On Antimagic Total Labelings of Generalized Petersen Graph

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Abstract. A *total labeling* of graph G with p vertices and q edges is an one-to-one mapping from $V(G) \cup E(G)$ onto $\{1, 2, \cdots, p+q\}$. If the edge-weights (resp. vertex-weights) form an arithmetic progression starting from a and having common difference d, then the labeling is called (a, d)-*edge* (resp. *vertex*) - *antimagic total labeling*. In this paper we consider such labeling applied to generalized Petersen graph.

1 Introduction

All graphs considered here are finite, simple and undirected. The graph G has a vertex set $V(G)$ and edge set $E(G)$ and we let $|V(G)| = p$ and $|E(G)| = q$. For a general reference for graph theoretic notions, see [11] and [17].

A *total labeling* on a graph G with p vertices and q edges is a one-to-one mapping from $V(G) \cup E(G)$ onto the set of integers $1, 2, \cdots, p+q$. The edge-weight of an edge uv under a total labeling is the sum of labels uv and the vertices u, v incident with uv. Similarly, the vertex-weight of a vertex u under a total labeling is defined as the sum of label of u and the labels of all edges incident to u. If the edge-weights (resp. vertex-weights) form an arithmetic progression starting from a and having common difference d, then the labeling is called (a, d)-*edge* (resp. *vertex*) - *antimagic total labeling*. These labelings were introduced by Simanjuntak *et al* in 2000 [15] and Baca *et al* in 2003 [2], respectively.

In this paper we deal with such labelings applied to generalized Petersen graph. A generalized Petersen graph $P(n, m)$, $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, is a 3-regular graph with $2n$ vertices $u_0, u_1, u_2, \cdots, u_{n-1}, v_0, v_1, v_2, \cdots, v_{n-1}$ and edges $\{u_i u_{i+1}\}$, $\{u_i v_i\}$, $\{v_i v_{i+m}\}$, for all $i \in \{0, 1, 2, \cdots, n-1\}$, where

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the subscripts are reduced modulo n. Generalized Petersen graphs was first defined by Watkins [18]. Various graph labelings heve been considered for generalized Petersen graphs; see for instances [1, 3, 4, 9, 13, 14].

2 An (a, d)-edge-antimagic total labeling

Bodendiek and Walther [8] introduced (a, d)-vertex-antimagic edge labelings which they called the (a, d)-antimagic labeling. Simanjuntak *et al.* [15] modified the definition of (a, d)-vertex-antimagic edge labeling and introduced an (a, d)-edge-antimagic total labeling as follows. An *(a, d)-edge-antimagic total labeling* on graph G is a one-to-one mapping from $E(G) \cup V(G)$ onto the set $\{1, 2, \cdots, p+q\}$ so that the set of edge-weight of all edges in G is $\{a, a+d, \cdots, a+(q-1)d\}$, for two positive integers $a > 0$ and $d \geq 0$. If all vertices receive p smallest labels, then the labeling is called *super (a, d)-edge-antimagic total.* The graphs that admit an (a, d)-edge-antimagic total labeling (resp. a super (a, d)-edge-antimagic total labeling) are said to be *(a, d)-edge-antimagic total* (resp. *super (a, d)-edge-antimagic total*).

A number of studies on (a, d)-edge-antimagic total labeling graphs has been investigated. In [15] and [5] it were proved that all cycles and paths have a (a, d)-edge-antimagic total labeling for some values of a and d.

The (a, d)-edge-antimagic total labeling for wheels, fans, complete graphs and complete bipartite graphs can be found in [6].

Ngurah and Baskoro [14] showed that every generalized Petersen graphs $P(n, m)$ $n \geq 3$, $1 \leq m < \frac{n}{2}$, has a $(4n+2, 1)$-edge-antimagic total labeling and a $(8n+2, 1)$-edge-antimagic total labeling. Furthermore, in [15] it was studied the duality of (a, d)-edge-antimagic total labeling as follows.

Proposition 1 *[15] If λ is an (a, d)-egde-antimagic total labeling of G then its dual labeling λ' is an $(3p+3q+3-a-(q-1)d, d)$-egde-antimagic total labeling.*

In [5], Baca *et al.* provided relationships between (a, d)-edge-antimagic total labeling and edge-magic total labeling as follows.

Proposition 2 *[5] Let G be a graph which admits total labeling and whose edge labels constitute an arithmetic progression with difference d. Then the following are equivalent.*
(i). G has an edge-magic total labeling with magic constant k,
(ii). G has a $(k-(q-1)d, 2d)$-edge-antimagic total labeling.

In [14], we have described the super edge-magic total labeling of $P(n, 1)$, n odd, $n \geq 3$ with $k = \frac{1}{2}(11n+3)$. Fukuchi [9] proved that for n odd,

$n \geq 3$, $P(n,2)$ has super edge-magic total labeling. From these results and Proposition 2 , for n odd, $n \geq 3$ the graph $P(n,m)$, $m = 1,2$ has a super $(\frac{1}{2}(5n+5), 2)$-edge-antimagic total labeling.

