Creating new super edge-magic total labelings from old ones *

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Abstract. In this paper, we study the properties of super edge-magic total graphs. In particular, we propose some algorithms to construct new super edge-magic total graphs from the old ones. We also construct a super edge-magic total labeling on certain disconnected graphs, namely $P_n \cup P_{n+1}$, $nP_2 \cup P_n$, and $nP_2 \cup P_{n+2}$.

Key words and phrases: *edge-magic total labeling, super edge-magic total labeling, graph, dual labeling, the magic constant*

1 Introduction

In this paper we consider finite undirected graphs without loops and multiple edges. $V(G)$ and $E(G)$ stand for the vertex set and edge set of graph G, respectively. We denote by P_n a path on n vertices. The general references for graph-theoretic ideas can be seen in [10] and [18].

We denote by (p, q)-*graph* G a graph with p vertices and q edges. *An edge-magic total labeling* on a (p, q)-graph G is a bijection $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., p+q\}$ with the property that, for each edge xy of G, $\lambda(x) + \lambda(xy) + \lambda(y) = k$, for a fixed positive integer k. Moreover, λ is a *super edge-magic total labeling* if it has the property that the vertex labels are the integers $1, 2, \ldots, p$, the smallest possible labels.

A (p, q)-graph G is called *edge-magic total* (*super edge-magic total*) if there exists an edge-magic (super edge-magic, respectively) total labeling of G. We shall follow [19] to call $\lambda(x) + \lambda(xy) + \lambda(y)$ the *edge sum* of xy, and k the *magic constant* of the graph G.

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Edge-magic total graphs were first discussed by Kotzig and Rosa [13] (under the name of graph with magic valuation). Super edge-magic graphs were introduced by Enomoto et al. [4].

A number of classification studies on edge-magic total (resp. super edge-magic total) graphs has been intensively investigated. In [13] and [9] it is proved that every cycle C_n and caterpillar are edge-magic total. Kotzig and Rosa [14] showed that no complete graph K_n with $n > 6$ is edge-magic total and give some edge-magic total labeling for K_n, $3 \leq n \leq 6$, $n \neq 4$. Wallis et al. [19] showed that all paths P_n and all n-suns are edge-magic total. In [5] and [1] are exhibited the relationships between super edge-magic total labelings and other well studied classes of labelings (harmonius, cordial, graceful and antimagic).

Some conjectures remain open, namely that all trees are edge-magic total [13] (super edge-magic total [4]) and all wheels W_n are edge-magic total if $n \not\equiv 3 \pmod 4$. Enomoto et al. [4] have checked (by a computer) that their conjectures are true for all trees with less than or equal 15 vertices and wheels W_n for $n \leq 30$. Recently, Philips et al. [15] showed that a wheel W_n for $n \equiv 0$ or $1 \pmod 4$ is edge-magic total. Slamin et al. [17] proved that for $n \equiv 6 \pmod 8$, every wheel W_n has an edge-magic total labeling.

For disconnected graphs, in [13] there is proved that nP_2 is super edge-magic total if and only if n is odd. Kotzig [12] showed that if G is a trichromatic graph and G is edge-magic total then a disjoint union of n (n odd) identical copies of G is also edge-magic total. In particular, if $G = P_3$ then graph nP_3, for n odd, is edge-magic total. In [20] it is shown that nP_3 is super edge-magic total when n is odd. Yegnanarayanan [20] also conjectured that for all n, nP_3 has an edge-magic total labeling. Baskoro and Ngurah [2] proved that nP_3 is super edge-magic total for n even, $n \geq 4$.

Figueroa-Centeno et al. [6] showed that $P_3 \cup nP_2$ is super edge-magic total for every $n \geq 1$; $P_2 \cup P_n$ is super edge-magic total for every $n \geq 3$; $mK_{1,n}$ is super edge-magic total for m odd and for every $n \geq 1$; and graph mP_n is super edge-magic total for any m and for any odd n. The super edge-magic total characterization of the $nP_3 \cup kP_2$ and $K_{1,m} \cup K_{1,n}$ can be found in [11]. More comprehensive information on edge-magic total and super edge-magic total graphs is given in [8].

