) + f(y_i) + (\sum X_1 - |V(G)| - 1) + f(e_i) \\ &= s(f) + \sum X_1 - |V(G)| - 1, \end{aligned}$$

where $s(f)$ is the supermagic sum of f.

Therefore f' is an H-magic labeling of $G * eH$. □

An example of the labelling obtained in the above proof is displayed in Figure 2.

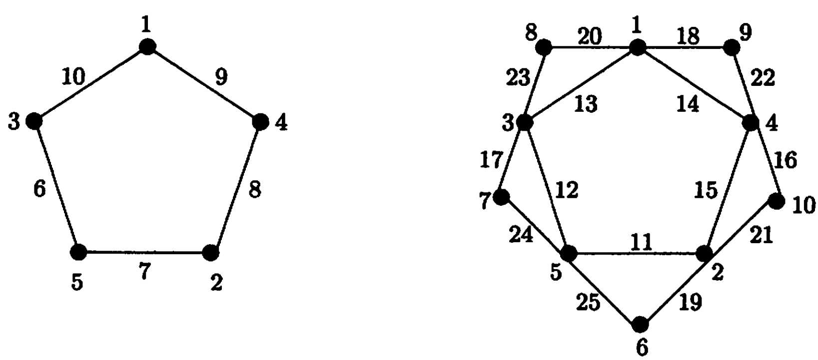

Fig. 2. A supermagic labeling of C_5, and the C_3–magic labeling of $C_5 * eC_3$.

References

1. B.D. Acharya and S.M. Hedge, Strongly indexable graphs, *Discrete Mathematics*, **93**, (1991) 275–299.
2. H. Enomoto, A. Llado, T. Nakimigawa and G. Ringel, Super edge-magic graphs. *SUT Journal of Mathematics 34*, No 2 (1998), 105-109.
3. D. Craft and E.H. Tesar, On a question by Erdös about edge-magic graphs, *Discrete Mathematics*, **207**, (1999) 271–276.
4. J. A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics* 5 # DS6.
5. R. Godbold and P.J. Slater, All cycles are edge-magic, *Bull. Inst. Comb. Appl.*, **22** (1998) 93-97.
6. A. Kotzig, On well spread sets of integers, *Publications du CRM-161*, (1972), (83 pages).
7. A. Kotzig, A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13**,(4) (1970) 451-461.
8. A. Kotzig, A. Rosa, Magic valuations of complete graphs, *Centre de Recherches Mathematiques, Universite de Montreal* (Intern Report) (1972) (8 pages)
9. A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (International Symposium, Rome, July 1966)* New York and Dunod Paris, (1967) 349-355.