Super (a, d)-vertex-antimagic total labelings

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Abstract. Let $G = (V, E)$ be a graph with v vertices and e edges. A (a, d)-*vertex-antimagic total labeling* is a bijection α from $V(G) \cup E(G)$ to the set of consecutive integers $1, 2, ..., v + e$, such that the weights of the vertices form an arithmetic progression with the initial term a and the common difference d. If $\alpha(V(G)) = \{1, 2, \dots, v\}$ then we call the labeling a *super* (a, d)-*vertex antimagic total*. We study basic properties of such labelings and show how to construct such labelings for some families of graphs, such as paths, cycles and generalised Petersen graphs. We also show that such labeling do not exist for certain families of graphs, such as cycles with at least one tail, trees with even number of vertices and all stars.

1 Introduction

In this paper we consider simple and connected graphs. For a graph $G = G(V, E)$ we will denote the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. We also denote $v = |V(G)|$ and $e = |E(G)|$.

A *labeling* α of a graph G is a mapping that assigns elements of a graph to set of numbers (usually positive integers). If the domain of the mapping is the set of vertices (respectively, the set of edges) then we call the labeling *vertex labeling* (respectively, *edge labeling*). If the domain is $V \cup E$ then we call the labeling a *total labeling*. For a further explanation of vertex, edge and total labelings, see [8], [17].

The *vertex-weight* $wt(x)$ of a vertex $x \in V$, under a labeling $\alpha : V \cup E \to \{1, 2, ..., v + e\}$, is the sum of values $\alpha(xy)$ assigned to all edges incident to a given vertex x together with the value assigned to x itself.

A bijection $\alpha : V \cup E \to \{1, 2, ..., v + e\}$ is called an (a, d)-*vertex-antimagic total* (in short, (a, d)-VAT) *labeling of* G if the set of vertex-weights of all vertices in G is $\{a, a + d, a + 2d, \dots, a + (v - 1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed nonnegative integers.

If $d = 0$ then we call α a *vertex-magic total labeling.* The concept of the vertex-magic total labeling was introduced by MacDougall *et al.* [14] in 2002. In this paper, they also showed that some families of graphs cannot have SVMT labeling, for example, graphs that have a vertex of degree 1; also wheels, fans, friendship graphs, ladders and complete bipartite graphs. For other results in vertex magic total labeling, see [4], [10], [12], [13], [14].

An (a, d)-vertex-antimagic total labeling α is called a *super* (a, d)-*vertex-antimagic total* (in short, *super* (a, d)-*VAT*) labeling if $\alpha(V)) = \{1, 2, ..., v\}$ and $\alpha(E) = \{v + 1, v + 2, ..., v + e\}$.

On the other hand, Bača *et al.* [2] investigated basic properties of (a, d)-VAT labelings and constructed such labelings for some families of graphs. They also studied dual labeling and a relationship between SVMT labeling and an (a, d)-antimagic graph. Furthermore $(a, 1)$-VAT labelings for the family of quartic graphs R_n have been described in [1], and $(a, 1)$-*VAT* labelings of the complete graph K_n, $n = 2$ or $n > 5$, $n \not\equiv 0 \pmod 4$, and of the complete bipartite graph $K_{n,n}, n \geq 3$, follow from Stewart's results [16] (see [2]).

In this paper we study basic properties of super (a, d)-VAT labeling and we show how to construct such labelings for certain families of graphs, including complete graphs, complete bipartite graphs, cycles, paths and generalised Petersen graphs. We also show that some families of graphs do not admit super (a, d)-VAT labelings.

