Edge-three cordial graphs arising from complete graphs

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Abstract

A graph G is said to be E_k**-Cordial** if there is an edge labeling $f : E(G) :\longrightarrow \{0, 1, \cdots, k-1\}$ such that, at each vertex v, the sum modulo k of the labels on the edges incident with v is $f(v)$ and it satisfies the inequalities $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$, where $v_f(s)$ and $e_f(t)$ are, respectively, the number of vertices labeled with s and the number of edges labeled with t. The map f is then called an E_k-cordial labeling of G.

This paper investigates E_3-cordiality of snakes, one point unions, path unions and coronas involving complete graphs.

A graph G is said to be E_k**-Cordial** if there is an edge labeling f : $E(G) :\longrightarrow \{0, 1, \cdots, k-1\}$ such that, at each vertex v, the sum modulo k of the labels on the edges incident with v is $f(v)$ and it satisfies the inequalities $| v_f(i) - v_f(j) | \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$, where $v_f(s)$ and $e_f(t)$ are, respectively, the number of vertices labeled with s and the number of edges labeled with t. The map f is then called an E_k-cordial labeling of G. This concept was introduced by Cahit and Yilmaz. They proved that the following graphs are E_3-cordial: P_n $(n \geq 3)$; stars S_n if and only if $n \not\equiv 1 \pmod 3$; K_n $(n \geq 3)$; C_n $(n \geq 3)$; friendship graphs; and fans F_n $(n \geq 3)$. They also prove that S_n $(n \geq 2)$ is E_k-cordial if and only if $n \not\equiv 1 \bmod k$ when k is odd or $n \not\equiv 1 \bmod 2k$ when k is even and $k \neq 2$.

Complete Graphs

One can easily see that K_2 is not E_3-cordial. Cahit and Yilmaz proved that K_n is E_3-cordial for all $n \geq 3$. We give here E_3-cordial labelings different from those given by them and later develope some more labelings and use all of them in path unions as well as one point unions of complete graphs.

Theorem: For all $n \geq 3$, the complete graph K_n is E_3-cordial.

Proof: If we have an E_3-labeling of a complete graph K_m we can develope an E_3-labeling of K_{m+3} first taking K_3 on three new vertices and assigning the labels $0, 1, 2$ to the edges of this K_3. The vertices automatically get the labels $0, 1, 2$, that is, K_3 is E_3-cordial with label numbers $v_f(0, 1, 2) = (1, 1, 1) = e_f(0, 1, 2)$.

Join all the edges from the vertex of K_3 with the label i to vertices of K_m and assign the label i to them. The resulting labeling of the copy of K_{m+3} is E_3-cordial.

For K_4 we assign the labels $0, 1, 2$ to the edges of K_4, as shown in the following figure. For K_5 we in fact three E_3-cordial labelings.

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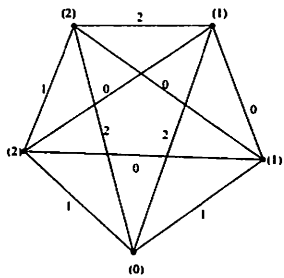
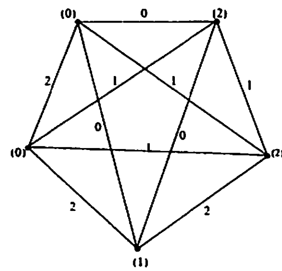
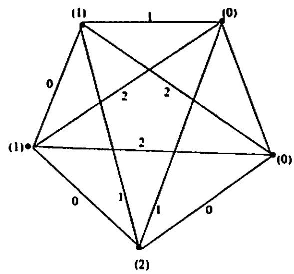
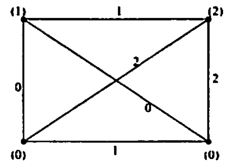


Labeling B



Labeling C





After we develope these basic labelings we get E_3-cordial labelings of K_n with label numbers shown in the following table. Hence K_n is E_3-cordial for all $n \geq 3$.

n	$v_f(0)$	$v_f(1)$	$v_f(2)$	$e_f(0)$	$e_f(1)$	$e_f(2)$
$3x$	x	x	x	$\frac{3x^2-x}{2}$	$\frac{3x^2-x}{2}$	$\frac{3x^2-x}{2}$
$3x+1$	$x+1$	x	x	$\frac{3x^2+x}{2}$	$\frac{3x^2+x}{2}$	$\frac{3x^2+x}{2}$
$3x+2$	$x+1$	$x+1$	x	$\frac{3x^2+3x}{2}$	$\frac{3x^2+3x}{2}$	$\frac{3x^2+3x}{2}+1$
$3x+2$	x	$x+1$	$x+1$	$\frac{3x^2+3x}{2}+1$	$\frac{3x^2+3x}{2}$	$\frac{3x^2+3x}{2}$
$3x+2$	$x+1$	x	$x+1$	$\frac{3x^2+3x}{2}$	$\frac{3x^2+3x}{2}+1$	$\frac{3x^2+3x}{2}$

Snakes, One Point Unions and Path Unions of Complete Graphs

By the snake $S(K_n, m)$ we mean the graph whose cut-vertex-block graph is a path of length $m-1$ and whose blocks are all isomorphic to K_n. By the one point union $K_n^{(m)}$ we mean the graph obtained by taking vertex disjoint union of m copies of K_n all having precisely one vertex in common. By the path uniOn $P_m(K_n)$ we mean the graph obtained by taking a path of length $m-1$ and attaching a copy of K_n at each of its vertices.

Theorem: The snake $S(K_{3x}, m)$, one point union $K_{3x}^{(m)}$ and the path union $P_m(K_{3x})$ is E_3-cordial.

Proof: Consider $S(K_{3x}, m)$. If $m = 1$, it is just K_{3x} and it has a E_3-cordial labeling A. We now construct two more labelings which are non-E_3-cordial.

labeling B: First take a copy of K_3 with edge labels $2, 0, 2$. Follow the previous construction to get a labeling B of K_{3x} with the label numbers

$v_B(0,1,2) = (x-1, x, x+1), e_B(0,1,2) = \left(\frac{3x^2-x}{2}, \frac{3x^2-x}{2}-1, \frac{3x^2-x}{2}+1\right)$

labeling C: First take a copy of K_3 with edge labels $1,0,1$. Follow the previous construction to get a labeling C of K_{3x} with the label numbers $v_C(0,1,2) = (x-1, x+1, x), e_C(0,1,2) = \left(\frac{3x^2-x}{2}, \frac{3x^2-x}{2}+1, \frac{3x^2-x}{2}-1\right)$

If $m = 2$, Take two copies of K_{3x}, with the labeling A and join them at a vertex having the label 0 in each. The vertex-label numbers of the resulting labeling are $v_f(0,1,2) = (2x-1, 2x, 2x)$. The edges are labeled equitably.

Now let $m = 3p + r, r = 0,1,2$. If $r = 0$, consider $S(K_{3x}, 3p)$. From left to right repeatedly assign the labelings C, B, A to the blocks, such that the blocks with labelings C and A are joined to the block B at vertex having the label 1 in them with a vertex of B having the label 2 in B. Similarly, a vertex of A having the label 0 is identified with a vertex of C having the label 1 in it. The edges are labeled equitably. The vertex-label numbers of the resulting labeling are $v_f(0,1,2) = (p(3x-1)+1, p(3x-1), p(3x-1))$

If $r = 1$, Take the previous snake $S(K_{3x}, 3p)$ whose last block has the labeling A. To this attach a copy of K_{3x} with the lableing A at a vertex having the label 0 in both. The vertex-label numbers are $v_f(0,1,2) = (p(3x-1)+x, p(3x-1)+x, p(3x-1)+x)$.

If $r = 2$, Take the previous snake $S(K_{3x}, 3p)$ whose last block has the labeling A. To last block of this snake attach a copy of $S(K_{3x}, 2)$ with its E_3-cordial lableing, at a vertex having the label 0 in the snake of length $3p$ and a vertex having the label 1 in $S(K_{3x}, 2$. The vertex-label numbers are $v_f(0,1,2) = (p(3x-1)+2x-1, p(3x-1)+2x, p(3x-1)+2x)$.

Hence $S(K_{3x}, m)$ is E_3-cordial.

