Relationship between adjacency matrices and super (a, d)-edge-antimagic-total labeling of graphs

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Abstract. Let $G = G(v, e)$ be a finite simple graph with v vertices and e edges. An (a, d)-*edge-antimagic-vertex* (EAV) *labeling* is a one-to-one mapping f from $V(G)$ onto the integers 1,2,...,v with the property that for every $xy \in E(G)$, the edge-weight set $\{f(x)+f(y)|x, y \in V\}=\{a, a+d, a+2d, ..., a+(e-1)d\}$, for some positive integers a and d.An (a, d)-*edge-antimagic-total labeling* is a one-to-one mapping f from $V(G)\cup E(G)$ onto the integers 1,2,...,$v+e$ with the property that, for every $xy \in E(G)$, the edge-weight set $\{f(x)+f(y)+f(xy)|x, y \in V(G), xy \in E(G)\} = \{a, a+d, a+2d, ..., a+(e-1)d\}$. Such labeling is called *super (a,d)-edge-antimagic total labeling* if $f(V(G)) = \{1, 2, ..., v\}$. In this paper we investigate the relationship between the adjacency matrix, (a, d)-edge-antimegic vertex labeling and super (a, d)-edge-antimagic total labeling and show how to manipulate this matrix to construct new (a, d)-edge-animagic vertex labeling and new super (a, d)-edge-antimagic total graphs.

1 Introduction

In this paper, we consider finite simple undirected graphs. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We put $v = |V(G)|$ and $e = |E(G)|$. For simplicity, we denote $V(G)$ by V and $E(G)$ by E.

A *labeling* of a graph G is a mapping that carries a set of graph elements into a set of numbers (usually positive integers), called *labels*. Edge magic total labeling was introduced by Kotzig and Rosa in 1970 [5]. Many results on graph labeling, including edge magic total labeling, have been discovered since then. A recent survey of graph labelings can be found in [4].

An (a,d)-*edge-antimagic-vertex* (EAV) *labeling* is a one-to-one mapping f from V onto the integers 1,2,...,v with the property that for every $xy \in E$, the edge-weight set $\{f(x)+f(y)|x,y \in V\}$=$\{a, a+d, a+2d, ..., a+(e-1)d\}$, for some positive integers a and d. A graph that has an (a,d)-edge-antimagic-vertex labeling is called (a,d)-*edge-antimagic vertex graph.* An (a,d)-*edge-antimagic-total* (*EAT*) *labeling* is a one-to-one mapping f from $V \cup E$ onto the integers 1,2,...,v+e with the property that, for every $xy \in E$, the edge-weights set $\{f(x)+f(y)+f(xy)|x,y \in V, xy \in E\} = \{a, a+d, a+2d, ..., a+(e-1)d\}$. Such a labeling is called *super* (a,d)-*edge-antimagic total* (SEAT) *labeling* if $f(V) = \{1,2,...,v\}$. A graph that has a super (a,d)-edge-antimagic-total labeling is called a *super* (a,d)-*edge-antimagic-total graph* Note that a super edge-magic-total labeling is a special case of a super (a,d)-edge-antimagic-total labeling when $d=0$.

In this paper, we investigate the relationship between the adjacency matrix and EAV labeling of graphs and show how to manipulate this matrix to construct new EAV graphs. Moreover, we use these results to construct similar results for SEAT graphs.

2 Adjacency matrix

Let $G = (v,e)$ be a graph and f be a SEAT labeling of G. Let $V = \{x_1, x_2, ..., x_v\}$ be the set of vertices in G with the labels $1,2,...,v$. A symmetric matrix $A = (a_{ij}), i,j = 1,...,v$ is called *an adjacency matrix of G* if

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between } x_i \text{ and } x_j, \\ 0 & \text{otherwise.} \end{cases}$$

If G is an EAV graph, then the rows and columns of A can be labeled using 1,2,...,v such that every skew-diagonal (diagonal of A which is traversed in the "northeast" direction) line of matrix A has either zero or two "1" elements. The set $\{f(x)+f(y) : x,y \in V\}$ in a skew-diagonal line generates a sequence of integers of difference d. If $d=1$ then the non-zero skew-diagonal lines form a band of consecutive integers. If $d=2$ then the non-zero skew-diagonal lines form a band of difference 2 but with a zero skew-diagonal line in between. We have similar skew-diagonal line bands for d=3,4,... . We denote such a skew-diagonal band as a *d-band.*