2 Basic properties

Suppose that graph G has a super (a, d)-VAT labeling. Let S_v be the sum of all the vertex labels and let S_e be the sum of all the edge labels. If δ is the smallest degree in G then the minimum possible vertex-weight is $1 + (v + 1) + (v + 2) + ... + (v + \delta)$. Then

$$a \geq 1 + v\delta + \frac{\delta(\delta + 1)}{2}. \tag{1}$$

If Δ is the largest degree of G then the maximum possible vertex-weight is $v + (v + e) + (v + e - 1) + ... + (v + e - (\Delta - 1))$.
Consequently,

$$a + (v - 1)d \leq v + \Sigma_{i=0}^{\Delta-1}(v + e - i). \tag{2}$$

From (1) and (2), we conclude that

$$d \leq 1 + \frac{\Delta(2v + 2e - \Delta + 1) - \delta(2v + \delta + 1)}{2(v - 1)}. \tag{3}$$

The sum of all the vertex labels and all the edge labels used to calculate the vertex-weights is

$$S_v + 2S_e = \frac{v(v+1)}{2} + 2ve + e(e+1). \tag{4}$$

The sum of all the vertex-weights is

$$\Sigma_{x \in V} wt(x) = av + \frac{vd(v-1)}{2}. \tag{5}$$

Combining Equations (4) and (5), we obtain

$$a = \frac{1}{2}(v + 1 - (v-1)d) + 2e + \frac{e(e+1)}{v}. \tag{6}$$

3 New labelings from old

If a regular graph possesses an (a,d)-VAT labeling then we can create a new labeling from it.

Let $\alpha : V \cup E \to \{1, 2, ..., v+e\}$ be an (a,d)-VAT labeling for a graph G. Define the *dual labeling of* α on $V \cup E$ as follows [2]:

$$\alpha'(x) = v + e + 1 - \alpha(x), \text{ for any vertex } x \in V,$$

$$\alpha'(xy) = v + e + 1 - \alpha(xy), \text{ for any edge } xy \in E.$$

Clearly, the labeling α' is also a bijection from the set $V \cup E$ into $\{1, 2, ..., v+e\}$. We say that α' is the *dual* of α.

Proposition 1 *[2] The dual of an (a,d)-VAT labeling of a graph G is an (a',d)-VAT labeling for some a' if and only if G is regular.*

The following theorem gives a relationship between a vertex-antimagic edge labeling and a super vertex-antimagic total labeling for a regular graph.

Theorem 1 *An (a,d)-vertex-antimagic edge labeling of G is a super $(a', d+1)$-VAT labeling or a super $(a'', d-1)$-VAT labeling for some a' and a'', if and only if G is regular.*

Proof. Suppose λ is an (a,d)-vertex-antimagic edge labeling of G and let $wt_\lambda(x)$ be the vertex-weight of x under the edge labeling λ. Then $W = \{wt_\lambda(x) : x \in V\} = \{a, a+d, a+2d, \ldots, a+(v-1)d\}$ is the set of vertex-weights of G. Let x_i be the vertex of V such that $wt_\lambda(x_i) = a + (i-1)d$, for $i = 1, 2, \ldots, v$. We will distinguish two cases.

Case 1. Define a new mapping α in G by

$$\alpha(x_i) = i + e, \text{ for } i = 1, 2, \dots, v$$

$$\alpha(xy) = \lambda(xy), \text{ for all } xy \in E.$$

Under the labeling α, the vertices and edges of G use each integer from the set $\{e+1, e+2, \dots, e+v\}$ and $\{1, 2, \dots, e\}$, respectively.
Then $wt_\alpha(x_i) = wt_\lambda(x_i) + \alpha(x_i) = a + (i-1)d + i + e = a + e + 1 + (i-1)(d+1)$, for $i = 1, 2, \dots, v$, i.e., the weights of the vertices constitute an arithmetic progression with the difference $d+1$ and with the initial term $a+e+1$. Thus the labeling α is a $(a+e+1, d+1)$-VAT labeling.

In the light of Proposition 1, the dual of an α labeling is an $(a', d+1)$-VAT labeling if and only if G is regular. We can see that the vertices of G under the dual labeling of α, use exactly all the integers from the set $\{1, 2, \dots, v\}$. Therefore, the dual labeling is a super $(a', d+1)$-VAT labeling.