The next theorem appeared in [3] will be useful in the next section.

Theorem 1 *[3] Let a (p,q)-graph G be a super edge-magic total. Let λ be a super edge-magic total labeling of G with the magic constant k. Then, the labeling λ' defined:*

$$\lambda'(x) = p + 1 - \lambda(x),\ \forall x \in V(G),\ \textit{and}$$

$$\lambda'(xy) = 2p + q + 1 - \lambda(xy),\ \forall xy \in E(G)$$

is a super edge-magic total labeling with the magic constant $k' = 4p + q + 3 - k$.

The labeling λ' is called a *dual super labeling of* λ on G.

2 Main Results

In this section, we construct new super edge-magic total labelings from old ones. We also give a construction of a super edge-magic total labeling on finite disconnected graphs, especially $P_n \cup P_{n+1}$, $nP_2 \cup P_n$, and $nP_2 \cup P_{n+2}$.

2.1 Expanding super-edge magic total graphs

In this section we give an expansion technique on super edge-magic total graphs. Let us start with a necessary and sufficient condition for a graph of being super edge-magic total in the following lemma.

Lemma 1. *[5] A (p,q)-graph G is super edge-magic total if and only if there exists a bijective function $f : V(G) \to \{1, 2, ..., p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic total labeling of G with the magic constant $k = p + q + s$, where $s = min(S)$ and $S = \{k - (p + 1), k - (p + 2), \cdots, k - (p + q)\}$.*

Furthermore, in order to know what possible values of the magic constant for graph G to be super edge-magic total, E.T. Baskoro et al. [3] proved the following lemma.

Lemma 2. *[3] Let a (p,q)-graph G be super edge-magic total. Then, the magic constant k of G satisfies $p + q + 3 \leq k \leq 3p$.*

Theorem 2 *Let $p \geq 2$. Let a (p,q)-graph G be a super edge-magic total with the magic constant k and $k \geq 2p + 2$. Let m be any positive integer and $m \leq 3p + 2 - k$. Then a new graph, formed from G by adding exactly one vertex adjacent to m distinct vertices $z_1, z_2, ..., z_m$ of G labeled by $k - 2p - 1, k - 2p, ..., k - 2p - 2 + m$ respectively, is super edge-magic total with the magic constant $k_1 = k + m + 1$.*

Proof. By Lemma 1 there exists a vertex labeling λ on G such that $S = \{k - (p + q), k - (p + q - 1), ..., k - (p + 1)\}$. Let x_0 be the new vertex. Let G_1 be the new graph. In G_1, define a vertex labeling in the following way.

$$\lambda_1(u) = \lambda(u) \text{ for } u \in V(G),$$
$$\lambda_1(x_0) = p + 1.$$

Let $S_1 = \{\lambda_1(u)+\lambda_1(v) : \forall uv \in E(G_1)\}$. Clearly, $S_1 = S \cup \{\lambda_1(x_0)+\lambda(z_i) : 1 \leq i \leq m\}$, where $x_0 z_i$ are the new edges. So, we have $S_1 = \{k-(p+q), ..., k-(p+1)\} \cup \{k-(p+1)+1, ..., k-(p+1)+m\}$. This implies that the new graph is super edge-magic total with the magic constant $k_1 = k-(p+q)+p+1+q+m = k+m+1$ (by Lemma 1). The theorem holds only if the highest label of the vertex adjacent to x_0 is less than or equal to p, namely $k-2p-2+m \leq p$. So, $m \leq 3p+2-k$. □

If the magic constant k of a (p,q)-graph G is exactly $2p+2$, then m can be equal to p (by Theorem 2). In this case, we add one vertex adjacent to all vertices of G. The resulting graph has the magic constant $k+p+1$. Thus, the following corollary holds.