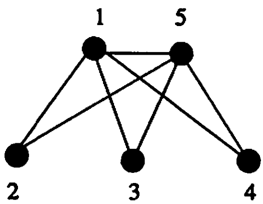
Observation 1 *The number of edges of maximal* (a,d)*-EAV graph is* $\lceil \frac{v-1}{d} \rceil + \lceil \frac{v-2}{d} \rceil$.

Consequently, a graph that has d-band for $d > 2$ cannot be connected. In this paper we only deal with the case $d=1$.

A *maximal EAV graph* is a graph that has an EAV labeling and has the maximum possible number of edges. If G has a maximal $(a,1)$-EAV labeling then $a = 3$. By checking the number of edges possible in an adjacency matrix of an EAV graph, we obtain the following observation.
If $d = 1$ then the number of edges of a maximal $(a,1)$-EAV graph is $2v-3$. The same maximal number of edges of a super edge magic total graph was obtained by Enomoto *et al* [2].

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Maximal EAV



Non–maximal EAV

Fig. 1. Examples of adjacency matrices of maximal and non-maximal EAV graphs with $d = 1$.

3 Constructing a new graph from an old one

In this section we construct new $(a^*,1)$-EAV graphs from an existing $(a,1)$-EAV graph, based on its adjacency matrices manipulation. For simplicity, we use EAV graph instead of $(a,1)$-EAV graph in the rest of the paper.

3.1 Constructing equivalent EAV graphs

Considering the adjacency matrix of an EAV graph G of order v, if we move the element "1" along the skew-diagonal line, we obtain another graph that also has an EAV labeling with the same d and the same set of edge-weights. This process can be repeated several times. Thus, if we have a non-maximal (respectively, maximal) graph G then we can obtain another non-maximal (respectively, maximal) graph G^*. If the graph G^* is obtained by the previous technique of moving a "1" element, then we say that G and G^* are *EAV-equivalent*. Figure 2 shows an example of generating a new maximal (3,1)-EAV graph from an old one.

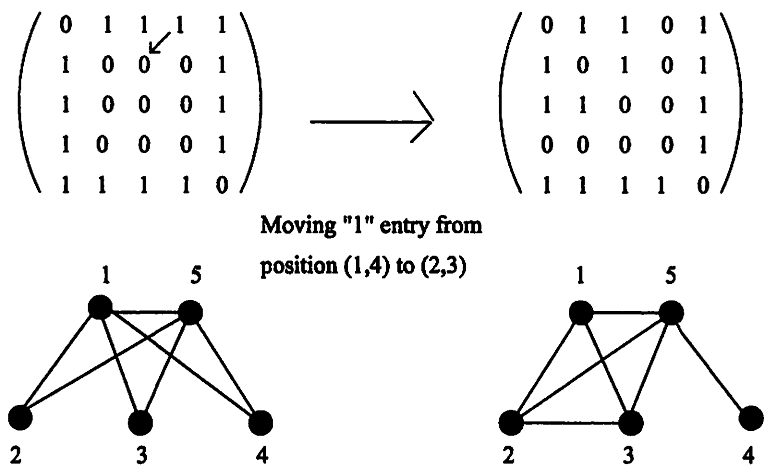


Fig. 2. Generating a new EAV graph.

In Figure 2, graph G^* is obtained from graph G by moving the element "1" from position (1,4) to position (2,3) in the same skew-diagonal line. By using this technique several times, we can obtain new graphs that have (a, d)-EAV labelings and the same edge-weights set as G. Two EAV labeled graphs that have the same edge-weights set are called *edge-weight equivalent*. By counting all the possibilities of moving the "1" elements (including disconnected graph results), we have the following observation.