Case 2. We construct a new mapping β of G by

$$\beta(x_i) = v + e + 1 - i, \text{ for } i = 1, 2, \dots, v$$

$$\beta(xy) = \lambda(xy), \text{ for all } xy \in E.$$

Under the labeling β, the vertices and edges of G receive the integers $e+1, e+2, \dots, e+v$ and $1, 2, \dots, e$, respectively. Then
$wt_\beta(x_i) = wt_\lambda(x_i) + \beta(x_i) = a + (i-1)d + v + e + 1 - i = a + v + e + (i-1)(d-1)$ for $i = 1, 2, \dots, v$, i.e., the labeling β is a $(a+v+e, d-1)$-VAT labeling. According to Proposition 1, the dual of a β labeling is an $(a'', d-1)$-VAT labeling if and only if G is regular.

Evidently, under dual labeling of β, the values of the vertices are $1, 2, \dots, v$. This implies that the dual labeling of β is a super $(a'', d-1)$-VAT labeling. □

Stewart [16] showed that the complete graph K_n has a super-magic edge labeling when $n = 2$ or $n > 5$ and $n \not\equiv 0 \pmod 4$. From [16], we know that the complete bipartite graph $K_{n,n}$ is super-magic for all $n \geq 3$. In our terminology a super-magic labeling is a super $(a, 0)$-vertex-antimagic edge labeling.

The complete graph K_n is of course a regular graph. Consequently, from Theorem 1, it follows the following that:

Corollary 5 *If $n = 2$ or $n > 5$ and $n \not\equiv 0$ (mod 4), then the complete graph K_n has a super $(a', 1)$-VAT labeling.*

Since the complete bipartite graph $K_{n,n}$ is also a regular graph, the following results is obvious.

Corollary 6 *There is a super $(a',1)$-VAT labeling of $K_{n,n}$, for all $n \geq 3$.*

4 Generalized Petersen graph

Let n and m be positive integers, $n \geq 3$ and $1 \leq m < \frac{n}{2}$. The generalized Petersen graph $P(n,m)$ is a graph that consists of an outer-cycle $y_0, y_1, y_2, \ldots, y_{n-1}$, a set of n spokes y_ix_i, $0 \leq i \leq n-1$, and n edges x_ix_{i+m}, $0 \leq i \leq n-1$, where all the subscripts are taken modulo n. The standard Petersen graph is the instance $P(5,2)$. Generalized Petersen graphs were first defined by Watkins [18]. From (3), it follows that if $P(n,m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$, has a super (a,d)-VAT labeling then $d < 5$.

The following proposition was proved in [5]

Proposition 2 *[5] A generalized Petersen graph $P(n,m)$ has an $(a,1)$-vertex-antimagic edge labeling if and only if n is even, $n \geq 4$, $1 \leq m \leq \frac{n}{2}-1$ and $a = \frac{7n+4}{2}$.*

Since $P(n,m)$ is regular of degree $r = 3$, by Theorem 1 we have

Corollary 7 *For n even, $n \geq 4$, $1 \leq m \leq \frac{n}{2}-1$, every generalized Petersen graph $P(n,m)$ has a super $(a',2)$-VAT labeling and a super $(a'',0)$-VAT labeling.*

The next theorem gives a super $(a,1)$-VAT labeling of $P(n,m)$ for n odd.

Theorem 2 *For n odd, $n \geq 3$, $1 \leq m < \frac{n}{2}$, every generalized Petersen graph $P(n,m)$ have a super $(a,1)$-VAT labeling.*

Proof. Let the generalized Petersen graph $P(n,m)$ has $V(P(n,m)) = \{x_0, x_1, \ldots, x_{n-1}\} \cup \{y_0 . y_1, \ldots, y_{n-1}\}$ and $E(P(n,m)) = \{y_ix_i, y_iy_{i+1}, \quad x_ix_{i+m} : i = 0,1,2,\ldots,n-1\}$ with the indices taken modulo n. Now, consider two cycles of $P(n,m)$; the outer-cycle $y_0, y_1, \ldots, y_{n-1}$ and the inner-cycle $x_0, x_m, x_{2m}, \ldots, x_{(n-1)m}$. Rename the inner cycle vertices: $x_0^* = x_0, x_1^* = x_m, x_2^* = x_{2m}, \ldots, x_{n-1}^* = x_{(n-1)m}$. Then we have the inner-cycle $x_0^*, x_1^*, \ldots, x_{n-1}^*$.