In the next two theorems, we construct a $(\frac{1}{2}(9n+5), 2)$- edge-antimagic total labeling of $P(n,m)$ for n odd and $m = 1,2$.

Theorem 1 *For odd n, $n \geq 3$, every generalized Petersen graph $P(n,1)$ has a $(\frac{9n+5}{2}, 2)$-edge-antimagic total labeling.*

Proof Consider the labeling f such that

$$f(u_i) = \begin{cases} \frac{1}{2}(2n+2+i), & \text{for } i \equiv 0 \ (mod\ 2), \\ \frac{1}{2}(3n+2+i), & \text{for } i \equiv 1 \ (mod\ 2). \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{2}(7n+3+i), & \text{for } i \equiv 0 \ (mod\ 2), i \neq n-1, \\ \frac{1}{2}(6n+3+i), & \text{for } i \equiv 1 \ (mod\ 2), \\ 3n+1, & \text{for } i = n-1. \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+2+i, & \text{for } i \neq n-1, \\ 2n+1, & \text{for } i = n-1. \end{cases}$$

$$f(u_i v_i) = \begin{cases} 4n+2+i, & \text{for } i \neq n-1, \\ 4n+1, & \text{for } i = n-1. \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} 3+i, & \text{for } i \neq n-2, n-1, \\ 3+i-n, & \text{for } i = n-2, n-1. \end{cases}$$

Then the edge-weights of all edges in $P(n,1)$ under the labeling f are

$$w_f(u_i u_{i+1}) = \begin{cases} \frac{1}{2}(9n+5)+(2+2i), & \text{for } 0 \leq i \leq n-2, \\ \frac{1}{2}(9n+5), & \text{for } i = n-1. \end{cases}$$

$$w_f(v_i v_{i+1}) = \begin{cases} \frac{1}{2}(13n+1)+(6+2i), & \text{for } 0 \leq i \leq n-3, \\ \frac{1}{2}(13n+1)+4, & \text{for } i = n-1, \\ \frac{1}{2}(13n+1)+2, & \text{for } i = n-2. \end{cases}$$

$$w_f(u_i v_i) = \begin{cases} \frac{1}{2}(17n+1)+(4+2i), & \text{for } 0 \leq i \leq n-2, \\ \frac{1}{2}(17n+1)+2, & \text{for } i = n-1. \end{cases}$$

So, the set of edge-weights of $P(n,1)$ is $\{\frac{1}{2}(9n+5), \frac{1}{2}(9n+5)+2, \cdots, \frac{1}{2}(21n+1)\}$. □

Theorem 2 *For odd n, $n \geq 3$, every generalized Petersen graph $P(n,2)$ has a $(\frac{9n+5}{2}, 2)$-edge-antimagic total labeling.*

Proof Label the vertices and edges of $P(n,2)$ in the following way.

$$g(u_i) = f(u_i) \quad \text{and} \quad g(u_iu_{i+1}) = f(u_iu_{i+1}).$$

$$g(v_iv_{i+2}) = \begin{cases} \frac{1}{2}(2n-2-i), & \text{for } i \equiv 0 \ (mod\ 2), \\ \frac{1}{2}(n-2-i), & \text{for } i \equiv 1 \ (mod\ 2), i \neq n-2, \\ n, & \text{for } i = n-2. \end{cases}$$

Case : $n \equiv 1 \ (mod\ 4)$.

$$g(v_i) = \begin{cases} \frac{1}{4}(16n-i), & \text{for } i \equiv 0 \ (mod\ 4), \\ \frac{1}{4}(13n-i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{4}(14n-i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{4}(15n-i), & \text{for } i \equiv 3 \ (mod\ 4). \end{cases}$$

$$g(u_iv_i) = \begin{cases} \frac{1}{4}(18n+2+i), & \text{for } i \equiv 0 \ (mod\ 4), \\ \frac{1}{4}(17n+2+i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{4}(16n+2+i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{4}(19n+2+i), & \text{for } i \equiv 3 \ (mod\ 4). \end{cases}$$

Case : $n \equiv 3 \ (mod\ 4)$.

$$g(v_i) = \begin{cases} \frac{1}{4}(16n-i), & \text{for } i \equiv 0 \ (mod\ 4), \\ \frac{1}{4}(15n-i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{4}(14n-i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{4}(13n-i), & \text{for } i \equiv 3 \ (mod\ 4). \end{cases}$$

$$g(u_iv_i) = \begin{cases} \frac{1}{4}(18n+2+i), & \text{for } i \equiv 0 \ (mod\ 4), \\ \frac{1}{4}(19n+2+i,), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{4}(16n+2+i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{4}(17n+2+i), & \text{for } i \equiv 3 \ (mod\ 4). \end{cases}$$

The edge-weights of $P(n,2)$ are

$$w_g(u_iu_{i+1}) = w_f(u_iu_{i+1}).$$

$$w_g(v_iv_{i+2}) = \begin{cases} \frac{1}{2}(17n-3)-i, & \text{for } i \equiv 0 \ (mod\ 2), \\ \frac{1}{2}(15n-3)-i, & \text{for } i \equiv 1 \ (mod\ 2), i \neq n-2, \\ \frac{1}{2}(17n+1), & \text{for } i = n-2. \end{cases}$$