Observation 2 *The number of non isomorphic edge-weight equivalent maximal labeled $(a, 1)$-EAV graphs on v vertices is*

- $(\frac{v-3}{2}!)^4(\frac{v-1}{2})^3$*, for v odd,*
- $(\frac{v-2}{2}!)^4\frac{v}{2}$*, for v even.*

Figure 3 contains all the possible maximal EAV graphs on 5 vertices, except one graph that already mentioned in Figure 2. There are 48 different possibilities for a maximal EAV graph on 6 vertices.

Using the adjacency matrix A for a maximal EAV graph G with v vertices and $e = 2v - 3$ edges, we can find the other maximal EAV graphs with

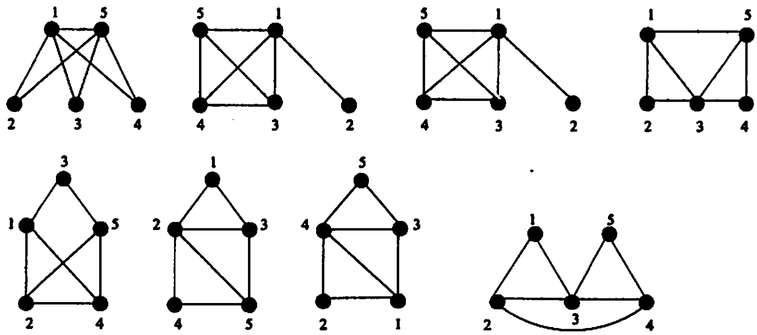


Fig. 3. Maximal EAV graphs on 5 vertices.

the same number of vertices and the same equivalent-edge weights set. Let EW_G be an edge-weights set of an EAV graph G' that has equivalent-edge weights set which is the same as in G.

3.2 Constructing new larger EAV graphs

Given an EAV graph G, there are several ways to obtain a larger EAV graph, for example,

- by adding some edges
- by combining two (or more) given EAV graphs
- by adding some vertices and edges

The results are presented in the following theorems.

Theorem 1 *Any non-maximal EAV graph can be extended to a maximal EAV graph.*

Proof.

If G is a non-maximal EAV graph of order v then its adjacency matrix A has v rows and v columns but only some $p < 2v - 3$ non-empty skew-diagonal lines. By adding element "1" in $2v - 3 - p$ empty skew-diagonal lines, we obtain a maximal EAV graph. □

Figure 4 illustrates a maximal EAV labeling extending a non-maximal EAV graph of order 5. We can see that P_5 is not a maximal EAV graph. It has only 4 edges, the maximal EAV graph on 5 vertices has 7 edges. To extend P_5 as a maximal EAV graph, we need 3 more edges. In this case, we only have one possible way to enter all four "1" elements in the adjacency matrix of P_5.

Theorem 2 *Let G_1 and G_2 be any EAV graphs of order v and w respectively. Then there exists an EAV graph of order $v+w$ which contain G_1 and G_2 as induced subgraphs. The number of additional edges needed is* $2v-1+\min\{wt(e_i) : e_i \in E(G_2)\} - \max\{wt(e_j) : e_j \in E(G_1)\}$.

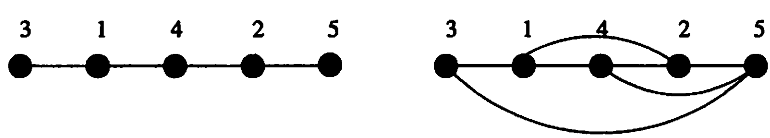
Proof.

Recall that the weight of an edge xy under a labeling α is $wt(xy) = \alpha(x) + \alpha(y)$. Let G_1 and G_2 be EAV graphs of order v and w, respectively, and with the number of edges e and f, respectively. Let A and B be the adjacency matrices of G_1 and G_2, respectively. Since A and B are adjacency matrices of EAV graphs then each of them has a skew-diagonal line bands. Create a new adjacency matrix C as

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Considering the set of skew-diagonal lines in C, we have several empty skew-diagonal line bands in the middle. If we put two "1" elements in every skew-diagonal line of the set of these empty skew-diagonal bands and define new vertex labels corresponding to the new arrangement then we obtain an EAV graph with $v+w$ vertices. □

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$



Non–maximal graph. Maximal graph.

Fig. 4. Expanding non-maximal EAV graph on 5 vertices.