Define the total labeling β for the outer-cycle and the inner-cycle as follows.

$$\beta(x_i^*) = i + 1 \text{ for } i = 0, 1, \ldots, n-1$$

$$\beta(y_i) = n + 1 + i \text{ for } i = 0, 1, \ldots, n-1$$

$$\beta(y_i y_{i+1}) = \begin{cases} 3n - \frac{i}{2} & \text{for } i \text{ even} \\ \frac{5n-i}{2} & \text{for } i \text{ odd} \end{cases}$$

$$\beta(x_i^* x_{i+1}^*) = \begin{cases} 4n - \frac{i}{2} & \text{for } i \text{ even} \\ \frac{7n-i}{2} & \text{for } i \text{ odd} \end{cases}$$

We can see that

$$\beta(y_{i-1}y_i) + \beta(y_i) + \beta(y_i y_{i+1}) = \frac{13n+3}{2}$$

and

$$\beta(x_{i-1}^* x_i^*) + \beta(x_i^*) + \beta(x_i^* x_{i+1}^*) = \frac{15n+3}{2}$$

for $i = 0, 1, \ldots, n-1$, where all the subscripts are taken modulo n.

If we complete the labels for spokes by

$$\beta(y_i x_i) = 4n + 1 + i, \text{ for } i = 0, 1, \ldots, n-1$$

then the vertex-weights of $P(n, m)$ are

$$wt_\beta(y_i) = \frac{21n+5}{2} + i$$

and

$$wt_\beta(x_i) = \frac{23n+5}{2} + i$$

for $i = 0, 1, \ldots, n-1$.

Thus the total labeling β is a super $\left(\frac{21n+5}{2}, 1\right)$-VAT labeling. □

Note that the generalized Petersen graph $P(n, 1)$ is known as a prism. Bača and Holländer [3] proved

Proposition 3 *[3] If n is odd, $n \geq 3$, then the prism $P(n, 1)$ has a $\left(\frac{5n+5}{2}, 2\right)$-vertex-antimagic edge labeling.*

Proposition 4 *[3] If n is even, $n \geq 4$, then the prism $P(n, 1)$ has a $\left(\frac{7n+4}{2}, 1\right)$-vertex-antimagic edge labeling and a $\left(\frac{3n+6}{2}, 3\right)$-vertex-antimagic edge labeling.*

For prism $P(n, 1)$, by Theorem 1, we can obtain two corollaries when $d = 3$ and $d = 4$.

Corollary 8 *For n odd, $n \geq 3$, every prism $P(n, 1)$ has a super $(a', 3)$-VAT labeling.*

Corollary 9 *For n even, $n \geq 4$, every prism $P(n, 1)$ has a super $(a', 4)$-VAT labeling.*

5 Cycles and Paths

Next we define the *edge-weight* of an edge xy ($w(xy)$), under a vertex labeling to be the sum of the vertex labels corresponding to the vertices x and y. Under a total labeling, we would also add the label of xy.

An (a,d)-*edge-antimagic total* (in short, (a,d)-EAT) labeling is defined as a one-to-one mapping from $V \cup E$ onto the set $\{1,2,\ldots,v+e\}$ so that the set of the edge-weights of all theedges in G is equal to $\{a, a+d, a+2d, \ldots, a+(e-1)d\}$, for two integers $a > 0$ and $d \geq 0$.

An (a,d)-EAT labeling f is called *super* (a,d)-EAT if $f(V) = \{1,2,\ldots,v\}$ and, consequently $f(E) = \{v+1, v+2, \ldots, v+e\}$.