Case : $n \equiv 1 \ (mod\ 4)$.

$$w_g(u_iv_i) = \begin{cases} \frac{1}{2}(19n+3+i), & \text{for } i \equiv 0 \ (mod\ 4), \\ \frac{1}{2}(18n+3+i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{2}(17n+3+i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{2}(20n+3+i), & \text{for } i \equiv 3 \ (mod\ 4). \end{cases}$$

Case : $n \equiv 3 \ (mod\ 4)$.

$$w_g(u_iv_i) = \begin{cases} \frac{1}{2}(19n+3+i), & \text{for } i \equiv 0 \ (mod\ 4), \\ \frac{1}{2}(20n+3+i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{2}(17n+3+i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{2}(18n+3+i), & \text{for } i \equiv 3 \ (mod\ 4). \end{cases}$$

Thus, the set of edge-weight over all edges in $P(n,2)$ is $\{a, a+d, a+2d, \cdots, a+(q-1)d\}$, where $a = \frac{1}{2}(9n+5)$ and $d = 2$. □

Note that, the labeling defined in Theorems 1 and 2 are *self dual*.

3 An (a, d)-vertex-antimagic total labeling

Baca *et al.* [2] introduced a new type of graph labeling which called (a, d)-vertex-antimagic total labeling. They defined an *(a, d)-vertex-antimagic total labeling* on graph G as a one-to-one mapping from $V(G) \cup E(G)$ onto the set of integers $1, 2, 3, \cdots, p+q$ such that the set of vertex-weights is $\{a, a+d, a+2d, \cdots, a+(v-1)d\}$ for some integers $a > 0$ and $d \geq 0$. An (a, d)-vertex-antimagic total labeling is called *super* if $E(G)$ receive p smallest labels. A graph is called (a, d)-vertex-antimagic total (resp. super (a, d)-vertex-antimagic total) if it admits an (a, d)-vertex-antimagic total labeling (resp. a super (a, d)-vertex-antimagic total labeling).

The (a, d)-vertex-antimagic total labeling is natural extension of the notion of a vertex-magic total labeling introduced by MacDougall *et al.* in [12]. Gray *et al.*[10] examined existence of vertex-magic total labelings of trees, forests and galaxies. Slamin and Miller [16] described a vertex-magic total labeling for $P(n, m)$ when n and m are coprime. In [7] is given a vertex-magic total labeling for the generalized Petersen graphs $P(n, m)$ for all $n > 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$.

Proposition 3 *[2] Let G be a regular graph of degree r. Then G has an (a, d)-vertex-antimagic total labeling if and only if G has an (a', d)-vertex-antimagic total labeling where $a' = (r+1)(p+q+1) - a - (p-1)d$.*

Baca *et al.* [3], proved that for n odd, $n \geq 3$, the prism D_n has a (a, d)-vertex-antimagic total labeling for $(a, d) \in \{(\frac{15n+5}{2}, 1), (\frac{11n+7}{2}, 3), (\frac{21n+5}{2}, 1), (\frac{17n+7}{2}, 3)\}$. Every the prism D_n with even cycles admits a (a, d)-vertex-antimagic total labeling for $(a, d) \in \{(\frac{13n+6}{2}, 2), (\frac{9n+8}{2}, 4), (\frac{19n+6}{2}, 2),$

$(\frac{15n+8}{2}, 4)\}$. Additionally, by the use of the results in [1] and [13], Baca *et al.* showed that for $n \geq 4$, n even, $1 \leq m \leq \frac{n}{2} - 1$, the generalized Petersen graph $P(n,m)$ has a (a, 2)-vertex-antimagic total labeling, for $a \in \{\frac{13n+6}{2}, \frac{19n}{2} + 3\}$. For $n \geq 8$, $n \equiv 0 \bmod 4$, the generalized Petersen graph $P(n,2)$ has a $(\frac{9n}{2} + 4, 4)$-vertex-antimagic total labeling and a $(\frac{15n}{2} + 4, 4)$-vertex-antimagic total labeling. Note that, the prism D_n is exactly the same with generalized Petersen graph $P(n,1)$.

Theorem 1 proves the Conjecture 1, proposed in [3], for $d = 2$.

