The size multipartite Ramsey numbers for paths

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Abstract. For graphs $G_1, G_2, \cdots, G_k$, the (generalized) *size multipartite Ramsey number* $m_j(G_1, G_2, \cdots, G_k)$ is the least natural number m so that any colouring of the edges of $K_{j\times m}$ with k colours will yield a copy of G_i in the ith colour for some i. In this note, we determine the exact value of the size multipartite Ramsey number $m_j(P_s, P_t)$ for $s = 2, 3$ and all integers $t \geq 2$, where P_t denotes a path on t vertices.

1 Introduction

Recently, Burger and van Vuuren [3] studied one of generalisations of the classical Ramsey number as follows. Let $K_{n\times l}$ denote a complete, balanced, multipartite graph consisting of n partite sets and l vertices per partite set. Let j, l, n, s and t be natural numbers with $n, s \geq 2$. Then the *size multipartite Ramsey number* $m_j(K_{n\times l}, K_{s\times t})$ is the smallest natural number ζ such that an arbitrary colouring of the edges of $K_{j\times\zeta}$, using two colours red and blue, necessarily forces a red $K_{n\times l}$ or a blue $K_{s\times t}$ as subgraph.

In this paper, we generalize this concept by releasing completeness requirement in the forbidden graphs as follows. Let $j \geq 2$ be a natural number. For graphs $G_1, G_2, \cdots, G_k$, the (generalized) *size multipartite Ramsey number* $m_j(G_1, G_2, \cdots, G_k)$ is the smallest natural number m so that any colouring of the edges of $K_{j\times m}$ with k colours will yield a copy of G_i in the ith colour for some i. The existence of all numbers $m_j(G_1, G_2, \cdots, G_k)$ for $j = 2$ follows from a result of Erdös and Rado [4]. For the case of $k = 2$, with G_1, G_2 are complete balanced multipartite graphs, the numbers can be derived from result Burger and van Vuuren [3]. The exact values of bipartite Ramsey numbers $b(P_s, P_t) = m_2(P_s, P_t)$ of two paths can be obtained from

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a special case of some results of Gyárfás and Lehel [6], and Faudree and Schelp [5]. Furthermore, Hattingh and Henning [7] determined the exact values of bipartite Ramsey numbers $b(P_m, K_{1,n})$. In this paper, we establish the exact values of the size multipartite Ramsey numbers $m_j(P_s, P_t)$ of two paths with $s = 2, 3$.

2 Main results

In this note, we prove the following theorem.

Theorem 1. *If* $n \geq 3$ *then* $m_j(P_s, P_n) = \lceil \frac{n}{j} \rceil$ *for* $s = 2, 3$.

Proof. Let $k = \lceil \frac{n}{j} \rceil$. If all edges of $F = K_{j \times (k-1)}$ are colored by blue then F contains neither red P_2 (and P_3) nor blue P_n for $n \geq 3$. Therefore, $m_j(P_s, P_n) \geq k$ for $s = 2, 3$ and $n \geq 3$. It easy to see that $m_j(P_2, P_n) \leq k$, and so $m_j(P_2, P_n) = k$. Now, we prove that $m_j(P_3, P_n) \leq k$. Let all edges of $F = K_{j \times k}$ be colored by red or blue, so that F contains no red P_3. To show that F contains a blue path P_n on n vertices, consider the following three cases.

Case 1. $j = 2$.
Let $V_1 = \{a_1, a_2, ..., a_k\}$ and $V_2 = \{b_1, b_2, ..., b_k\}$ be the partite sets of F. If all edges of F are blue then the proof is complete. Now, suppose F contains r red edges, $r \leq k$. Since there is no red P_3, these red edges are independent. Without loss of generality, we may assume that the r red edges are: $a_1b_1, a_2b_2, \cdots, a_rb_r$. If r is odd then
$a_1b_2a_3b_4 \cdots a_{r-2}b_{r-1}a_rb_1a_2b_3a_4 \cdots b_{r-2}a_{r-1}b_ra_{r+1}b_{r+1}a_{r+2}b_{r+2} \cdots a$
is a blue path with at least n vertices in F. Now, if r is even then we have a blue path $a_1b_2a_3b_4 \cdots a_{r-3}b_{r-2}a_{r-1}b_ra_{r-2}b_{r-3}a_{r-4} \cdots b_3a_2b_1a_r$ $b_{r-1}a_{r+1}b_{r+1} \cdots a_kb_k$ with at least n vertices in F.