These labelings are natural extensions of the notion of an edge-magic labeling which was introduced by Kotzig and Rosa [11] and of the notion of super edge-magic labeling which was defined by Enomoto *et al.* in [7].

The following theorem is proved in [6].

Proposition 5 *[6] The cycle C_n has a super (a,d)-EAT labeling if and only if either*
(i) $d \in \{0,2\}$ and n is odd, $n \geq 3$, or
(ii) $d = 1$ and $n \geq 3$.

The following theorem gives a relationship between (a,d)-EAT and (a,d)-VAT labelings for cycles.

Theorem 3 *For cycles and only for cycles, a super (a,d)-EAT labeling is equivalent to a super (a',d)-VAT labeling.*

Proof. Let the cycle C_n be defined as follows: $V(C_n) = \{x_0, x_1, \ldots, x_{n-1}\}$ and $E(C_n) = \{x_i, x_{i+1} : i = 0,1,\ldots,n-1\}$, with the indices taken modulo n. Suppose that a bijection f from $V(C_n) \cup E(C_n)$ onto the set $\{1,2,\ldots,2n\}$ is super (a,d)-EAT. This means that $\{w_f(x_ix_{i+1}) : w_f(x_ix_{i+1}) = f(x_i) + f(x_{i+1}) + f(x_ix_{i+1}), i = 0,1,\ldots,n-1\} = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$ is the set of edge-weights of C_n.

Define a new mapping α by

$$\alpha(x_ix_{i+1}) = f(x_i) \text{ for } i = 0,1,\ldots,n-1$$

$$\alpha(x_{i+1}) = f(x_ix_{i+1}) \text{ for } i = 0,1,\ldots,n-1.$$

It can be seen that $\alpha(V) = \{n+1, n+2, \ldots, 2n\}$ and $\alpha(E) = \{1,2,\ldots,n\}$. Moreover $w_f(x_ix_{i+1}) = f(x_i)+f(x_{i+1})+f(x_ix_{i+1}) = \alpha(x_ix_{i+1})+\alpha(x_{i+1}x_{i+2})+\alpha(x_{i+1}) = wt_\alpha(x_{i+1})$ for all $i = 0,1,\ldots,n-1$, i.e., the edge-weight $w_f(x_ix_{i+1})$ is equivalent to the vertex-weight $wt_\alpha(x_{i+1})$ for all $i = 0,1,\ldots,n-1$. So, labeling α is (a,d)-VAT.

We construct the dual labeling α' by

$$\alpha'(x) = 2n + 1 - \alpha(x) \text{ for any vertex } x \in V(C_n)$$

$$\alpha'(xy) = 2n + 1 - \alpha(xy) \text{ for any edge } xy \in E(C_n).$$

Since the cycles are regular graphs it follows from Proposition 1 that the dual labeling α' is (a', d)-VAT. Again it is readily verified that $\alpha'(V) = \{1, 2, \ldots, n\}$ and $\alpha'(E) = \{n+1, n+2, \ldots, 2n\}$. This guarantees that α' is super (a', d)-VAT labeling. □

In light Theorem 3 and Proposition 5 we can claim

Theorem 4 *The cycle C_n has super (a, d)-VAT labeling if and only if either*
(i) $d \in \{0, 2\}$ and n is odd, $n \geq 3$, or
(ii) $d = 1$ and $n \geq 3$.

Next we turn our attention to a super (a, d)-VAT labeling of path P_n, $n \geq 3$. Let the path P_n be defined as follows: $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : i = 1, 2, \ldots, n-1\}$. From (3) it follows that if P_n, $n \geq 2$, has a super (a, d)-VAT labeling then $d < 4$.

Theorem 5 *For the path P_n, $n \geq 3$ and $d \in \{0, 1\}$ there is no super (a, d)-VAT labeling.*

Proof. The fact that P_n does not have any super $(a, 0)$-VAT labeling was already proved in [15].