Theorem 3 *For $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, every generalized Petersen graph $P(n,m)$ has a self-dual $(8n+3, 2)$-vertex-antimagic total labeling.*

Proof The desired vertex-antimagic total labeling of $P(n,m)$, $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, can be described by the following formula.

$$\lambda(u_i) = 3n+1+i, \text{ for } 0 \leq i \leq n-1 \text{ and } \lambda(u_i u_{i+1}) = 1+i, \text{ for } 0 \leq i \leq n-1.$$

$$\lambda(v_i) = \begin{cases} n+m+i, & \text{for } 0 \leq i \leq n-m, \\ m+i, & \text{for } n-m+1 \leq i \leq n-1. \end{cases}$$

$$\lambda(u_i v_i) = \begin{cases} 4n+1, & \text{for } i = 0, \\ 5n+1-i, & \text{for } 1 \leq i \leq n-1. \end{cases}$$

$$\lambda(v_i v_{i+m}) = \begin{cases} 2n+m+i, & \text{for } 0 \leq i \leq n-m, \\ n+m+i, & \text{for } n-m+1 \leq i \leq n-1. \end{cases}$$

Under the labeling λ the vertex-weights of $P(n,m)$ are

$$w_\lambda(u_i) = 8n+3+2i, \text{ for } 0 \leq i \leq n-1.$$

$$\begin{aligned} w_\lambda(v_i) &= 10n+2m+1+2i, \text{ for } 0 \leq i \leq n-m, \\ &= 8n+2m+1+2i, \text{ for } n-m+1 \leq i \leq n-1. \end{aligned}$$

Thus, the set of all vertex-weights of $P(n,m)$ is $\{a, a+d, a+2d, \cdots, a+(2n-1)d\}$, where $a = 8n+3$ and $d = 2$. It is easy to verify that this labeling is *self-dual*. This completes the proof of the theorem. □

Theorem 4 *For n odd, $n \geq 5$, every generalized Petersen graph $P(n,2)$ has a super $(\frac{15n+5}{2}, 1)$-vertex-antimagic total labeling.*

Proof Consider the labeling h such that,

$$h(u_i) = \begin{cases} \frac{1}{2}(8n-i), & \text{for } i \equiv 0 \ (mod\ 2), \\ \frac{1}{2}(7n-i), & \text{for } i \equiv 1 \ (mod\ 2). \end{cases}$$

$$h(v_i) = \begin{cases} \frac{1}{2}(10n-i), & \text{for } i \equiv 0 \ (mod\ 2), \\ \frac{1}{2}(9n-i), & \text{for } i \equiv 1 \ (mod\ 2). \end{cases}$$

$$h(u_iu_{i+1}) = \begin{cases} 2n+1, & \text{for } i = 0, \\ \frac{1}{2}(6n+2-i), & \text{for } i \equiv 0 \ (mod\ 2), i \neq 0, \\ \frac{1}{2}(5n+2-i), & \text{for } i \equiv 1 \ (mod\ 2). \end{cases}$$

$$h(u_iv_i) = \begin{cases} \frac{1}{2}(2+i), & \text{for } i \equiv 0 \ (mod\ 2), \\ \frac{1}{2}(n+2+i), & \text{for } i \equiv 1 \ (mod\ 2). \end{cases}$$

Case $n \equiv 1 \ \ (mod\ \ 4)$.

$$h(v_iv_{i+2}) = \begin{cases} n+1, & \text{for } i = 0, \\ \frac{1}{4}(5n+4-i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{4}(6n+4-i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{4}(7n+4-i), & \text{for } i \equiv 3 \ (mod\ 4), \\ \frac{1}{4}(8n+4-i), & \text{for } i \equiv 0 \ (mod\ 4), i \neq 0. \end{cases}$$

Case $n \equiv 3 \ \ (mod\ \ 4)$.

$$h(v_iv_{i+2}) = \begin{cases} n+1, & \text{for } i = 0, \\ \frac{1}{4}(7n+4-i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{4}(6n+4-i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{4}(5n+4-i), & \text{for } i \equiv 3 \ (mod\ 4), \\ \frac{1}{4}(8n+4-i), & \text{for } i \equiv 0 \ (mod\ 4), i \neq 0. \end{cases}$$

Labeling h gives vertex-weights, w_h :

$$w_h(u_i) = \begin{cases} \frac{1}{2}(17n+3) + (2-i), & \text{for } i = 0, 1, \\ \frac{1}{2}(19n+3) + \frac{1}{2}(4-2i), & \text{for } i = 2, 3, 4, \cdots n-1. \end{cases}$$

$$w_h(v_i) = \begin{cases} \frac{1}{2}(15n+5) + \frac{1}{2}(2-i), & \text{for } i = 0, 2, \\ \frac{1}{2}(16n+4) + \frac{1}{2}(3-i), & \text{for } i = 1, 3, 5, \cdots n-2, \\ \frac{1}{2}(17n+3) + \frac{1}{2}(4-i), & \text{for } i = 4, 6, 8, \cdots n-1. \end{cases}$$

Hence, the set of vertex-weights is $\{\frac{1}{2}(15n+5), \frac{1}{2}(15n+7), \cdots \frac{1}{2}(19n+3)\}$. Consequently, h is a super $(\frac{1}{2}(15n+5), 1)$-vertex-antimagic total labeling.
□

In Light Proposition 3 we have

Corollary 1 *For odd n, $n \geq 5$, every generalized Petersen graph $P(n,2)$ has a $(\frac{21n+5}{2},1)$-vertex-antimagic total labeling.*