Case 2. $j = 3$.
If all edges of F are blue then it is finished. Let V_1, V_2 and V_3 be the partite sets of F. Now, assume, without loss of generality, there exist r, s and t red edges connecting V_1 to V_2, V_1 to V_3, and V_2 to V_3, respectively. By considering these red edges, partition V_1, V_2 and V_3 as follow: $V_1 = R_1 \cup X \cup S_1$, $V_2 = R_2 \cup Y \cup T_2$ and $V_3 = S_3 \cup Z \cup T_3$, where $|R_1| = |R_2| = r$, $|S_1| = |S_3| = s$ and $|T_2| = |T_3| = t$ so that all edges connecting R_1 to R_2, S_1 to S_3 and T_2 to T_3 are red. Next, without loss of generality, assume $r \leq s \leq t$. This implies that $|Z| \leq |Y| \leq |X|$. Observe that there exist three independent blue paths: (i) path ${}_aP_b$ of $2r$ vertices connecting all vertices of R_1 and some of S_3 with the initial vertex $a \in R_1$ and the terminal vertex $b \in S_3$, (ii) path ${}_cP_d$ of $2r$ vertices connecting all vertices of R_2 and some of T_3 with the initial vertex $c \in R_2$ and the terminal vertex $d \in T_3$, (iii)

path ${}_eP_f$ of $2s$ vertices connecting all vertices of S_1 and some of T_2 with the initial vertex $e \in T_2$ and the terminal vertex $f \in S_1$, see Fig.1.(i). We can the join all these paths into one larger blue path ${}_aP_f{:=}{}_aP_{bc}P_{de}P_f$. This path has $4r + 2s$ vertices, see Fig.1.(ii).

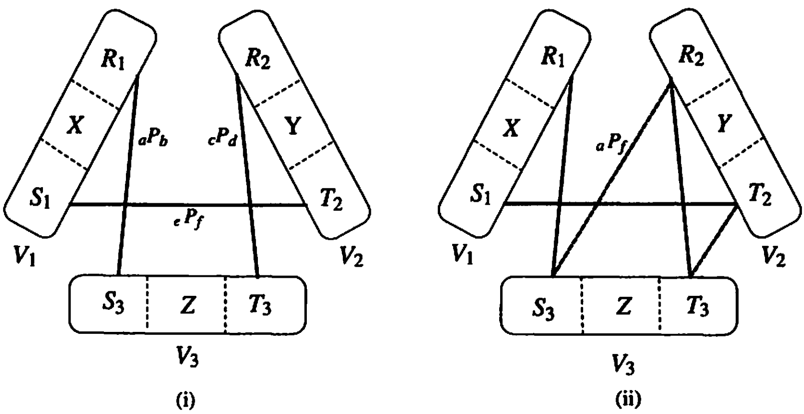

Fig. 1. The three blue paths form a larger blue path starting from vertex $a \in R_1$ and ending at $f \in S_1$.

Let denote by A, B and C the subsets of T_2, S_3 and T_3, respectively, which contain all vertices not in the above three blue paths. Then, we have $|Y| + |A| = |X|$ and $|B| + |Z| + |C| = |X| + |B| = |X| + (s - r)$, and $(s - r) = |Y| - |Z| \leq |X|$. We will show that there exists a blue path connecting X, $Y \cup A$ and $B \cup Z \cup C$ with at least $3|X| + (s - r)$ vertices.

Partition the sets $C = C_1 \cup C_2$ such that C_2 consists of all end-vertices of red edges connecting A and C, and so $|C_2| = |A| = (t - s)$ and $|C_1| = |B| = (s - r)$. Partition the sets $X = X_1 \cup X_2$ such that $|X_2| = |C_2|$; Clearly $|X_1| = |Y|$. Suppose $D = B \cup Z \cup C_1$. Note that $|X_2| = |A| = |C_2|$. Let $C_2 = \{a_1, a_2, \cdots, a_m\}$, $X_2 = \{b_1, b_2, \cdots, b_m\}$, and $A = \{c_1, c_2, \cdots, c_m\}$, where $m = t - s$. Then we obtain a blue path $a_1b_1c_1a_2b_2c_2 \cdots a_mb_mc_m$. This path has $3(t - s)$ vertices, and is denoted by ${}_{a_1}P_{c_m}$. Since fa_1 is a blue edge then by joining the two paths ${}_aP_f$ and ${}_{a_1}P_{c_m}$ we have a blue path with $4r + 3t - s$ vertices. This resulting path, denote by ${}_aP_{c_m}$, starts from a and ends at c_m. Next, we consider the following three subcases.