Suppose, to the contrary, that γ is a super $(a, 1)$-VAT labeling of P_n. Using equation (6) we find $a = 3n - 2$. However, the maximum weights of end vertices v_1 and v_n can be obtained as sum of the biggest possible vertex labels and edge labels as follows:

$wt_\gamma(v_1) = n + (2n - 1) = 3n - 1$ and $wt_\gamma(v_n) = (n - 1) + (2n - 2) = 3n - 3 < a$ or

$wt_\gamma(v_1) = n + (2n - 2) = a$ and $wt_\gamma(v_n) = (n - 1) + (2n - 1) = a$.

We have a contradiction. Thus, P_n does not have any super $(3n - 2, 1)$-VAT labeling. □

Theorem 6 *The path P_n, $n \geq 3$, has a super $(a, 2)$-VAT labeling if and only if n is odd.*

Proof. From (6) we have that for a super $(a, 2)$-VAT labeling of P_n the smallest vertex-weight is $a = \frac{5n-3}{2}$.

If n is even this contradicts the fact that a is an integer.

For n odd we define the bijection

$\beta : V(P_n) \cup E(P_n) \to \{1, 2, \ldots, n\} \cup \{n+1, n+2, \ldots, 2n-1\}$ in the following way:

$$\beta(v_1) = n$$
$$\beta(v_i) = i - 1 \text{ for } i = 2, 3, \ldots, n$$
$$\beta(v_i v_{i+1}) = \begin{cases} \frac{3n+i}{2} & \text{for } i \text{ odd} \\ n + \frac{i}{2} & \text{for } i \text{ even.} \end{cases}$$

The vertex-weights form the arithmetic progression $\frac{5n-3}{2}, \frac{5n+1}{2}, \ldots, \frac{9n-7}{2}$. Thus P_n has the super $(\frac{5n-3}{2}, 2)$-VAT labeling for n odd. □

Theorem 7 *Every path P_n, $n \geq 3$, has a super $(a, 3)$-VAT labeling.*

Proof. We discuss two cases.
Case 1. n odd.

We construct a labeling φ in which the vertices receive labels

$$\varphi(v_1) = 1$$
$$\varphi(v_n) = n$$
$$\varphi(v_i) = n - i + 1 \text{ for } i = 2, 3, \ldots, n-1$$

and the edges receive labels

$$\varphi(v_i v_{i+1}) = \begin{cases} 2n - 1 - i & \text{for } i \text{ odd} \\ 2n + 1 - i & \text{for } i \text{ even.} \end{cases}$$

We can see that the labeling φ is super labeling and the vertex-weights form the arithmetic progression with difference 3, namely $2n-1, 2n+2, \ldots, 5n-4$.
Case 2. n even.

Define the labeling $\psi : V(P_n) \cup E(P_n) \to \{1, 2, \ldots, 2n-1\}$ where

$$\psi(v_1) = n - 2$$
$$\psi(v_n) = n$$
$$\psi(v_i) = \begin{cases} 2i - 3 & \text{for } i = 2, 3, \ldots, \frac{n}{2} + 1 \\ 2(n-i) & \text{for } i = \frac{n}{2} + 2, \frac{n}{2} + 3, \ldots, n-1 \end{cases}$$

and

$$\psi(v_i v_{i+1}) = \begin{cases} n + 2i - 1 & \text{for } i = 1, 2, \ldots, \frac{n}{2} \\ 3n - 2i & \text{for } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n-1 \,. \end{cases}$$

We conclude that the total labeling ψ extends to a super $(2n-1, 3)$-VAT. □

We summarise our results for path as follows.

Theorem 8 *The path P_n has a super (a, d)-VAT labeling if and only if either*

(i) $d = 2$ and n is odd, $n \geq 3$, or

(ii) $d = 3$ and $n \geq 3$.

6 Families of Trees and Unicyclic Graphs

Recall that in this paper we consider only connected graphs. In this section we present a super (a, d)-VAT labeling of slightly more general graphs, in particular trees instead of paths and unicyclic graphs instead of cycles. We shall start by unicyclic graphs.