Now consider the one point union $K_{3x}^{(m)}$. Write $m = 3p + r, r = 0,1,2$. If $r = 0$, take $3p$ copies of K_{3x} with the labeling A. Identify vertex with label 0 in p copies, with the vertex with the label 1 in next p copies. Finally with the vertex with the label 2 in the remaining p copies. The edges are labeled equitably and the vertex label numbers are same as thaose of $S(K_{3x}, 3p)$. The central vertex gets the label 0.

If $r = 1$, take one more copy with the labeling A. attach it to the earlier copy of $K_{3x}^{(3p)}$, identifying a vertex with the lable 1 with the central vertex with the label 0. The edges are labeled equitably and the vertex label numbers are same as thaose of $S(K_{3x}, 3p+1)$. The central vertex gets the label 1.

If $r = 2$, take one more copy with the labeling A. attach it to the earlier copy of $K_{3x}^{(3p+1)}$, identifying a vertex with the lable 2 with the central vertex with the label 1. The edges are labeled equitably and the vertex label numbers are same as those of $S(K_{3x}, 3p+2)$. The central vertex gets the label 0.

Finally we consider $P_m(K_{3x})$. If $m = 1$, we just have K_{3x}, which is E_3-cordial.

If $m = 2$, take one copy of K_{3x} with the labeling A. Join, by an edge, a vertex of this copy with the label 1 to a vertex with the label 2 of a copy of K_{3x} having the labeling B. Assign the label 2 to the joining edge. As a result the vertex of A will change its label from 1 to 0 and the vertex of B will change its label from 2 to 1. The label numbers of the resulting labeling are $v_f(0,1,2) = (2x, 2x, 2x), e_f(0,1,2) = (3x^2 - x, 3x^2 - x, 3x^2 - x + 1)$.

If $m = 3p$, take a path on $3p$ points with the edge labels $1, 2, 0, 1, 2, 0, \cdots, 1, 2$. As a result the vertices on this path get the labels $1, 0, 2, \cdots, 1.0, 2$. We attach the copy of K_{3x} with the E_3-cordial labeling A at each points such that, the vertices of K_{3x} where they are attached have $0, 0, 1, 0, 0, 1, \cdots, 0, 0, 1$ as their original labels in K_{3x}. These labels change to $1, 0, 0, \cdots, 1, 0, 0$. The label numbers of the resulting labeling are

$$\begin{aligned} v_f(0,1,2) &= p\{(x-1, x+1, x) + (x, x, x) + (x+1, x-1, x)\} \\ &= (3px, 3px, 3px) \\ e_f(0,1,2) &= p(\alpha, \alpha, \alpha) + (p-1, p, p) \\ &= (p\alpha + p - 1, p\alpha + p, p\alpha + p) \end{aligned}$$

where $\alpha = \dfrac{3x^2 - x}{2}$.

If $m = 3p + 1$, attach one more edge with the label 0 and join a copy of K_{3x} with the labeling A at a vertex with label 0. The label numbers are

$$\begin{aligned} v_f(0,1,2) &= p\{(x-1, x+1, x) + (x, x, x) + (x+1, x-1, x)\} + (x, x, x) \\ &= (3px + x, 3px + x, 3px + x) \\ e_f(0,1,2) &= ((p+1)(\alpha, \alpha, \alpha)) + (p, p, p) \\ &= ((p+1)\alpha + p, (p+1)\alpha + p, (p+1)\alpha + p) \end{aligned}$$

If $m = 3p+2$, take the labeled copy of $P_{3p}(K_{3x}$. Attach, by an adge with the label 0, the labeled copy of $P_2(K_{3x})$. The label numbers are

$$\begin{aligned} v_f(0,1,1) &= p\{(x-1, x+1, x) + (x, x, x) + (x+1, x-1, x)\} + (2x, 2x, 2x) \\ &= (3px + 2x, 3px + 2x, 3px + 2x) \\ e_f(0,1,2) &= ((p+2)(\alpha, \alpha, \alpha)) + (p, p, p+1) \\ &= ((p+2)\alpha + p, (p+2)\alpha + p, (p+2)\alpha + p + 1) \end{aligned}$$

Hence $P_m(K_{3x})$ is E_3-cordial for all $x, m \in \mathbb{N}$. •

Remark: We have a E_3-cordial labeling A of K_{3x+1} with the label numbers $v_A(0,1,2) = (x+1, x, x), e_A(0,1,2) = \left(\dfrac{3x^2+x}{2}, \dfrac{3x^2+x}{2}, \dfrac{3x^2+x}{2}\right)$. If we assign same labeling to all the m copies of K_{3x+1}, taking care that they are attached only at a vertex with the label 0, then we get a E_3-cordial labeling of $S(K_{3x+1}, m)$ as well as $K_{3x+1}^{(m)}$.

For the path union $P_m(K_{3x+1}), m \not\equiv 2 \bmod 3$, we take a path of length $m-1$ with edge labels $1,2,0,1,2,0,\cdots$ and attach a copy of K_{3x+1} with its E_3-cordial labeling such that the vertx where it is attached has original label 0 in K_{3x+1}. One can easily check that this gives a E_3-cordial labeling.

For $m = 3p+2$, take a labeled copy of $P_{3p+1}(K_{3x+1})$. Attach to this one more edge with the label 1. To the last vertex, which has the label 1 on this elongated path, attach a copy of K_{3x+1} at a vertex having the label 1 in K_{3x+1}. The resulting labeling clearly labels the edges equitably. The vertex numbers of this labeling are

$$\begin{aligned} v_f(0,1,2) &= (x,x+1,x)+(x+1,x,x)+(x,x,x+1)+\cdots(p \text{ times }) \\ &\quad +(x,x+1,x)+(x+1,x-1,x+1) \\ &= \{p(3x+1)+2x+1, p(3x+1)+2x, p(3x+1)+2x+1\} \end{aligned}$$

Hence $P_m(K_{3x+1})$ is E_3-cordial for all $m, x \in \mathbb{N}$. •

Theorem: The snake $S(K_{3x+2}, m)$, one point union $K_{3x+2}^{(m)}$ and path union $P_m(K_{3x+2})$ are all E_3-cordial for all $x, m \in \mathbb{N}$.

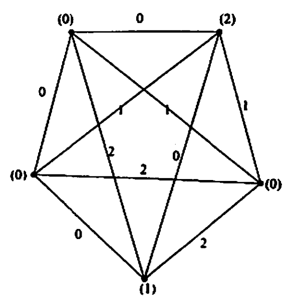
Proof: We already know that there is an E_3-cordial labeling A of K_{3x+2} with label numbers

$$\begin{aligned} v_A(0,1,2) &= (x+1,x,x+1), \\ e_A(0,1,2) &= \left(\frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}+1, \frac{3x^2+3x}{2}\right). \end{aligned}$$

There is also an E_3-cordial labeling B of K_{3x+2} with label numbers

$$\begin{aligned} v_B(0,1,2) &= (x+1,x+1,x), \\ e_B(0,1,2) &= \left(\frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}+1\right). \end{aligned}$$

We construct another labeling which is not E_3-cordial as follows:



If we start with the labeling given by the figure above and follow the construction of previous section, we get a Non-E_3-cordial labeling C of K_{3x+2}

with label numbers $v_f(0,1,2) = (x+2, x, x)$,
$e_C(0,1,2) = \left(\frac{3x^2+3x}{2}+1, \frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}\right)$.

When we attach copies of K_{3x+2}, we attach them only at a vertex with the label 0. The labeling $A+B$ produces a labeling with label numbers $v_f(0,1,2) = (2x+1, 2x+1, 2x+1), e_f(0,1,2) = (3x^2+3x, 3x^2+3x+1, 3x^2+3x+1)$, which is clearly a E_3-cordial labeling of $S(K_{3x+2}, 2)$.

The labeling $A+B+C$ produces a labeling with label numbers $v_f(0,1,2) = (3x+2, 3x+1, 3x+1), e_f(0,1,2) = (\alpha, \alpha, \alpha)$, where $\alpha = \frac{3(3x^2+3x)}{2}+1$. Clearly this is a E_3-cordial labeling of $S(K_{3x+2}, 3)$.