Using a different composition, we can obtain different larger EAV graphs with different number of vertices. Let G_1 and G_2 be EAV graphs of order

v and w. Define a new graph $G^* = (G_1 * G_2)_{x_1,..,x_r;y_1,...,y_r}$ for some integer $r < min\{v, w\}$, as $G^* = (V(G^*), E(G^*))$ where $V(G^*) = V(G_1) \cup V(G_2)$, with $x_i = y_i$, $i = 1, ..., r$ for $x_i \in V(G_1), y_i \in V(G_2)$ and $E(G^*) = E(G_1) \cup E(G_2)$.

Theorem 3 *Let G_1 and G_2 be two EAV graphs of order v and w, respectively. Then there exist EAV graphs of orders $v+w-1$ and $v+w-2$ which contain G_1 and G_2 as subgraphs.*

Proof.

Let $A = (a_{ij})$ and $B = (b_{kl})$ be adjacency matrices of G_1 and G_2, respectively. Generate a new graph $G^* = (G_1 * G_2)_{x_1,..,x_r;y_1,...,y_r}$, $r \in \{1, 2\}$.
Case 1. The order of new graph is $v + w - 1$.

We obtain the adjacency matrix of G^* is C as follows

$$C = \begin{pmatrix} A^* & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & B^* \end{pmatrix}.$$

where A^* is a $(v-2) \times (v-2)$ submatrix consisting of the first $v-2$ rows and columns of A, B^* is a $(w-2) \times (w-2)$ submatrix consisting of the last $w-2$ last rows and columns of B, and X is 1×1 matrix with entry $x = 0$. Then we have a new larger adjacency matrix C with order $v + w - 1$. If we put two "1" elements in every skew-diagonal line of the set of the empty skew-diagonal bands and define new vertex labels corresponding to the new arrangement then we obtain an EAV graph G^* with $v + w - 1$ vertices.

Case 2. The order of new graph is $v + w - 2$. Let $A = (a_{ij})$ and $B = (b_{kl})$

be adjacency matrices of G_1 and G_2, respectively. Create a new symmetric diagonal block matrix C as follows

$$C = \begin{pmatrix} A^* & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & B^* \end{pmatrix}.$$

where A^* and B^* are the same matrices as in the previous case and

$$X = \begin{pmatrix} 0 & x_{12} \\ x_{21} & 0 \end{pmatrix}.$$

is a 2×2 matrix with entry $x_{21} = \max(\ a_{v-1,v}, b_{12})$ and $x_{12} = \max(\ a_{v,v-1}, b_{21})$. Then we have a new larger adjacency matrix C with order $v+w-2$. If we put two "1" elements in every skew-diagonal line of the set of the empty skew-diagonal bands and define new vertex labels corresponding to the new arrangement then we obtain an EAV graph with $v + w - 2$ vertices. □

Note that we can enlarge the graph by as little as one vertex using G_1 as the original graph and $G_2 = K_1$.

Theorems 3 and 4 can be generalised as follows.

Theorem 4 *Let $G_i, i = 1, ..., p$ be EAV graphs of order $v_i, i = 1, ..., p$ respectively. Then there are EAV graphs of orders ω, $\sum_{i=1}^{p} v_i - 2(p-1) \leq \omega \leq \sum_{i=1}^{p} v_i$, each containing $G_i, i = 1, ..., p$, as induced subgraphs.*

Using the above theorems, we obtain the following corollary.

Corrolary 1 *Every EAV graph has an EAV supergraph.*

Enomoto *et al.* [3] proved that every graph can be embedded in a connected SEMT graph as an induced graph. Similarly, we have the following results.

Theorem 5 *Every graph can be embedded in a connected EAV graph as an induced graph.*

Proof:

Let G be an arbitrary graph with adjacency matrix A. Suppose that there are more than two "1" elements in a skew diagonal line. Note that the number of nonzero elements in one skew-diagonal line must be even. For ilustration, suppose that there are four "1" elements in one skew-diagonal line, in positions (i, k), (j, l), (k, i) and (l, j). Add one additional row between rows $i-1$ and i, rows $j-1$ and j, rows $k-1$ and k, and rows $l-1$ and l. Add also one additional column between columns $k-1$ and k, columns $l-1$ and l, columns $i-1$ and i, and columns $j-1$ and j. Repeat this process until there are only two "1" elements in every non-empty skew-diagonal line. Add to the empty skew-diagonal line with a "1" element in arbitrary place, until we have a full d-band of skew-diagonals. Denote the resulting adjacency matrix as B. Then the graph that is represented by B is an EAV graph. □