Let G be a graph where $v = e$. From (6) we have that $a = \frac{7v+3-(v-1)d}{2}$. If v is even then a is integer only for d odd.

Theorem 9 *For every cycle with at least one tail and even number of vertices there is no super $(a, 1)$-VAT labeling.*

Proof. Let G be a cycle with at least one tail. Suppose that α is a super $(a, 1)$-VAT labeling of G for $a = 3v + 2$ (see (6)). By assumption, G has at least one vertex of degree 1, say x_p. Then the maximum possible vertex-weight of x_p can be obtained by the biggest value of vertex and the biggest value of edge, i.e., $wt_\alpha(x_p) = v + 2v = 3v$. However $wt_\alpha(x_p) < a$ and we have a contradiction. □

Now, we consider a super (a, d)-VAT labeling for tree where $e = v - 1 \geq 1$. Applying equation (6) we have $a = \frac{7v-5-(v-1)d}{2}$. If v is even then a is integer only for d odd.

Theorem 10 *For every tree with even number of vertices there is no super $(a, 1)$-VAT labeling.*

Proof. Let G be a tree with $e = v - 1$ and v be even. G has at least two vertices of degree one, say x_p and x_q. Suppose, to the contrary, that β is a super $(a, 1)$-VAT labeling of G for $a = 3v - 2$. Considering the extreme values of the labeling of vertices and edges, the largest vertex-weights for x_p and x_q are

$wt_\beta(x_p) = v + (2v - 1) = 3v - 1$ and $wt_\beta(x_q) = (v - 1) + (2v - 2) = 3v - 3 < a$ or

$wt_\beta(x_p) = v + (2v - 2) = a$ and $wt_\beta(x_q) = (v - 1) + (2v - 1) = a$.

It is obvious that both cases give a contradiction. □

Let x_0 denote the central vertex of star S_n, $n \geq 1$, and x_i, $1 \leq i \leq n$ be its leaves. In light of Theorem 10, the star S_n for n odd has not any super $(a, 1)$-VAT labeling. More generally, we have the following theorem.

Theorem 11 *For star $S_n, n \geq 3$, there is no super (a,d)-VAT labeling for any d.*

Proof. Suppose that φ is a super (a,d)-VAT labeling of star S_n. From inequality (3) it follows that $d \leq \frac{3n^2+3n-4}{2n}$. The smallest vertex-weight of the central vertex x_0 under the labeling φ is

$$\min\ (wt_\varphi(x_0)) = 1 + (n+2) + (n+3) + \cdots + (2n+1) = \frac{3n^2+3n+2}{2}$$

and the largest vertex-weight of a leave x_i is

$$\max\ (wt_\varphi(x_i)) = (n+1) + (2n+1) = 3n+2.$$

Clearly, $\min\ (wt_\varphi(x_0)) - \max\ (wt_\varphi(x_i)) \leq d$ and thus

$$(n+1)(3n^2 - 9n + 4) \leq 0.$$

The last inequality holds only for two integers: $n = 1$ and $n = 2$. It means that S_n has a super $(4,1)$-VAT labeling only for $n = 1$ ($\alpha(x_0) = 1$, $\alpha(x_1) = 2$, $\alpha(x_0x_1) = 3$) and a super (a,d)-VAT labeling for $n = 2$ (see Theorem 8). □

7 Open Problems

We list here two problems for further investigation.

Open Problem 1 *For the complete graph K_n and complete bipartite graph $K_{n,n}$, determine (if there is) a super (a,d)-VAT labeling for every feasible value of $d > 1$.*

Open Problem 2 *For the generalized Petersen graph $P(n,m)$, find (if there is) a construction of a super (a,d)-VAT labeling for*
(i) n even, $n \geq 4$, $1 \leq m \leq \frac{n}{2} - 1$, and $d \in \{3,4\}$
(ii) n odd, $n \geq 3$, $2 \leq m < \frac{n}{2}$, and $d \in \{0,2,3,4\}$.

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