Theorem 5 *For odd n, $n \geq 7$, every generalized Petersen graph $P(n,3)$ has a super $(\frac{15n+5}{2},1)$-vertex-antimagic total labeling.*

Proof We consider three possible cases.
Case 1 : for $n \equiv 1 \pmod 6$.
Label the vertices and edges of $P(n,3)$ in the following way.

$$\alpha_1(u_i) = \begin{cases} \frac{1}{3}(12n - i), & \text{for } i \equiv 0 \ (mod\ 3), \\ \frac{1}{3}(10n - i), & \text{for } i \equiv 1 \ (mod\ 3), \\ \frac{1}{3}(11n - i), & \text{for } i \equiv 2 \ (mod\ 3). \end{cases}$$

$$\alpha_1(v_i) = \begin{cases} \frac{1}{3}(15n - i), & \text{for } i \equiv 0 \ (mod\ 3), \\ \frac{1}{3}(13n - i), & \text{for } i \equiv 1 \ (mod\ 3), \\ \frac{1}{3}(14n - i), & \text{for } i \equiv 2 \ (mod\ 3). \end{cases}$$

$$\alpha_1(u_iu_{i+1}) = h(u_iu_{i+1})$$

$$\alpha_1(u_iv_i) = \begin{cases} \frac{1}{3}(3 + i), & \text{for } i \equiv 0 \ (mod\ 3), \\ \frac{1}{3}(2n + 3 + i), & \text{for } i \equiv 1 \ (mod\ 3), \\ \frac{1}{3}(n + 3 + i), & \text{for } i \equiv 2 \ (mod\ 3). \end{cases}$$

$$\alpha_1(v_iv_{i+3}) = \begin{cases} n + 1, & \text{for } i = 0, \\ \frac{1}{6}(7n + 6 - i), & \text{for } i \equiv 1 \ (mod\ 6), \\ \frac{1}{6}(8n + 6 - i), & \text{for } i \equiv 2 \ (mod\ 6), \\ \frac{1}{6}(9n + 6 - i), & \text{for } i \equiv 3 \ (mod\ 6), \\ \frac{1}{6}(10n + 6 - i), & \text{for } i \equiv 4 \ (mod\ 6), \\ \frac{1}{6}(11n + 6 - i), & \text{for } i \equiv 5 \ (mod\ 6), \\ \frac{1}{6}(12n + 6 - i), & \text{for } i \equiv 0 \ (mod\ 6), i \neq 0. \end{cases}$$

Then the vertex-weights under labeling α_1 are

$$w_{\alpha_1}(u_i) = w_h(u_i).$$

$$w_{\alpha_1}(v_i) = \begin{cases} \frac{1}{2}(15n + 5) + \frac{1}{3}(3 - i), & \text{for } i = 0, 3, \\ \frac{1}{6}(47n + 21 - 2i), & \text{for } i = 1, 4, 7, 10, \cdots, n - 3, \\ \frac{1}{6}(49n + 21 - 2i), & \text{for } i = 2, 5, 8, 11, \cdots, n - 2, \\ \frac{1}{6}(51n + 21 - 2i), & \text{for } i = 6, 9, 12, \cdots, n - 1. \end{cases}$$

Hence, α_1 extends to a super $(\frac{15n+5}{2},1)$-vertex-antimagic total labeling.
Case 2 : for $n \equiv 3 \pmod 6$. Label the vertices and edges of $P(n,3)$ in the following way.

$$\alpha_2(u_i)=\begin{cases}\frac{1}{3}(12n-i), & \text{for } i\equiv 0\ (mod\ 3),\\ \frac{1}{3}(11n+1-i), & \text{for } i\equiv 1\ (mod\ 3),\\ \frac{1}{3}(10n+2-i), & \text{for } i\equiv 2\ (mod\ 3).\end{cases}$$

$$\alpha_2(v_i)=\begin{cases}\frac{1}{3}(15n-i), & \text{for } i\equiv 0\ (mod\ 3),\\ \frac{1}{3}(14n+1-i), & \text{for } i\equiv 1\ (mod\ 3),\\ \frac{1}{3}(13n+2-i), & \text{for } i\equiv 2\ (mod\ 3).\end{cases}$$

$$\alpha_2(u_iu_{i+1})=h(u_iu_{i+1})$$

$$\alpha_2(u_iv_i)=\begin{cases}\frac{1}{3}(3+i), & \text{for } i\equiv 0\ (mod\ 3),\\ \frac{1}{3}(n+2+i), & \text{for } i\equiv 1\ (mod\ 3),\\ \frac{1}{3}(2n+1+i), & \text{for } i\equiv 2\ (mod\ 3).\end{cases}$$

$$\alpha_2(v_iv_{i+3})=\begin{cases}n+1, & \text{for } i=0,\\ \frac{1}{6}(8n+7-i), & \text{for } i\equiv 1\ (mod\ 6),\\ \frac{1}{6}(10n+8-i), & \text{for } i\equiv 2\ (mod\ 6),\\ \frac{1}{6}(9n+6-i), & \text{for } i\equiv 3\ (mod\ 6),\\ \frac{1}{6}(11n+7-i), & \text{for } i\equiv 4\ (mod\ 6),\\ \frac{1}{6}(7n+8-i), & \text{for } i\equiv 5\ (mod\ 6),\\ \frac{1}{6}(12n+6-i), & \text{for } i\equiv 0\ (mod\ 6), i\neq 0.\end{cases}$$