Now write $m = 3q+r, r = 0,1,2$. If $r = 0$, use the sequence $A+B+C$ repeatedly q times. If $r = 1$, assign the labeling A to the remaining block. If $r = 2$, use labeling $A+B$ to the remaining two blocks. Following table shows the label numbers of the resulting labeling:

m	$v_f(0)$	$v_f(1)$	$v_f(2)$	$e_f(0)$	$e_f(1)$	$e_f(2)$
$3q$	$q(3x+1)+1$	$q(3x+1)$	$q(3x+1)$	$q\alpha$	$q\alpha$	$q\alpha$
$3q+1$	$q(3x+1)$ $+x+1$	$q(3x+1)$ $+x$	$q(3x+1)$ $+x+1$	$q\alpha+$ $\frac{3x^2+3x}{2}$	$q\alpha+$ $\frac{3x^2+3x}{2}+1$	$q\alpha+$ $\frac{3x^2+3x}{2}$
$3q+2$	$q(3x+1)$ $+2x+1$	$q(3x+1)$ $+2x+1$	$q(3x+1)$ $+2x+1$	$q\alpha+$ $3x^2+3x$	$q\alpha+$ $3x^2+3x$	$q\alpha+$ $3x^2+3x$

Hence $S(K_{3x+2}, m)$ is E_3-cordial for all $x, m \in \mathbb{N}$.

Now consider the one point union $K_{3x+2}^{(m)}$. If $m = 3p$, assign the labelings A, B, C to p copies each such that the point where they are joined has the label 0 in each copy. If $m = 3p+1$, assign the labeling A to the additonal copy, again joining it at a vertex with the label 0. Finally, if $m = 3p+2$, assign the labeling B to the additonal copy, again joining it at a vertex with the label 0. The label numbers of the resulting labeling is same as given by the previous table.

Now consider $P_m(K_{3x+2})$. If $m = 1$, we have E_3-cordial labeling A of K_{3x+2}. If $m = 2$, Take an edge with the label 0. Attach the copy of K_{3x+2} with the labeling A at one end and a copy with the labeling B at the other end. The resulting labeling has the label numbers

$$v_f(0,1,2) = (2x+2, 2x+1, 2x+1),$$
$$e_f(0,1,2) = (3x^2+3x+1, 3x^2+3x+1, 3x^2+3x+1).$$

If $m = 3p$, Take a path on $3p-1$ edges. Label the edges from left as $2,1,0,2,1,0,\cdots,2,1$. From left attach repeatedly the copies of K_{3x+2} with the labelings A, B, C such that the copy with the labeling A is attached at a vertex with the label 0 in it, one with the labeling B is attached at a vertex with the label 2 in it and one with the labeling C is attached at a vertex with the label 0 in it. In each copy the edge labels remain same. However, the vertex labels change to $v_A(0,1,2) = (x, x, x+2), v_B(0,1,2) = (x+1, x+1, x), v_C(0,1,2) = (x+1, x+1, x)$. The resulting labeling has label numbers

$$\begin{aligned} v_f(0,1,2) &= (p(3x+2), p(3x+2), p(3x+2)) \\ e_f(0,1,2) &= p\left(\frac{9x^2+9x}{2}+1, \frac{9x^2+9x}{2}+1, \frac{9x^2+9x}{2}+1\right) + (p-1, p, p) \end{aligned}$$

If $m = 3p+1$, attach one additional edge with the label 0 and attach the copy of K_{3x+2} with the labeling A, at a vertex having the label 0 in it. The label numbers are

$$\begin{aligned} v_f(0,1,2) &= (p(3x+2, p(3x+2), p(3x+2)) + (x+1, x, x+1) \\ e_f(0,1,2) &= p\left(\frac{9x^2+9x}{2}+1, \frac{9x^2+9x}{2}+1, \frac{9x^2+9x}{2}+1\right) + \\ &\quad +(p,p,p) + (\alpha, \alpha+1, \alpha) \end{aligned}$$

where $\alpha = \dfrac{3x^2+3x}{2}$.

Finally, let $m = 3p+2$. If $p = 0$, we already have a E_3-cordial labeling of $P_2(K_{3x+2})$. Now if $p > 0$, first take a labeled copy of $P_{3p}(K_{3x+2})$. Attach to this a copy of $P_2(K_{3x+2})$ by an edge having the label 0. The label numbers of the resulting labeling are

$$\begin{aligned} v_f(0,1,2) &= (p(3x+2), p(3x+2), p(3x+2)) + (2x+2, 2x+1, 2x+1) \\ e_f(0,1,2) &= p\left(\frac{9x^2+9x}{2}+1, \frac{9x^2+9x}{2}+1, \frac{9x^2+9x}{2}+1\right) \\ &\quad +(p+1+2\alpha, p+2\alpha+1, p+2\alpha+1) \end{aligned}$$

Hence $P_m(K_{3x+2})$ is E_3-cordial for all $x, m \in \mathbb{N}$. •

Corona Graphs involving complete graphs

Let G_1, G_2 be two finite graphs. The **corona graph** $G_1 \circ G_2$ is defined as follows: Let $V(G_1) = \{u_1, \cdots, u_n\}$. Take n copies of the graph G_2. Denote them as $H_1, \cdots, H_n$. Now

$$\begin{aligned} V(G_1 \circ G_2) &= V(G_1) \bigcup_{i=1}^{n} V(H_i) \\ E(G_1 \circ G_2) &= E(G_1) \bigcup_{i=1}^{n} E(H_i) \bigcup_{i=1}^{n} \{g_i z \mid z \in V(H_i)\} \end{aligned}$$

The graphs G_1 and G_2 are called the component graphs of the corona graph. Clearly, $G_1 \circ G_2$ is not isomorphic to $G_2 \circ G_1$.

Theorm: The corona graph $K_{3x} \circ K_n$ is E_3-cordial for all natural numbers x and $n \geq 3$.

Proof: **Case 1:** $n = 3y, y \geq 1$. Clearly the number of vertices is $3x + 9xy$ and the number of edges is $\frac{9x^2 - 3x}{2} + \frac{3x(9y^2 - 3y)}{2} + 9xy$. From the results of previous section K_{3x} has a E_3-cordial labeling g with

$$\begin{aligned} v_g(0,1,2) &= (x, x, x), \\ e_g(0,1,2) &= \left(\frac{3x^2 - x}{2}, \frac{3x^2 - x}{2}, \frac{3x^2 - x}{2}\right) \end{aligned}$$

Each copy H of K_{3y} has a similar labeling h.

Create the labeling f of the edges of $K_{3x} \circ K_{3y}$ as follows:

$$\begin{aligned} f(e) &= g(e) \quad \text{if } e \in K_{3x} \\ &= h(e) \quad \text{if } e \in H \end{aligned}$$

The edges joining the vertices of K_{3x} to the vertices of H are labeled as follows: All the edges joining a vertex of K_{3x} having the label 0 to vertices of a copy of K_{3y} are labeled 0. All the edges joining a vertex of K_{3x} having the label 1 to vertices of a copy of K_{3y} are labeled 1. All the edges joining a vertex of K_{3x} having the label 2 to vertices of a copy of K_{3y} are labeled 2. Since the original vertex of K_{3x} receives $3y$ edges with same label its label does not change. In each copy of H joined to a vertex with label 0, labels of all the vertices remain unchanged. In each copy of H joined to a vertex with label 1, vertex label 0 becomes 1. vertex label 1 becomes 2 and vertex label 2 becomes 0. In each copy of H joined to a vertex with label 2, vertex label 0 becomes 2. vertex label 1 becomes 0 and vertex label 2 becomes 1.

Hence the label numbers of the resulting labeling f are

$$\begin{aligned} v_f(0,1,2) &= (x + 3xy, x + 3xy, x + 3xy) \\ e_f(0,1,2) &= (\alpha, \alpha, \alpha) \end{aligned}$$

where $\alpha = \frac{3x^2 - x + 9xy^2 - 3xy}{2} + 3xy$. Hence $K_{3x} \circ K_{3y}$ is E_3-cordial.