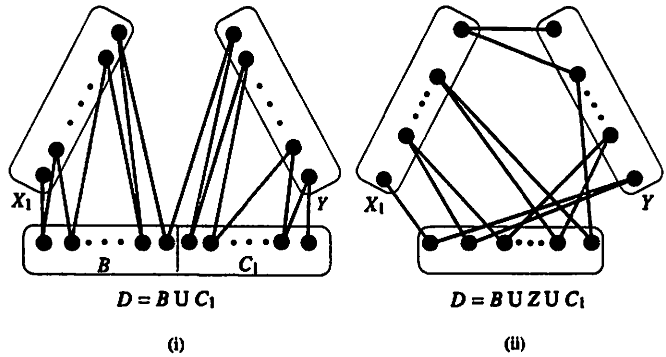

Fig. 2. (i) A blue path ${}_gP_h$ (ii) A blue path ${}_uP_v$

Subcase 2.1. $|Z| = 0$.
Since $|Z| = 0$ then $|D| = |B \cup C_1| = 2|Y| = 2|X_1| = 2(s-r)$. Then, we obtain a blue path by connecting all vertices in X_1 with a half of D alternatingly, and continuing connecting the other half of D with all the vertices in Y alternatingly. This path starts at some vertex $g \in X_1$ and ends at $h \in C_1$, and is denoted by ${}_gP_h$ (see Fig.2.(i)). Note that this path has $4(s-r)$ vertices. Since $c_m g$ is a blue edge then by joining the two paths ${}_aP_{c_m}$ and ${}_gP_h$ we have a blue path with $3(s+t)$ vertices. This resulting path uses all vertices of F, and so F contains a blue path with at least n vertices.

Subcase 2.2. $0 < |Z| < |Y|$.
Since $|Z| < |Y|$ then $|D| = |B \cup Z \cup C_1| < 2|Y|$. Then, we obtain a blue path ${}_uP_v$ connecting all vertices in X_1 with all vertices in Y through all vertices in D one by one each time, until all the vertices in D have been totally used. If there are still some vertices in X_1 (and so in Y) left then connect directly these remaining vertices alternatingly, see Fig.2.(ii). Since $c_m u$ is a blue edge then by joining the two paths ${}_aP_{c_m}$ and ${}_uP_v$, we have a blue path with $3(|Y| + r + t)$ vertices. This resulting path contains all the vertices of F, and so F has contains a blue path with at least n vertices.

Subcase 2.3. $|Z| = |Y| \neq 0$.
Since $|Z| = |Y|$, then $s-r = 0$ and so $|D| = |Z|$. Then we obtain a blue path ${}_wP_z$ connecting all vertices in D, X_1, and Y alternatingly, where $w \in D$ and $z \in Y$. Since $c_m w$ is a blue edge then by joining the two paths ${}_aP_{c_m}$ and ${}_wP_z$, we have a blue path with $3(|Y| + r + t)$ vertices. This resulting path will contains all the vertices of F.

Case 3. $j \geq 4$.
Let $V_1, V_2, \cdots, V_j$ be the partite sets of F. Trivially, if all edges of F are blue then it is finished. If j even by Case 1 we have $\frac{j}{2}$ blue paths connecting all vertices V_1 to V_2, V_3 to V_4, $\cdots$, V_{j-1} to V_j. Each path has $2k$ vertices. Since F has no a red P_3 then we can concatenate these $\frac{j}{2}$ paths into one blue path of kj vertices. This final path will have at least n vertices. Ifj is odd then by Case 1 we obtain $\frac{j-3}{2}$ blue paths connecting all vertices V_1 to V_2, V_3 to V_4, $\cdots$, V_{j-4} to V_{j-3} independently. Each path has $2k$ vertices. By using the method in Case 2 we get another blue path connecting all vertices in V_{j-2}, V_{j-1} and V_j. Again, since F has no red P_3, we can join all these paths into one with at least n vertices.□

Corollary 1. *If $n \geq 3$ then $m_j(P_s, C_n) = \lceil \frac{n}{j} \rceil$ for $s = 2, 3$.*

Proof. Let ${}_xP_y$ be the final blue path obtained in the proof of Theorem 1. This path consists of at least n vertices. Since xy is a blue edge then by joining the two vertices x and y, we have a blue cycle C_n with at least n vertices.□

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