Corollary 3 *Let a (p,q)-graph G be a super edge-magic total with the magic constant $k = 2p+2$. Then, a new graph formed from G by adding one vertex adjacent to all vertices of G is a super edge-magic total with the magic constant $k_1 = k+p+1$.* □

Theorem 3 *Let a (p,q)-graph G be a super edge-magic total with the magic constant k and $k \geq 2p+2$. If n is odd and $n = 6p+5-2k$ then the new graph, formed from G and path P_n by joining all vertices of P_n to a vertex x_0 of G labeled by $k-2p-1$, is super edge-magic total with the magic constant $k_1 = k+3n-1$.*

Proof. By Lemma 1 there exists a vertex labeling λ on G such that $S = \{k-(p+q), k-(p+q-1), \cdots, k-(p+1)\}$. Let P_n be the path on n vertices $u_1, u_2, ..., u_n$. Let G_1 be a new graph formed from G and P_n. Define a vertex labeling $\lambda_1 : V(G_1) \rightarrow \{1, 2, ..., p, p+1, ..., p+n\}$ in the following way.

$$\lambda_1(u_i) = \begin{cases} p + \frac{i+1}{2}, & \text{for odd } i \\ p + \frac{n+1+i}{2}, & \text{for even } i \end{cases} \quad \forall u_i \in V(P_n)$$

$$\lambda_1(y) = \lambda(y), \forall y \in V(G).$$

Let $S_1 = \{\lambda_1(x) + \lambda_1(y) : \forall xy \in E(G_1)\}$. Clearly, $S_1 = S \cup \{\lambda(x_0) + \lambda_1(u_i) : 1 \leq i \leq n\} \cup \{\lambda_1(u_i) + \lambda_1(u_{i+1}) : 1 \leq i \leq n-1\}$, where $x_0 u_i$ are the edges connecting the path with graph G, and $u_i u_{i+1}$ are the edges of P_n. Since $n = 6p+5-2k$, then it can be easily verified that $S_1 = \{k-(p+q), \cdots, k-(p+1), k-(p+1)+1, \cdots, 5p+4-k, 5p+5-k, \cdots, 5p+4-k+n-1\}$. By Lemma 1, G_1 is super edge-magic total with the magic constant $k_1 = k-(p+q)+p+n+q+n+n-1 = k+3n-1$. □

Theorem 4 *Let a (p,q)-graph G be a super edge-magic total with the magic constant k and $k \geq 2p+2$. If n is even and $n = 6p+4-2k$ then the new* *graph, formed from G and path P_n by joining all vertices of P_n to a vertex x_0 of G labeled by $k-2p-1$, is super edge-magic total with the magic constant $k_1 = k+3n-1$.*

Proof. Let G_1 be the graph formed from G and P_n. Construct the bijection $\lambda_1 : V(G_1) \to \{1, 2, ..., p, p+1, ..., p+n\}$ as follows.

$$\lambda_1(u_i) = \begin{cases} p + \frac{i+1}{2}, & \text{for odd } i \\ & \\ p + \frac{n+i}{2}, & \text{for even } i \end{cases} \quad \forall u_i \in V(P_n)$$

$$\lambda_1(y) = \lambda(y), \forall y \in V(G).$$

We can proceed analogously as in the proof Theorem 3 and similarly show that the labeling λ_1 extends to a super edge-magic total labeling of G_1. □

Theorem 5 *Let a (p,q)-graph G be a super edge-magic total with the magic constant k and $k \geq 2p+2$. If $n = 3p+3-k$ then the new graph, formed from G and star $K_{1,n-1}$ by joining all vertices of $K_{1,n-1}$ to a vertex x_0 of G labeled by $k-2p-1$, is super edge-magic total with the magic constant $k_1 = k+3n-1$.*

Proof. By Lemma 1 there exists a vertex labeling λ on G such that $S = \{k-(p+q), k-(p+q-1), ..., k-(p+1)\}$. Let $K_{1,n-1}$ has the vertex-set $\{u_1, u_2, \cdots, u_n\}$ with the edge-set $\{u_1u_i : i = 2, 3, \cdots, n\}$. Let G_1 be the new graph formed from G and $K_{1,n-1}$ by joining all vertices of $K_{1,n-1}$ to a vertex x_0 of G. In G_1, define a vertex labeling in the following way.