The vertex-weights under labeling α_2 are

$$w_{\alpha_2}(u_i)=w_h(u_i).$$

$$w_{\alpha_2}(v_i)=\begin{cases}\frac{1}{6}(47n+21+in), & \text{for } i=0,2,\\ \frac{1}{2}(15n+5)+\frac{1}{2}(3-i), & \text{for } i=1,3,\\ \frac{1}{6}(49n+23-2i), & \text{for } i=4,7,10,\cdots,n-2,\\ \frac{1}{6}(47n+25-2i,) & \text{for } i=5,8,11,\cdots,n-1,\\ \frac{1}{6}(51n+21-2i), & \text{for } i=6,9,12,\cdots,n-3.\end{cases}$$

Hence, α_2 extends to a super $(\frac{15n+5}{2},1)$-vertex-antimagic total labeling

.

Case 3 : for $n\equiv 5\ \ (mod\ 6)$. Label the vertices and edges of $P(n,3)$ in the following way.

$$\alpha_3(u_i)=\begin{cases}\frac{1}{3}(12n-i), & \text{for } i\equiv 0\ (mod\ 3),\\ \frac{1}{3}(11n-i), & \text{for } i\equiv 1\ (mod\ 3),\\ \frac{1}{3}(10n-i), & \text{for } i\equiv 2\ (mod\ 3).\end{cases}$$

$$\alpha_3(v_i)=\begin{cases}\frac{1}{3}(15n-i), & \text{for } i\equiv 0\ (mod\ 3),\\ \frac{1}{3}(14n-i), & \text{for } i\equiv 1\ (mod\ 3),\\ \frac{1}{3}(13n-i), & \text{for } i\equiv 2\ (mod\ 3).\end{cases}$$

$$\alpha_3(u_iu_{i+1}) = h(u_iu_{i+1})$$

$$\alpha_3(u_iv_i) = \begin{cases} \frac{1}{3}(3+i), & \text{for } i \equiv 0 \ (mod\ 3), \\ \frac{1}{3}(n+3+i), & \text{for } i \equiv 1 \ (mod\ 3), \\ \frac{1}{3}(2n+3+i), & \text{for } i \equiv 2 \ (mod\ 3). \end{cases}$$

$$\alpha_3(v_iv_{i+3}) = \begin{cases} n+1, & \text{for } i = 0, \\ \frac{1}{6}(11n+6-i), & \text{for } i \equiv 1 \ (mod\ 6), \\ \frac{1}{6}(10n+6-i), & \text{for } i \equiv 2 \ (mod\ 6), \\ \frac{1}{6}(9n+6-i), & \text{for } i \equiv 3 \ (mod\ 6), \\ \frac{1}{6}(8n+6-i), & \text{for } i \equiv 4 \ (mod\ 6), \\ \frac{1}{6}(7n+6-i), & \text{for } i \equiv 5 \ (mod\ 6), \\ \frac{1}{6}(12n+6-i), & \text{for } i \equiv 0 \ (mod\ 6), i \neq 0. \end{cases}$$

The set of vertex-weights over all vertices in $P(n,3)$ is

$$w_{\alpha_3}(u_i) = w_h(u_i).$$

$$w_{\alpha_3}(v_i) = \begin{cases} \frac{1}{2}(15n+5) + \frac{1}{3}(3-i), & \text{for } i = 0, 3, \\ \frac{1}{6}(49n+21-2i), & \text{for } i = 1, 4, 7, 10, \cdots, n-1, \\ \frac{1}{6}(47n+21-2i), & \text{for } i = 2, 5, 8, 11, \cdots, n-3, \\ \frac{1}{6}(51n+21-2i), & \text{for } i = 6, 9, 12, \cdots, n-2. \end{cases}$$

Hence, α_3 extends to a super $(\frac{15n+5}{2}, 1)$-vertex-antimagic total labeling

We conclude that $P(n,3)$ is a super $(\frac{15n+5}{2}, 1)$-vertex-antimagic total. □

In light Proposition 3 we have

Corollary 2 *For n odd, $n \geq 7$, every generalized Petersen graph $P(n,3)$ has a $(\frac{21n+5}{2}, 1)$-vertex antimagic total labeling .*

Theorem 6 *For n odd, $n \geq 9$, every generalized Petersen graph $P(n,4)$ has a super $(\frac{15n+5}{2}, 1)$-vertex-antimagic total labeling .*

Proof We consider two possible cases.
Case 1 : for $n \equiv 1 \pmod 4$.
Label the vertices and edges of $P(n,4)$ in the following way.