Case 2: $n = 3y + 1, y \geq 1$. Clearly the number of vertices is $6x + 9xy$ and the number of edges is $\frac{9x^2 - 3x}{2} + \frac{3x(9y^2 + 3y)}{2} + 3x(3y + 1)$. From the results of previous section K_{3x} has a E_3-cordial labeling g with

$$\begin{aligned} v_g(0,1,2) &= (x, x, x), \\ e_g(0,1,2) &= \left(\frac{3x^2 - x}{2}, \frac{3x^2 - x}{2}, \frac{3x^2 - x}{2}\right) \end{aligned}$$

Each copy H of K_{3y+1} has a similar labeling h with label numbers

$$\begin{aligned} v_h(0,1,2) &= (y+1, y, y), \\ e_h(0,1,2) &= \left(\frac{3y^2+y}{2}, \frac{3y^2+y}{2}, \frac{3y^2+y}{2}\right) \end{aligned}$$

Create the labeling f of the edges of $K_{3x} \circ K_{3y+1}$ as follows:

$$\begin{aligned} f(e) &= g(e) \quad \text{if } e \in K_{3x} \\ &= h(e) \quad \text{if } e \in H \end{aligned}$$

The edges joining the vertices of K_{3x} to the vertices of H are labeled as follows: All the edges joining a vertex of K_{3x} having the label 0 to vertices of a copy of K_{3y+1} are labeled 0. All the edges joining a vertex of K_{3x} having the label 1 to vertices of a copy of K_{3y+1} are labeled 1. All the edges joining a vertex of K_{3x} having the label 2 to vertices of a copy of K_{3y+1} are labeled 2. Since the original vertices of K_{3x} receives $3y+1$ edges with the same label, the vertex label 0 in K_{3x} does not change but the vertex labels 1 and 2 are interchanged. In each copy of H joined to a vertex with label 0, labels of all the vertices remain unchanged. In each copy of H joined to a vertex with label 1, vertex label 0 becomes 1. vertex label 1 becomes 2 and vertex label 2 becomes 0. In each copy of H joined to a vertex with label 2, vertex label 0 becomes 2. vertex label 1 becomes 0 and vertex label 2 becomes 1.

Hence the label numbers of the resulting labeling f are

$$\begin{aligned} v_f(0,1,2) &= (x+x(3y+1), x+x(3y+1), x+x(3y+1)) \\ e_f(0,1,2) &= (\alpha, \alpha, \alpha) \end{aligned}$$

where $\alpha = \dfrac{3x^2 - x + 9xy^2 + 3xy}{2} + x(3y+1)$. Hence $K_{3x} \circ K_{3y+1}$ is E_3-cordial.

Case 3: $n = 3y+2$. Clearly the number of vertices in $K_n \circ K_m$ is $3x+3x(3y+2) = 9x+9xy$ and the number of edges is $\dfrac{9x^2-3x}{2} + \dfrac{3x(3y+2)(3y+1)}{2} + 3x(3y+2)$.

In this case the previous technique does not work since each copy of K_{3y+2} has one extra edge with the label 2. We note that $K_{3x} \circ K_{3y+2}$ can be looked upon as the graph obtained by attaching a copy of K_{3y+3} to each vertex of K_{3x}. As before take a E_3-cordial labeling g of K_{3x} and a E_3-cordial labeling h of each copy of K_{3y+3} with its label numbers

$$\begin{aligned} v_h(0,1,2) &= (y+1, y+1, y+1) \\ e_h(0,1,2) &= \left(\frac{3y^2+5y+2}{2}, \frac{3y^2+5y+2}{2}, \frac{3y^2+5y+2}{2}\right) \end{aligned}$$

For $0 \leq i \leq 2$, identify a vertex with label i of K_{3x} with a vertex with label i of a copy of K_{3y+3}. The labels 0 in the vertices of K_{3x} remain unchanged, the labels 1 become 2 and the labels 2 become 1. The edge labels of course remain same. The label numbers of the resulting labeling f are

$$\begin{aligned} v_f(0,1,2) &= (3x+3xy, 3x+3xy, 3x+3xy) \\ e_f(0,1,2) &= (\alpha, \alpha, \alpha) \end{aligned}$$

where $\alpha = \dfrac{3x^2-x}{2} + \dfrac{3x(3y^2+5y+2)}{2}$. Hence $K_{3x} \circ K_{3y+2}$ is E_3-cordial. •

We now consider $K_n \circ K_{3y}$. The case $K_{3x} \circ K_{3y}$ has been covered by the previous theorem.

Theorem: The corona graph $K_n \circ K_{3y}$ is E_3-cordial for all $n \geq 3$.

Proof: **Case 1**: $n = 3x+1$. Clearly the number of vertices in $K_n \circ K_{3y}$ is $(3x+1)(3y+1)$ and the number of edges is $\dfrac{9x^2+3x}{2} + (3x+1)\dfrac{9y^2-3y}{2} + (3x+1)(3y)$.

Take a copy of K_{3x+1} with its E_3- cordial labeling g with the label numbers

$$\begin{aligned} v_g(0,1,2) &= (x+1, x, x), \\ e_g(0,1,2) &= \left(\frac{3x^2+x}{2}, \frac{3x^2+x}{2}, \frac{3x^2+x}{2}\right). \end{aligned}$$

Take $3x+1$ copies of $H = K_{3y}$ with its E_3- cordial labeling h with label numbers

$$\begin{aligned} v_h(0,1,2) &= (y, y, y), \\ e_h(0,1,2) &= \left(\frac{3y^2-y}{2}, \frac{3y^2-y}{2}, \frac{3y^2-y}{2}\right). \end{aligned}$$

Define the labeling f of $K_{3x+1} \circ K_{3y}$ as follows: $f(e) = g(e)$ if $e \in K_{3x+1}$, $f(e) = h(e)$ if e is an edge in some copy of H. First keep aside one extra vertex w of K_{3x+1} with label 0. For the remaining $3x$ vertices of K_{3x+1}, the edges joining a vertex a (say) to the vertices of K_{3y} are assigned label $g(a)$. The label $g(a)$ of a remains in tact. On the other hand in the copy of K_{3y} attached to this vertex the vertex labels undergo transformation $h(b) \to h(b) + g(a) \bmod 3$. Clearly this labels the vertices of that copy equitably. Now the extra vertex w of K_{3y+1} is remaining from which the $3y$ edges going to the copy of K_{3y} are yet to be labeled. For a vertex v of K_{3y} we define $f(wv) = 1$ if $h(v) = 0$, $f(wv) = 2$, if $h(v) = 1$ and $f(wv) = 0$, if $h(v) = 2$. Clearly $f(w) = g(w)$ and this creates exactly y vertices with label $i, 0 \leq i \leq 2$ each with the transformation

$$h(v) = 0 \quad \to \quad f(v) = 1$$

$$h(v)=1 \quad \rightarrow \quad f(v)=0$$
$$h(v)=2 \quad \rightarrow \quad f(v)=2$$

One can see that this labeling f has label numbers

$$\begin{aligned} v_f(0,1,2) &= (x+1+y(3x+1), x+y(3x+1), x+y(3x+1)), \\ e_f(0,1,2) &= (\alpha,\alpha,\alpha) \end{aligned}$$

where $\alpha = \dfrac{3x^2+x}{2} + (3x+1)\dfrac{3y^2-y}{2} + 3yx + y$. Hence $K_{3x+1} \circ K_{3y}$ is E_3-cordial.

Case 2: $n = 3x+2$. Clearly, the number of vertices in $K_{3x+2} \circ K_{3y}$ is $(3x+2)+(3x+2)3y$ and the number of edges is $\dfrac{9x^2+9x}{2} + 1 + (3x+2)\dfrac{9y^2-3y}{2} + (3x+2)3y$.

Take a copy of K_{3x+2} with its E_3- cordial labeling g with the label numbers

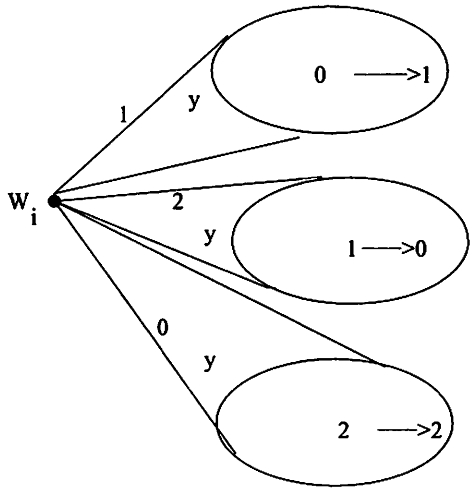
$$\begin{aligned} v_g(0,1,2) &= (x+1, x+1, x), \\ e_g(0,1,2) &= \left(\frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}+1\right). \end{aligned}$$

Take $3x+2$ copies of $H = K_{3y}$ with its E_3- cordial labeling h with label numbers

$$\begin{aligned} v_h(0,1,2) &= (y,y,y), \\ e_h(0,1,2) &= \left(\frac{3y^2-y}{2}, \frac{3y^2-y}{y}, \frac{3y^2-y}{2}\right). \end{aligned}$$

Define the labeling f of $K_{3x+2} \circ K_{3y}$ as follows: $f(e) = g(e)$ if $e \in K_{3x+2}$, $f(e) = h(e)$ if e is an edge in some copy of H. Now keep aside the extra vertices w_1, w_2 of K_{3x+2} with label $0, 1$ respectively. For the remaining $3x$ vertices of K_{3x+1}, the edges joining a vertex a (say) to the vertices of K_{3y} are assigned label $g(a)$. The label $g(a)$ of a remains in tact. On the other hand in the copy of K_{3y} attached to this vertex the vertex labels undergo transformation $h(b) \rightarrow h(b) + g(a)$. Clearly this labels the vertices of that copy equitably. Now the extra two vertices w_1, w_2 of K_{3y+1} are remaining from which the $3y$ edges going to the copy of K_{3y} are yet to be labeled.