$$\lambda_1(u_i) = p+i, \forall u_i \in V(K_{1,n-1}),$$
$$\lambda_1(y) = \lambda(y), \forall y \in V(G).$$

Let $S_1 = \{\lambda_1(x) + \lambda_1(y) : \forall xy \in E(G_1)\}$. Clearly, $S_1 = S \cup \{\lambda_1(x_0) + \lambda_1(u_i) : 1 \leq i \leq n\} \cup \{\lambda_1(u_1) + \lambda_1(u_i) : 2 \leq i \leq n\}$, where x_0u_i are the edges connecting the star with graph G, and u_1u_i are the edges of $K_{1,n-1}$. Since $n = 3p+3-k$ then $k-(p+1)+n = 2p+2$, and so $S_1 = \{k-(p+q), ..., k-(p+1), k-(p+1)+1, ..., 2p+2, 2p+3, ..., 2p+2+n-1\}$. Thus, by Lemma 1, G_1 is super edge-magic total with the magic constant $k_1 = k-(p+q)+p+n+q+n+n-1 = k+3n-1$. □

2.2 $P_n \cup P_{n+1}$

In this section we prove that graph $P_n \cup P_{n+1}$ is super edge-magic total. We denote that

$$V(P_n \cup P_{n+1}) = \{u_{1,i} | 1 \le i \le n\} \cup \{u_{2,j} | 1 \le j \le n+1\}, \text{ and}$$
$$E(P_n \cup P_{n+1}) = \{e_{1,i} | 1 \le i \le n-1\} \cup \{e_{2,j} | 1 \le j \le n\}$$

where $e_{1,i} = u_{1,i}u_{1,i+1}$, for $1 \le i \le n-1$, and $e_{2,j} = u_{2,j}u_{2,j+1}$, for $1 \le j \le n$.

Theorem 6 *For every odd n and $n \ge 3$, graph $P_n \cup P_{n+1}$ has a super edge-magic total labeling with the magic constant $k = 5n + 4$.*

Proof. Label the vertices and edges of $P_n \cup P_{n+1}$ in the following way

$$\lambda(u_{1,i}) = \begin{cases} \frac{i+1}{2}, & \text{for i=1,3,...,n} \\ n+2+\frac{i}{2}, & \text{for i=2,4,...,n-1} \end{cases}$$

$$\lambda(u_{2,j}) = \begin{cases} \frac{n+j}{2}+1, & \text{for j=1,3,...,n} \\ n+2, & \text{for j=n+1} \\ \frac{3(n+1)+j}{2}, & \text{for j=2,4,...,n-1} \end{cases}$$

$$\lambda(e_{1,i}) = \lambda(u_{1,i}u_{1,i+1}) = 4n+1-i, \text{ for } 1 \le i \le n-1.$$

$$\lambda(e_{2,j}) = \lambda(u_{2,j}u_{2,j+1}) = \begin{cases} 3n+1-j, & \text{for } 1 \le j \le n-1 \\ 3n+1, & \text{for j=n} \end{cases}$$

Clearly,
$\lambda(u_{1,i}) + \lambda(u_{1,i}u_{1,i+1}) + \lambda(u_{1,i+1}) = 5n+4$ for $1 \le i \le n-1$ and
$\lambda(u_{2,j}) + \lambda(u_{2,j}u_{2,j+1}) + \lambda(u_{2,j+1}) = 5n+4$ for $1 \le j \le n$.
Thus the labeling λ is super edge-magic total. □

By Theorem 1, we have

Corollary 4 *For every odd n and $n \ge 3$, the graph $P_n \cup P_{n+1}$ has a super edge-magic total labeling with the magic constant $k = 5n + 2$.*

2.3 $nP_2 \cup P_n$

In this section we prove that graph $nP_2 \cup P_n$ is super edge-magic total. The graph $nP_2 \cup P_n$ is a disjoint union of nP_2 and P_n. We denote that

$$V(nP_2 \cup P_n) = \{u_{1,i}, u_{2,i} | 1 \le i \le n\} \cup \{u_{3,j} | 1 \le j \le n\}, \text{ and}$$
$$E(nP_2 \cup P_n) = \{e_{1,i} | 1 \le i \le n\} \cup \{e_{2,j} | 1 \le j \le n-1\}$$

where $e_{1,i} = u_{1,i}u_{2,i}$, for $1 \le i \le n$ and $e_{2,j} = u_{3,j}u_{3,j+1}$, for $1 \le j \le n-1$.