$$\beta_1(u_i) = \begin{cases} \frac{1}{4}(16n-i), & \text{for } i \equiv 0 \ (mod\ 4), \\ \frac{1}{4}(13n-i), & \text{for } i \equiv 1 \ (mod\ 4), \\ \frac{1}{4}(14n-i), & \text{for } i \equiv 2 \ (mod\ 4), \\ \frac{1}{4}(15n-i), & \text{for } i \equiv 3 \ (mod\ 4). \end{cases}$$

$$\beta_1(v_i)=\begin{cases}\frac{1}{4}(20n-i), & \text{for } i\equiv 0\ (mod\ 4),\\ \frac{1}{4}(17n-i), & \text{for } i\equiv 1\ (mod\ 4),\\ \frac{1}{4}(18n-i), & \text{for } i\equiv 2\ (mod\ 4),\\ \frac{1}{4}(19n-i), & \text{for } i\equiv 3\ (mod\ 4).\end{cases}$$

$$\beta_1(u_iu_{i+1})=h(u_iu_{i+1})$$

$$\beta_1(u_iv_i)=\begin{cases}\frac{1}{4}(4+i), & \text{for } i\equiv 0\ (mod\ 4),\\ \frac{1}{4}(3n+4+i), & \text{for } i\equiv 1\ (mod\ 4),\\ \frac{1}{4}(2n+4+i), & \text{for } i\equiv 2\ (mod\ 4),\\ \frac{1}{4}(n+4+i), & \text{for } i\equiv 3\ (mod\ 4).\end{cases}$$

To label the edge v_iv_{i+4} consider the following two subcases.
Case $n\equiv 1\quad (mod\ 8)$.

$$\beta_1(v_iv_{i+4})=\begin{cases}\frac{1}{8}(9n+8-i), & \text{for } i\equiv 1\ (mod\ 8),\\ \frac{1}{8}(10n+8-i), & \text{for } i\equiv 2\ (mod\ 8),\\ \frac{1}{8}(11n+8-i), & \text{for } i\equiv 3\ (mod\ 8),\\ \frac{1}{8}(12n+8-i), & \text{for } i\equiv 4\ (mod\ 8),\\ \frac{1}{8}(13n+8-i), & \text{for } i\equiv 5\ (mod\ 8),\\ \frac{1}{8}(14n+8-i), & \text{for } i\equiv 6\ (mod\ 8),\\ \frac{1}{8}(15n+8-i), & \text{for } i\equiv 7\ (mod\ 8),\\ \frac{1}{8}(16n+8-i), & \text{for } i\equiv 0\ (mod\ 8), i\neq 0,\\ n+1, & \text{for } i=0.\end{cases}$$

Case $n\equiv 5\quad (mod\ 8)$.

$$\beta_1(v_iv_{i+4})=\begin{cases}\frac{1}{8}(13n+8-i), & \text{for } i\equiv 1\ (mod\ 8),\\ \frac{1}{8}(10n+8-i), & \text{for } i\equiv 2\ (mod\ 8),\\ \frac{1}{8}(15n+8-i), & \text{for } i\equiv 3\ (mod\ 8),\\ \frac{1}{8}(12n+8-i), & \text{for } i\equiv 4\ (mod\ 8),\\ \frac{1}{8}(9n+8-i), & \text{for } i\equiv 5\ (mod\ 8),\\ \frac{1}{8}(14n+8-i), & \text{for } i\equiv 6\ (mod\ 8),\\ \frac{1}{8}(11n+8-i), & \text{for } i\equiv 7\ (mod\ 8),\\ \frac{1}{8}(16n+8-i), & \text{for } i\equiv 0\ (mod\ 8), i\neq 0,\\ n+1, & \text{for } i=0.\end{cases}$$

Under the labeling β_1, the vertex-weights are

$$w_{\beta_1}(u_i)=w_h(u_i).$$

$$w_{\beta_1}(v_i) = \begin{cases} \frac{1}{2}(15n+5)+\frac{1}{4}(4-i), & \text{for } i = 0,4, \\ \frac{1}{4}(31n+14-i), & \text{for } i = 1,5,9,\cdots,n-4, \\ \frac{1}{4}(32n+14-i), & \text{for } i = 2,6,10,\cdots,n-3, \\ \frac{1}{4}(33n+14-i), & \text{for } i = 3,7,11,\cdots,n-2, \\ \frac{1}{4}(34n+14-i), & \text{for } i = 8,12,16,\cdots,n-1. \end{cases}$$

Hence, β_1 extends to a super $(\frac{15n+5}{2}, 1)$-vertex-antimagic total labeling

.