For a vertex v of K_{3y} we define $f(w_i v) = 1$ if $h(v) = 0$, $f(wv) = 2$, if $h(v) = 1$ and $f(wv) = 0$, if $h(v) = 2$. Clearly labels of w_1, w_2 are unchanged and this creates exactly $2y$ vertices with label $i, 0 \le i \le 2$ each with the



transformation

$$\begin{aligned} h(v) = 0 &\rightarrow f(v) = 1 \\ h(v) = 1 &\rightarrow f(v) = 0 \\ h(v) = 2 &\rightarrow f(v) = 2 \end{aligned}$$

One can see that this labeling f has label numbers

$$\begin{aligned} v_f(0,1,2) &= (x + y(3x+2) + 1, x + y(3x+2) + 1, x + y(3x+2)), \\ e_f(0,1,2) &= (\alpha + 1, \alpha, \alpha) \end{aligned}$$

where $\alpha = \dfrac{3x^2 + 3x}{2} + (3x+2)\dfrac{3y^2 - y}{2} + 3yx + 2y$.
Hence $K_{3x+1} \circ K_{3y}$ is E_3-cordial. •

These two theorems can be consolidated into one as follows:
Theorem: The corona graph $K_n \circ K_m$ is E_3-cordial if nm is divisible by 3.

Now only four cases remain, that is, $K_n \circ K_m$ where $n, m \equiv 1, 2(\bmod 3)$.
Theorem: The corona graph $K_n \circ K_m$ is E_3-cordial whenever $n \equiv m \bmod 3$.
Proof: We have already covered the case $n, m \equiv 0 \bmod 3$.
Case 1: $n, m \equiv 1 \bmod 3$. The number of vertices in $K_n \circ K_m$ is $(3x+1)(3y+2)$ and the number of edges is $\dfrac{9x^2+3x}{2} + (3x+1)\dfrac{9y^2+3y}{2} + (3x+1)(3y+1)$.

Let $n = 3x+1, m = 3y+1$. Take a copy of K_{3x+1} with its E_3-cordial labeling g with label numbers

$$v_g(0,1,2) = (x+1, x, x)$$

$$e_g(0,1,2) = \left(\frac{3x^2+x}{2}, \frac{3x^2+x}{2}, \frac{3x^2+x}{2}\right)$$

Take $3x+1$ copies of $H = K_{3y+1}$ with its E_3-cordial labeling h with similar label numbers. Define an edge labeling f of $K_{3x+1} \circ K_{3y+1}$ as follows:

The label $f(e)$ of an old edge $e \in K_{3x+1}$ is same as $g(e)$, the label $f(e)$ of an old edge e in a copy of K_{3x+1} is same as $h(e)$. Keep aside a vertex of K_{3x+1} with the label 0. From each of the $3x$ vertices $3y+1$ edges go to a copy of H. These edges are labeled as follows:

All the edges joining x vertices having label 0 are labeled 0. This creates $x(3y+1)$ new edges with label 0. In the x copies of H this creates $x(y+1)$ vertices with label 0 and xy vertices with labels 1 and 2 each.

All the edges joining x vertices having label 1 are labeled 1. This creates $x(3y+1)$ new edges with label 1. In the x copies of H this creates $x(y+1)$ vertices with label 1 and xy vertices with labels 0 and 2 each.

All the edges joining x vertices having label 2 are labeled 2. This creates $x(3y+1)$ new edges with label 2. In the x copies of H this creates $x(y+1)$ vertices with label 2 and xy vertices with labels 0 and 1 each.

Now one vertex w (say) of K_{3x+1} with the label 0 is remaining. We have to join K_{3y+1} to this vertex. This is same as one point union of K_{3y+2} with the previous graph, with the point of identification being w.

Take a copy of K_{3y+2} with its E_3-cordial labeling with the label numbers

$$\begin{aligned} v_h(0,1,2) &= (y, y+1, y+1) \\ e_h(0,1,2) &= \left(\frac{3y^2+3y}{2}+1, \frac{3y^2+3y}{2}, \frac{3y^2+3y}{2}\right) \end{aligned}$$

Identify one vertex with the label 0 in this with the vertex w. The resulting labeling f has the label numbers

$$\begin{aligned} v_f(0,1,2) &= ((3x+2x+y), (3x+2x+y+1), (3x+2x+y+1)) \\ e_f(0,1,2) &= (\beta, \beta, \beta) \end{aligned}$$

Where $\beta = \frac{3x^2+x}{2} + 3x\frac{3y^2+5y+2}{2} + y + 1 + \frac{3y^2+3y}{2} = \frac{3x^2+x}{2} + (3x+1)\frac{3y^2+5y+2}{2}$.

Case 2: $n, m \equiv 2 \bmod 3$. Clearly, the number of vertices in $K_n \circ K_m$ is $(3x+2)(3y+3)$ and the number of edges is $\frac{9x^2+9x}{2} + 1 + (3x+2)(\frac{9y^2+9y}{2} + 1) + (3x+2)(3y+2) = \frac{3(3x^2+3x)}{2} + (3x+2)\frac{9y^2+15y+6}{2} + 1$.

Let $n = 3x+2$ and $m = 3y+2$. This corona graph is the same as obtained by attaching a copy of K_{3y+3} at each vertex of K_{3x+2} by way of one point union. Take a copy of K_{3x+2} with its E_3-cordial labeling g with the label numbers

$$
\begin{aligned}
v_g(0,1,2) &= (x+1,x+1,x)\\
e_g(0,1,2) &= \left(\frac{3x^2+3x}{2},\frac{3x^2+3x}{2},\frac{3x^2+3x}{2}+1\right)
\end{aligned}
$$

Take $3x+2$ copies of K_{3y+3} with the E_3-cordial labeling h with each copy having label numbers

$$
\begin{aligned}
v_g(0,1,2) &= (y+1,y+1,y+1)\\
e_g(0,1,2) &= \left(\frac{3y^2+5y+2}{2},\frac{3y^2+5y+2}{2},\frac{3y^2+5y+2}{2}\right)
\end{aligned}
$$

While labeling the edges, all the edge labels assigned by g and h are retained. At each vertex with the label $i, 0 \leq i \leq 2$ of K_{3x+2} identify a vertex with the label i from a copy of K_{3y+3}. This will change the label i of that vertex to $2i$ modulo 3, that is 0 remains unchanged, the label 1 becomes 2 and the label 2 becomes 1. One can see that this creates a labeling f of $K_{3x+2} \circ K_{3y+2}$ with label numbers

$$
\begin{aligned}
v_f(0,1,2) &= ((y+1)(3x+2),(y+1)(3x+2),(y+1)(3x+2))\\
e_f(0,1,2) &= (\alpha,\alpha,\alpha+1)
\end{aligned}
$$

Where $\alpha = \frac{3x^2+3x}{2} + (3x+2)\frac{3y^2+5y+2}{2}$.

Hence $K_n \circ K_m$ is E_3-cordial whenever $n \equiv m \bmod 3$. •

Finally, we clear the remaining two cases in the following Theorem:

Theorem: The corona graph $K_n \circ K_m$ is E_3-cordial whenever $n = 3x+1, m = 3y+2$ or $n = 3x+2, m = 3y+1$.

Proof: **Case 1:** $n = 3x+1, m = 3y+2$.