3.3 Contracting an EAV subgraph

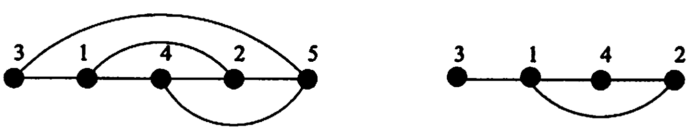
All theorems in Subsection 3.2 deal with the expansion of an EAV graph. On the other hand, we can also contract a maximal (respectively, non-maximal) EAV graph by choosing a submatrix of the adjacency matrix of a

EAV graph of a maximal (respectively, non-maximal) EAV graph G to have a subgraph G' that still have the EAV property. If the submatrix has a non-zero entry in position (1,1) then we have to add one column (say the first column) in the matrix or put zero value in the position (1,1). Then we still need to adjust the submatrix to make it onto a symmetric matrix and also adjust the labeling of all vertices and edges of the subgraph. However, we can only chose the submatrix so that the subgraph still remains connected.

Theorem 6 *Every EAV graph has an EAV subgraph.*

Figure 5 gives an example of constructing a subgraph that is still an EAV graph.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



An EAV graph on 5 vertices. An EAV subgraph on 4 vertices.

Fig. 5. Constructing non-maximal sub-EAV graph.

4 Relationship between adjacency matrix and SEAT labeling

Bača *et al.* [1] proved the following theorem.

Theorem 7 *If G has an (a,d)-EAV labeling then*

- *G has a $(a+|V|+1, d+1)$-SEAT labeling, and*
- *G has a $(a+|V|+|E|-1, d-1)$-SEAT labeling*

According to this theorem, if we have an (a,d)-EAV labeling for a graph G then we can add labels $\{v+1, v+2, ..., v+e\}$ to the edges of G in such a way that we obtain a super $(a+|V|+1, d+1)$-SEAT labeling and a $(a+|V|+|E|, d-1)$-SEAT labeling. In particular, if we start from an $(a,1)$-EAV labeling, we can obtain a $(a+|V|+1, 2)$-SEAT labeling or SEMT labeling. From an $(a,2)$-EAV labeling, we can get a $(a+|V|+1, 3)$-SEAT labeling and a $(a+|V|+|E|-1, 1)$-SEAT labeling. Note that in this paper we only consider the connected graph and an $(a,1)$-EAV graph.

The following theorem concerns EAV graph G with odd number of edges then we have the following result.

Theorem 8 *Let G be an $(a,1)$-EAV graph. If $e = |E|$ is odd then G has an $(a,1)$-SEAT labeling.*

Proof:

Suppose that G, with odd number of edges, has an $(a,1)$-EAV labeling α. Then the set of the edge-weights $\{w_\alpha(e_i) : i = 1, 2, ..., v\}$ consists of consecutive integers, namely, $W = \{a, a+1, ..., a+(e-1)\}$. Under the EAV labeling α, vertices of the graph have the consecutive labels 1,2,...,v. And

so we can label the edges of G using labels $v+1, v+2, ..., v+e$. To an edge e_i with weight $a+i-1$, $i = 1, 2, ..., e$ in $(a,1)$-EAV labeling, label the edges of G as follows

$$\alpha(e_i) = \frac{n-i+2}{2}, \text{ for } i \text{ odd}$$

$$\alpha(e_i) = n+1-\frac{i}{2}, \text{ for } i \text{ even}$$

.

Using this labeling the edge-weights set under the new labeling also consists of consecutive integers. □

If we delete all edge labels from every super edge magic total labeling then vertex labels will form an EAV graph. Thus we have the following result.