Case 2 : for $n \equiv 3 \pmod 4$.
Label the vertices and edges of $P(n,4)$ in the following way.

$$\beta_2(u_i) = \begin{cases} \frac{1}{4}(16n-i), & \text{for } i \equiv 0 \pmod 4, \\ \frac{1}{4}(15n-i), & \text{for } i \equiv 1 \pmod 4, \\ \frac{1}{4}(14n-i), & \text{for } i \equiv 2 \pmod 4, \\ \frac{1}{4}(13n-i), & \text{for } i \equiv 3 \pmod 4. \end{cases}$$

$$\beta_2(v_i) = \begin{cases} \frac{1}{4}(20n-i), & \text{for } i \equiv 0 \pmod 4, \\ \frac{1}{4}(19n-i), & \text{for } i \equiv 1 \pmod 4, \\ \frac{1}{4}(18n-i), & \text{for } i \equiv 2 \pmod 4, \\ \frac{1}{4}(17n-i), & \text{for } i \equiv 3 \pmod 4. \end{cases}$$

$$\beta_2(u_iu_{i+1}) = h(u_iu_{i+1})$$

$$\beta_2(u_iv_i) = \begin{cases} \frac{1}{4}(4+i), & \text{for } i \equiv 0 \pmod 4, \\ \frac{1}{4}(n+4+i), & \text{for } i \equiv 1 \pmod 4, \\ \frac{1}{4}(2n+4+i), & \text{for } i \equiv 2 \pmod 4, \\ \frac{1}{4}(3n+4+i), & \text{for } i \equiv 3 \pmod 4. \end{cases}$$

To label the edge v_iv_{i+4} consider following two subcases.
Case $n \equiv 3 \pmod 8$.

$$\beta_2(v_iv_{i+4}) = \begin{cases} \frac{1}{8}(11n+8-i), & \text{for } i \equiv 1 \pmod 8, \\ \frac{1}{8}(14n+8-i), & \text{for } i \equiv 2 \pmod 8, \\ \frac{1}{8}(9n+8-i), & \text{for } i \equiv 3 \pmod 8, \\ \frac{1}{8}(12n+8-i), & \text{for } i \equiv 4 \pmod 8, \\ \frac{1}{8}(15n+8-i), & \text{for } i \equiv 5 \pmod 8, \\ \frac{1}{8}(10n+8-i), & \text{for } i \equiv 6 \pmod 8, \\ \frac{1}{8}(13n+8-i), & \text{for } i \equiv 7 \pmod 8, \\ \frac{1}{8}(16n+8-i), & \text{for } i \equiv 0 \pmod 8, i \neq 0, \\ n+1, & \text{for } i = 0. \end{cases}$$

Case $n \equiv 7 \pmod 8$.

$$\beta_2(v_iv_{i+4}) = \begin{cases} \frac{1}{8}(15n+8-i), & \text{for } i \equiv 1 \ (mod\ 8), \\ \frac{1}{8}(14n+8-i), & \text{for } i \equiv 2 \ (mod\ 8), \\ \frac{1}{8}(13n+8-i), & \text{for } i \equiv 3 \ (mod\ 8), \\ \frac{1}{8}(12n+8-i), & \text{for } i \equiv 4 \ (mod\ 8), \\ \frac{1}{8}(11n+8-i), & \text{for } i \equiv 5 \ (mod\ 8), \\ \frac{1}{8}(10n+8-i), & \text{for } i \equiv 6 \ (mod\ 8), \\ \frac{1}{8}(9n+8-i), & \text{for } i \equiv 7 \ (mod\ 8), \\ \frac{1}{8}(16n+8-i), & \text{for } i \equiv 0 \ (mod\ 8), i \neq 0, \\ n+1, & \text{for } i = 0. \end{cases}$$

Under the labeling β_2 the vertex-weights are

$$w_{\beta_2}(u_i) = w_h(u_i).$$

$$w_{\beta_2}(v_i) = \begin{cases} \frac{1}{2}(15n+5) + \frac{1}{4}(4-i), & \text{for } i = 0, 4, \\ \frac{1}{4}(33n+14-i), & \text{for } i = 1, 5, 9, \cdots, n-4, \\ \frac{1}{4}(32n+14-i), & \text{for } i = 2, 6, 10, \cdots, n-3, \\ \frac{1}{4}(31n+14-i), & \text{for } i = 3, 7, 11, \cdots, n-2, \\ \frac{1}{4}(34n+14-i), & \text{for } i = 8, 12, 16, \cdots, n-1. \end{cases}$$

Hence, β_2 extends to a $(\frac{15n+5}{2}, 1)$-vertex-antimagic total labeling . We conclude that $P(n, 4)$ is a super $(\frac{15n+5}{2}, 1)$-vertex-antimagic total. □

By the duality property (Proposition 3) we have

Corollary 3 *For odd n, $n \geq 9$, every generalized Petersen graph $P(n, 4)$ has a $(\frac{21n+5}{2}, 1)$-vertex-antimagic total labeling .*

Theorems 4, 5 and 6 extend the results of Baca *et al.* in [3]. They showed that for odd n, $n \geq 3$ the prism D_n $(P(n, 1))$ has a $(\frac{15n+5}{2}, 1)$-vertex-antimagic total labeling.

We finish this section by giving the following conjecture.

Conjecture 1 *For n odd , $1 \leq m \leq \lfloor \frac{m-1}{2} \rfloor$, every generalized Petersen graphs $P(n, m)$ has a $(\frac{21n+5}{2}, 1)$-vertex-antimagic total labeling.*

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