The number of vertices in $K_{3x+1} \circ K_{3y+2}$ is $(3x+1)(3y+3)$ and the number of edges is

$$
\begin{aligned}
\mid E(K_{3x+1} \circ K_{3y+2}) \mid &= \frac{9x^2+3x}{2} + (3x+1)\frac{9y^2+9y+2}{2} + (3x+1)(3y+2)\\
&= \frac{3(3x^2+x)}{2} + (3x+1)\frac{9y^2+15y+6}{2}.
\end{aligned}
$$

We view this as the graph obtained by attaching a copy of K_{3y+3} at each vertex of K_{3x+1} by the way of one point union.

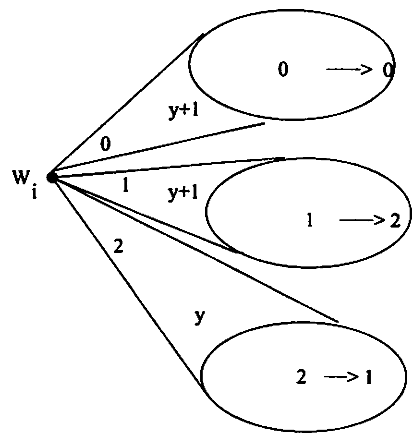
Take a copy of K_{3x+1} with its E_3-cordial labeling g having label numbers

$$
\begin{aligned}
v_g(0,1,2) &= (x+1,x,x)\\
e_g(0,1,2) &= (\frac{3x^2+x}{2},\frac{3x^2+x}{2},\frac{3x^2+x}{2})
\end{aligned}
$$

Take $3x$ copies of $H = K_{3y+3}$ with its E_3-cordial labeling h having label numbers

$$\begin{aligned} v_g(0,1,2) &= (y+1, y+1, y+1) \\ e_g(0,1,2) &= (\frac{3y^2+5y+2}{2}, \frac{3y^2+5y+2}{2}, \frac{3y^2+5y+2}{2}) \end{aligned}$$

While labeling the edges the values assigned by g and h are retained. Keep aside a vertex w of K_{3x+1} having the label 0. For the remaining $3x$ vertices attach a copy of H each to them, taking care of identifying a vertex with a label i in K_{3x+1} with a vertex with the label i in H. We note that in K_{3x+1} the label 0 remains unchanged, the label 1 changes to 2 and the label 2 changes to 1. So far we have created $3x(y+1)$ vertices of each label. and the edges have been labeled equitably.



Now consider w, the vertex of K_{3x+1} that we had kept aside. We Take a copy K of K_{3y+2} with its E_3-cordial labeling $\tilde{h}$ with its label numbers

$$\begin{aligned} v_{\tilde{h}}(0,1,2) &= (y+1, y+1, y) \\ e_{\tilde{h}}(0,1,2) &= \left(\frac{3y^2+3y}{2}, \frac{3y^2+3y}{2}, \frac{3y^2+3y}{2}+1\right) \end{aligned}$$

We keep in mind that in this labeling there is one extra edge with the label 2. Assign the label 0 to all the edges joining w to the $y+1$ vertices of K_{3y+2} having label 0, assign the label 1 to all the edges joining w to the $y+1$ vertices of K_{3y+2} having the label 1 and assign the label 2 to all the edges joining w to the y vertices of K_{3y+2} having the label 2. In H the vertices having the label 1 now get the label 2 and the vertices having the label 2 now have the label 1. The label of the vertex w has now become 1. The resulting labeling f has the label numbers

$$\begin{aligned} v_f(0,1,2) &= ((3x+1)(y+1),(3x+1)(y+1),(3x+1)(y+1)) \\ e_f(0,1,2) &= (\beta,\beta,\beta) \end{aligned}$$

Where $\beta = \dfrac{3x^2+x}{2} + 3x\dfrac{3y^2+5y+2}{2} + y + 1 + \dfrac{3y^2+3y}{2}$
$= \dfrac{3x^2+x}{2} + (3x+1)\dfrac{3y^2+5y+2}{2}$.

Case 2: $n = 3x+2, m = 3y+1$. The number of vertices in $K_{3x+2} \circ K_{3y+1}$ are $(3x+2)(3y+2)$ and the number of edges is $\dfrac{9x^2+9x+2}{2} + (3x+2)\dfrac{9y^2+3y}{2} + (3x+2)(3y+1) = \dfrac{3(3x^2+3x)}{2} + (3x+2)\dfrac{9y^2+9y}{2} + 3(x+1)$.

Take a copy of K_{3x+2} with its E_3-cordial labeling g with its label numbers

$$\begin{aligned} v_g(0,1,2) &= (x+1,x+1,x) \\ e_g(0,1,2) &= \left(\frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}, \frac{3x^2+3x}{2}+1\right) \end{aligned}$$

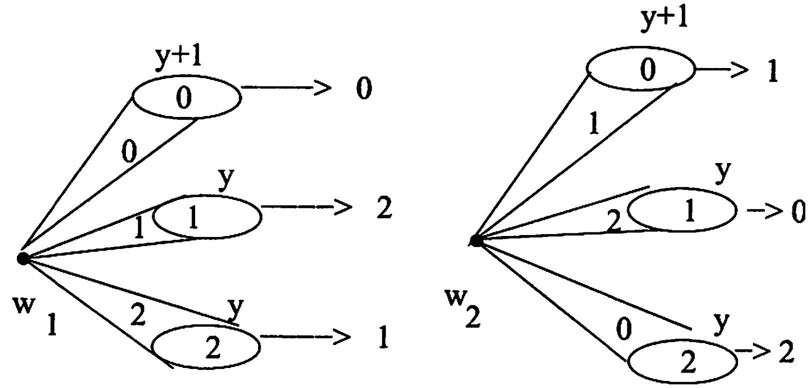
Take $3x+2$ copies of K_{3y+1} with its E_3-cordial labeling h with the label numbers

$$\begin{aligned} v_h(0,1,2) &= (y+1,y,y) \\ e_h(0,1,2) &= \left(\frac{3y^2+y}{2}, \frac{y^2+y}{2}, \frac{3y^2+y}{2}\right) \end{aligned}$$

While labeling the edges, the labels assigned by g and h are retained. Keep aside two vertices w_1, w_2 of K_{3x+2} having the labels 0 and 1 respectively. For each of the remaining $3x$ vertices, the edges joining that vertex v (say) to the vertices of the copy of K_{3y+1} are assigned the label $g(v)$. This means that a vertices with the label 0 produce $x(3y+1)$ edges with the label 0, the vertices with the label 1 produce $x(3y+1)$ edges with the label 1. However in the respective copies of K_{3y+1} the vertex label numbers change to $(y, y+1, y)$. Finally , the vertices with the label 2 produce $x(3y+1)$ edges with the label 2. However in the respective copies of K_{3y+1} the vertex label numbers change to $(y, y, y+1)$.

Now consider w_1 with $g(w_1) = 0$. For a vertex $v \in K_{3y+1}$ the edge w_1v is assigned the label $h(v)$. As a result we have created $y+1$ edges with the label 0 and y edges with the labels 1 and 2 each. However, in the copy of K_{3y+1} vertex labels 1 and 2 are interchanged. The label of w_1 remains unchanged.

Finally, consider w_2 with $g(w_2) = 1$. For a vertex $v \in K_{3y+1}$, the edge w_2v is assigned the label $h(v)+1$. As a result we have created $y+1$ edges with



the label 1 and y edges with the labels 0 and 2 each. However, in the copy of K_{3y+1} vertex labels 0 and 1 are interchanged. The label of w_2 changes to 2.

In this way we have created a labeling f of $K_{3x+2} \circ K_{3y+1}$ with the label numbers

$$\begin{aligned} v_f(0,1,2) &= (\alpha+1, \alpha, \alpha), \\ e_f(0,1,2) &= (\beta, \beta, \beta) \end{aligned}$$

where $\alpha = x + x(y+1) + 2xy + 2y + 1 = 3xy + 2x + 2y + 1$ and $\beta = \frac{3x^2+3x}{2} + x(3y+1) + (3x+2)\frac{3y^2+y}{2} + 2y + 1 = \frac{3x^2+3x}{2} + (3x+2)\frac{3y^2+3y}{2} + (x+1)$.