Theorem 9 *Every super edge magic total graph has an $(a,1)$-EAV labeling.*

5 Constructing new super (a,d)-edge-antimagic total graphs

Using Theorem 8 and 9, we can generalised the results of Section 3 by adding labels to all edges in a graph G to obtain the following results. Note that the way to add the edge labels depends on the SEAT that we construct. Since all the results in Section 3 hold for (a,d)-EAV graph for $d=1$ then the known results hold for super (a,d)-EAT graphs when $d \in \{0,2\}$, for every case, and when $d=1$ for e odd. We only consider all SEAT graphs that have an EAV labeling, and, in particular, $(a,1)$-EAV. For simplicity we use the term EAV-SEAT labeling and EAV-SEAT graph.

Theorem 10 *If G' is a graph with $e = 2v-3$ edges and G' has an adjacency matrix B such that there is a permutation matrix P and $B = P^{-1}AP$, where A is an adjacency matrix of a EAV-SEAT graph, then G' is also a EAV-SEAT graph.*

Since the basic idea of the proofs in this section is used repeatedly, we only prove one of the theorems.

Theorem 11 *Any non-maximal EAV-SEAT graph can be extended to a maximal EAV-SEAT graph.*

Proof:

Let G be a non-maximal (a,d)-EAV-SEAT graph of order v and $d \in \{0,2\}$ for any n, and $d=1$ for n odd. Then the adjacency matrix A has v rows and v columns but only p, $p < 2v-3$, non-empty skew-diagonal lines. By adding element "1" in $2v-3-p$ empty skew-diagonal lines, we obtain a maximal EAV graph with the set of edge-weights equal to $\{a, a+d, ..., a+2v-3d\}$.

Call the new EAV labeling α. Then $\alpha : V \to \{1, ..., 2v-3\}$. Next add the edge labels as follows

Case 1.

$$\beta(x_i) = \alpha(x_i) \text{ for every } i = 1, ..., v$$

$$\beta(xy) = 2v - 3 + i \text{ if the weight of the edge } xy \text{ under } \alpha \text{ is } a + id$$

Then we obtain a $(a',2)$-SEAT graph.

Case 2.

$$\beta(x_i) = \alpha(x_i) \text{ for every } i = 1, ..., v$$

$$\beta(xy) = 2v - 3 + v + e - i \text{ if the weight of the edge } xy \text{ under } \alpha \text{ is } a + id$$

Then we obtain a SEMT graph.

Case 3.

$$\beta(x_i) = \alpha(x_i) \text{ for every } i = 1, ..., v$$

Using the property of consecutive integers, we can obtain $\beta(xy)$ to complete the labels, and we can obtain $(a'', 1)$-SEAT. □

Theorem 12 *Let G_1 and G_2 be two EAV-SEAT graphs of order v and w, respectively. Then there exists a SEAT graph of order $v + w$ which contain G_1 and G_2 as induced subgraphs. The number of additional edges needed is $2v - 1 + \min\{wt(e_i) : e_i \in E(G_2)\} - \max\{wt(e_j) : e_j \in E(G_1)\}$.*

Theorem 13 *Let G_1 and G_2 be two non-maximal EAV-SEAT graphs of order v and w, respectively. Then there are SEAT graphs of orders $v+w-2$ and $v + w - 1$, each containing G_1 and G_2 as induced subgraphs.*

Note that for constructing an $(a, 1)$-SEAT graph, the number of edges of a new graph cannot be even. Theorems 14 and 15 can be generalised as follows.

Theorem 14 *Let $G_i, i = 1, ..., p$ be EAV-SEAT graphs of order $v_i, i = 1, ..., p$, respectively. Then there are SEAT graphs of orders $\sum_{i=1}^{p} v_i$, $\sum_{i=1}^{p} v_i - p + 1$ and $\sum_{i=1}^{p} v_i - 2(p - 1)$, each containing $G_i, i = 1, ..., p$, as induced subgraphs.*

Corrolary 2 *Every EAV-SEAT graph has a SEAT supergraph.*

Theorem 15 *Every graph can be embedded in a connected SEAT graph as an induced subgraph.*

Theorem 16 *Every EAV-SEAT graph contains a SEAT subgraph*

6 Conclusion

In this paper we presented a method, using adjacency matrix of an EAV graph, to create a new EAV graph. Moreover, we used this method to create a new SEAT-graph whenever the original SEAT graph also an EAV graph.

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