Theorem 7 *For every $n \ge 2$, the graph $nP_2 \cup P_n$ has a super edge-magic total labeling with the magic constant $k = 7n + 1$.*

Proof. Label the vertices and edges of $nP_2 \cup P_n$ in the following way

$$\begin{aligned}
\lambda(u_{1,i}) &= i, \text{ for } 1 \le i \le n \\
\lambda(u_{2,i}) &= 2n + i, \text{ for } 1 \le i \le n \\
\lambda(u_{3,j}) &= n + j, \text{ for } 1 \le j \le n \\
\lambda(e_{1,i}) &= \lambda(u_{1,i}u_{2,i}) = 5n - 2i + 1, \text{ for } 1 \le i \le n. \\
\lambda(e_{2,j}) &= \lambda(u_{3,j}u_{3,j+1}) = 5n - 2j, \text{ for } 1 \le j \le n - 1.
\end{aligned}$$

Clearly,
for every $1 \le i \le n$, $\lambda(u_{1,i}) + \lambda(u_{1,i}u_{2,i}) + \lambda(u_{2,i}) = 7n + 1$ and
for every $1 \le j \le n - 1$, $\lambda(u_{3,j}) + \lambda(u_{3,j}u_{3,j+1}) + \lambda(u_{3,j+1}) = 7n + 1$.
Thus the labeling λ is super edge-magic total. □

2.4 $nP_2 \cup P_{n+2}$

In this section we show that the graph $nP_2 \cup P_{n+2}$ is super edge-magic total. We denote that

$$V(nP_2 \cup P_{n+2}) = \{u_{1,i}, u_{2,i} | 1 \le i \le n\} \cup \{u_{3,j} | 1 \le j \le n+2\}, \text{ and}$$
$$E(nP_2 \cup P_{n+2}) = \{e_{1,i} | 1 \le i \le n\} \cup \{e_{2,j} | 1 \le j \le n+1\}$$

where $e_{1,i} = u_{1,i}u_{2,i}$, for $1 \le i \le n$ and $e_{2,j} = u_{3,j}u_{3,j+1}$, for $1 \le j \le n+1$.

Theorem 8 *For every $n \ge 1$, the graph $nP_2 \cup P_{n+2}$ has a super edge-magic total labeling with the magic constant $k = 7n + 6$.*

Proof. Label the vertices and edges of $nP_2 \cup P_{n+2}$ in the following way

$$\begin{aligned}
\lambda(u_{1,i}) &= i, \text{ for } 1 \le i \le n \\
\lambda(u_{2,i}) &= 2n + i + 2, \text{ for } 1 \le i \le n \\
\lambda(u_{3,j}) &= n + j, \text{ for } 1 \le j \le n + 2 \\
\lambda(e_{1,i}) &= \lambda(u_{1,i}u_{2,i}) = 5n - 2i + 4, \text{ for } 1 \le i \le n. \\
\lambda(e_{2,j}) &= \lambda(u_{3,j}u_{3,j+1}) = 5n - 2j + 5, \text{ for } 1 \le j \le n + 1.
\end{aligned}$$

Clearly,
$\lambda(u_{1,i}) + \lambda(u_{1,i}u_{2,i}) + \lambda(u_{2,i}) = 7n + 6$ for $1 \le i \le n$ and
$\lambda(u_{3,j}) + \lambda(u_{3,j}u_{3,j+1}) + \lambda(u_{3,j+1}) = 7n + 6$ for $1 \le j \le n + 1$.
Thus the labeling λ is super edge-magic total. □

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