This shows that in both the cases $K_n \circ K_m$ is E_3-cordial. •

Corona Graphs of Complete graphs with Cycles

Now we consider the corona $K_m \circ C_n$ of the complete graph K_m with the cycle C_n. This graph is same as the one obtained by attaching a copy of the wheel W_n at each vertex of K_m, identifying the hub of W_n with the vertex of K_m. We will first develope suitable labelings of the wheel W_n and use them while constructing a labeling of $K_m \circ C_n$. In the final construction, the label of the hub of W_n might change. However, the vertex-label numbers of the corona are just the sum of the corresponding numbers of those copies of W_n.
Lemmma: For every triple (a, b, c) of positive integers such that $n = a+b+c$, there exists a labeling of W_n and for suitable choice of a, b, c it is E_3-cordial.
Proof: Consider a cycle C_n. Assign the label 0 to a number of consecutive edges, the label 1 to next b number of edges and the label 2 to the remaining c number of edges. The vertex-numbers at this stage are (a, c, b). However, they are not consecutime on the cycle. Let w be the hub. For a spoke wv, assign the label i, if the label of v in C_n is i. The edge-label numbers of the resulting labeling f are $e_f(0,1,2) = (2a, b+c, b+c)$. On the cycle, the label 0 is assigned to a number of vertices, the label 1 is assigned to b number of

vertices and the label 2 is assigned to c number of vertices. The label of the hub is $(c+2b)$ modulo 3.

If $n = 3x$, let $(a,b,c) = (x,x,x)$. We then get a labeling A which assigns the label 0 to the hub and it has the label numbers $v_A(0,1,2) = (x+1,x,x), e_A(0,1,2) = (2x,2x,2x)$. Hence it is a E_3-cordial labeling.

If $n = 3x+1$, let $(a,b,c) = (x,x+1,x)$. We then get a labeling A which assigns the label 2 to the hub and it has the label numbers $v_A(0,1,2) = (x,x+1,x+1), e_A(0,1,2) = (2x,2x+1,2x+1)$. Hence it is a E_3-cordial labeling.

If $n = 3x+2$, let $(a,b,c) = (x+1,x,x+1)$. We then get a labeling A which assigns the label 1 to the hub and it has the label numbers $v_A(0,1,2) = (x+1,x+1,x+1), e_A(0,1,2) = (2x+2,2x+1,2x+1)$. Hence it is a E_3-cordial labeling.

Hence W_n is E_3-cordial for all $n \geq 3$. •

Theorem: The corona $K_m \circ C_n$ is E_3-cordial for all $n,m \in \mathbb{N}, n \geq 3$.

Proof: If $m = 1$, we just have W_n which is shown to be E_3-cordial in the previous lemma. Let $m \geq 2$.

Case 1: $n = 3x$. We first develpe four more labelings as follows:

Labeling B: Take $(a,b,c) = (x-1,x+1,x)$. The resulting labeling B assigns the label 2 to the hub and has the label numbers $v_B(0,1,2) = (x-1,x+1,x+1), e_B(0,1,2) = (2x-2,2x+1,2x+1)$. Clearly it is not a E_3-cordial labeling.

Labeling C: Take $(a,b,c) = (x-1,x,x+1)$. The resulting labeling C assigns the label 1 to the hub and has the label numbers $v_C(0,1,2) = (x-1,x+1,x+1), e_C(0,1,2) = (2x-2,2x+1,2x+1)$. Clearly it is not a E_3-cordial labeling.

Labeling D: Take $(a,b,c) = (x+2,x-1,x-1)$. The resulting labeling D assigns the label 0 to the hub and has the label numbers $v_D(0,1,2) = (x+3,x-1,x-1), e_D(0,1,2) = (2x+4,2x-2,2x-2)$. Clearly it is not a E_3-cordial labeling.

Labeling E: Take $(a,b,c) = (x,x,x)$. In this labeling the spokes are labeled differently. Leave three vertices on the cycle C_n, with the labels $0,1,2$ each. For the remaining vertices, a spoke wv is labeled i if the label of the vertex v on the cycle is i. Now, suppose the vertices kept aside are v_0, v_1, v_2 such that label if $v_i = i, 0 \leq i \leq 2$, then assign the label 0 to all three. The hub get the label 0. The label numbers of the resulting labeling E are $v_E(0,1,2) = (x+1,x,x), e_E(0,1,2) = (2x+2,2x-1,2x-1)$. Clearly it is not a E_3-cordial labeling.

If $m = 2$, take an edge with the label 0 and attach a copy of W_{3x} with the labeling C at one end and a copy with the labeling E at the other end. The attachment is done at the hubs. The label numbers of the resulting labeling f are $v_f(0,1,2) = (2x,2x+1,2x+1), e_f(0,1,2) = (4x+1,4x,4x)$ Hence $K_2 \circ C_{3x}$ is E_3-cordial.

If $m = 3p$, we have a E_3-cordial labeling g (say) of K_{3p} with the label numbers $v(0,1,2) = (p,p,p), e_g(0,1,2) = \left(\frac{3p^2-p}{2}, \frac{3p^2-p}{2}, \frac{3p^2-p}{2}\right)$. We do not have to worry about the edges of K_{3p}. Now attach a copy of W_{3x} with the labeling D to each vertex with the label 0. The hub does not change the label. Attach a copy of W_{3x} with the labeling C to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x} with the labeling B to each vertex with the label 2. The hub will change the label to 1. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_C(0,1,2)+v_B(0,1,2)+v_D(0,1,2)\} = (p(3x+1), p(3x+1), p(3x+1))$. The additional edge-label numbers are $Ae_f(0,1,2) = p(e_C(0,1,2) + e_B(0,1,2) + e_D(0,1,2)) = (6px, 6px, 6px)$. Hence $K_{3p} \circ C_{3x}$ is E_3-cordial.

If $m = 3p+1$, we have a E_3-cordial labeling g (say) of K_{3p} with the label numbers $v(0,1,2) = (p+1,p,p), e_g(0,1,2) = \left(\frac{3p^2+p}{2}, \frac{3p^2+p}{2}, \frac{3p^2+p}{2}\right)$. We do not have to worry about the edges of K_{3p+1}. Now attach a copy of W_{3x} with the labeling D to all but one vertices with the label 0. The hub does not change the label. For the additional vertex with the label 0, attach a copy of W_{3x} with the labeling A. Attach a copy of W_{3x} with the labeling C to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x} with the labeling B to each vertex with the label 2. The hub will change the label to 1. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_C(0,1,2) + v_B(0,1,2) + v_D(0,1,2)\} + v_A(0,1,2) = (p(3x+1)+x+1, p(3x+1)+x, p(3x+1)+x)$. The additional edge-label numbers are $Ae_f(0,1,2) = p\{e_C(0,1,2) + e_B(0,1,2) + e_D(0,1,2)\} + e_A(0,1,2) = (6px+2x, 6px+2x, 6px+2x)$. Hence $K_{3p+1} \circ C_{3x}$ is E_3-cordial.

If $m = 3p+2$, we have a E_3-cordial labeling g (say) of K_{3p+2} with the label numbers $v(0,1,2) = (p+1,p,p+1)$,
$e_g(0,1,2) = \left(\frac{3p^2+3p}{2}, \frac{3p^2+3p}{2}+1, \frac{3p^2+3p}{2}\right)$. We do not have to worry about the edges of K_{3p+2}. Now attach a copy of W_{3x} with the labeling D to all but one vertices with the label 0. The hub does not change the label. For the additional vertex with the label 0, attach a copy of W_{3x} with the labeling A, Attach a copy of W_{3x} with the labeling C to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x} with the labeling B to all but one vertices with the label 2. The hub will change the label to 1. To the remaining vertex with the label 2 attach a copy of W_{3x} with the labeling A. The hub changes its label to 2. The resulting labeling f will have vertex-label numbers
$v_f(0,1,2) = p\{v_C(0,1,2)+v_B(0,1,2)+v_D(0,1,2)\}+v_A(0,1,2)+(x,x,x+1) = (p(3x+1)+2x+1, p(3x+1)+2x, p(3x+1)+2x)$. The additional edge-label numbers are $Ae_f(0,1,2) = p\{e_C(0,1,2) + e_B(0,1,2) + e_D(0,1,2)\} + 2e_A(0,1,2) = (6px+4x, 6px+4x, 6px+4x)$. Hence $K_{3p+2} \circ C_{3x}$ is E_3-cordial.

Case 2: $n = 3x + 1$.
We already have the E_3-cordial labeling A of W_{3x+1}. Besides that we construct two more labelings of W_{3x+1}.
Labeling B: This E_3-cordial labeling is given by the triple $(x, x, x+1)$. The hub has the label 1 and the label numbers are $v_B(0,1,2) = (x, x+1, x+1), e_B(0,1,2) = (2x, 2x+1, 2x+1)$.
Labeling C: This non-E_3-cordial labeling is given by the triple $(x+1, x, x)$. The hub has the label 0 and the label numbers are $v_C(0,1,2) = (x+2, x, x), e_C(0,1,2)$
$= (2x+2, 2x, 2x)$.

If $m = 2$, assign the labeling C to one copy of W_{3x+1}, the labeling A to the second copy of W_{3x+1} and the label 2 to the edge joining the hubs. This produces a labeling f in which the hubs change their labels to 2 and 1 and the label numbers are $v_f(0,1,2) = (2x+1, 2x+2, 2x+1), e_f(4x+2, 4x+1.4x+2)$, that is $K_2 \circ C_{3x+1}$ is E_3-cordial.

If $m = 3p$, we have a E_3-cordial labeling g (say) of K_{3p} with the label numbers $v(0,1,2) = (p,p,p), e_g(0,1,2) = \left(\frac{3p^2-p}{2}, \frac{3p^2-p}{2}, \frac{3p^2-p}{2}\right)$. We do not have to worry about the edges of K_{3p}. Now attach a copy of W_{3x+1} with the labeling C to each vertex with the label 0. The hub does not change the label. Attach a copy of W_{3x+1} with the labeling B to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x+1} with the labeling A to each vertex with the label 2. The hub will change the label to 1. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_A(0,1,2) + v_B(0,1,2) + v_C(0,1,2)\} = (p(3x+2), p(3x+2), p(3x+2))$. The additional edge-label numbers are $Ae_f(0,1,2) = p(e_A(0,1,2) + e_B(0,1,2) + e_C(0,1,2)) = (p(6x+2), p(6x+2), p(6x+2))$. Hence $K_{3p} \circ C_{3x+1}$ is E_3-cordial.

If $m = 3p+1$, we have a E_3-cordial labeling g (say) of K_{3p+1} with the label numbers $v(0,1,2) = (p+1, p, p), e_g(0,1,2) = \left(\frac{3p^2+p}{2}, \frac{3p^2+p}{2}, \frac{3p^2+p}{2}\right)$. We do not have to worry about the edges of K_{3p+1}. Now attach a copy of W_{3x+1} with the labeling C to all but one vertices with the label 0. The hub does not change the label. For the additional vertex with the label 0, attach a copy of W_{3x+1} with the labeling B. Attach a copy of W_{3x+1} with the labeling B to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x+1} with the labeling A to each vertex with the label 2. The hub will change the label to 1. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_C(0,1,2) + v_B(0,1,2) + v_A(0,1,2)\} + v_B(0,1,2) =$
$(p(3x+2) + x, p(3x+2) + x + 1, p(3x+2) + x + 1)$. The additional edge-label numbers are $Ae_f(0,1,2) = p\{e_C(0,1,2) + e_B(0,1,2) + e_D(0,1,2)\} + e_B(0,1,2) = (p(6x+2) + 2x, p(6x+2) + 2x + 1, p(6x+2) + 2x + 1)$. Hence $K_{3p+1} \circ C_{3x+1}$ is E_3-cordial.

If $m = 3p+2$, we have a E_3-cordial labeling g (say) of K_{3p+2} with the label numbers $v_g(0,1,2) = (p+1, p, p+1)$ and $e_g(0,1,2) = \left(\frac{3p^2+3p}{2}, \frac{3p^2+3p}{2}+1, \frac{3p^2+3p}{2}\right)$. We do not have to worry about the edges of K_{3p+2}. Now attach a copy of W_{3x+1} with the labeling C to all but one vertices with the label 0. The hub does not change the label. For the additional vertex with the label 0, attach a copy of W_{3x+1} with the labeling B. This vertex changes its label to 1. Attach a copy of W_{3x+1} with the labeling B to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x+1} with the labeling A to all but one vertices with the label 2. The hub will change the label to 1. To the remaining vertex with the label 2 attach a copy of W_{3x+1} with the labeling C. The hub will change its label to 2. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_C(0,1,2)+v_B(0,1,2)+v_A(0,1,2)\}+v_B(0,1,2)+v_C(0,1,2) = (p(3x+2)+2x+1, p(3x+2)+2x+1, p(3x+2)+2x+2)$. The additional edge-label numbers are $Ae_f(0,1,2) = p\{e_C(0,1,2)+e_B(0,1,2)+e_A(0,1,2)\}+e_B(0,1,2)+e_C(0,1,2) = (6px+4x+2p+2, 6px+4x+2p+1, 6px+4x+2p+1)$. In the labeling of K_{3p+2} we had one additional edge with the label 1. But that balances with the additional edge labels. Hence $K_{3p+2} \circ C_{3x+1}$ is E_3-cordial.

Case 3: $n = 3x+2$.

We already have the E_3-cordial labeling A of W_{3x+2}. Besides that we construct two more labelings of W_{3x+2}.

Labeling B: This non-E_3-cordial labeling is given by the triple $(x, x+1, x+1)$. The hub has the label 0 and the label numbers are $v_B(0,1,2) = (x+1, x+1, x+1), e_B(0,1,2) = (2x, 2x+2, 2x+2)$.

Labeling C: This E_3-cordial labeling is given by the triple $(x+1, x+1, x)$. The hub has the label 2 and the label numbers are $v_C(0,1,2) = (x+1, x+1, x+1), e_C(0,1,2) = (2x+2, 2x+1, 2x+1)$.

If $m = 2$, assign the labeling B to one copy of W_{3x+2}, the labeling A to the second copy of W_{3x+2} and the label 0 to the edge joining the hubs. This produces a labeling f in which the hubs do not change their labels. The label numbers of the resulting labeling are $v_f(0,1,2) = (2x+2, 2x+2, 2x+2), e_f(4x+3, 4x+3, 4x+3)$, that is, $K_2 \circ C_{3x+1}$ is E_3-cordial.

If $m = 3p$, we have a E_3-cordial labeling g (say) of K_{3p} with the label numbers $v(0,1,2) = (p,p,p), e_g(0,1,2) = \left(\frac{3p^2-p}{2}, \frac{3p^2-p}{2}, \frac{3p^2-p}{2}\right)$. We do not have to worry about the edges of K_{3p}. Now attach a copy of W_{3x+2} with the labeling B to each vertex with the label 0. The hub does not change the label. Attach a copy of W_{3x+2} with the labeling A to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x+2} with the labeling C to each vertex with the label 2. The hub will change the label to 1. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_A(0,1,2)+v_B(0,1,2)+v_C(0,1,2)\} = (p(3x+3), p(3x+3), p(3x+3))$. The additional edge-label numbers are $Ae_f(0,1,2) = p(e_A(0,1,2) + e_B(0,1,2) + e_C(0,1,2)) = (p(6x+4), p(6x+4), p(6x+4))$. Hence $K_{3p} \circ C_{3x+1}$ is E_3-cordial.

If $m = 3p+1$, we have a E_3-cordial labeling g (say) of K_{3p+1} with the label numbers $v(0,1,2) = (p+1, p, p), e_g(0,1,2) = \left(\frac{3p^2+p}{2}, \frac{3p^2+p}{2}, \frac{3p^2+p}{2}\right)$. Again we do not have to worry about these edges. Now attach a copy of W_{3x+2} with the labeling B to all but one vertices with the label 0. The hub does not change the label. For the additional vertex with the label 0, attach a copy of W_{3x+2} with the labeling A. Attach a copy of W_{3x+2} with the labeling A to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x+2} with the labeling C to each vertex with the label 2. The hub will change the label to 1. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_A(0,1,2) + v_B(0,1,2) + v_C(0,1,2)\} + v_A(0,1,2) = (p(3x+3)+x+1, p(3x+3)+x+1, p(3x+3)+x+1)$. The additional edge-label numbers are $Ae_f(0,1,2) = p\{e_A(0,1,2)+e_B(0,1,2)+e_C(0,1,2)\}+e_A(0,1,2) = (p(6x+4)+2x+2, p(6x+4)+2x+1, p(6x+4)+2x+1)$. Hence $K_{3p+1} \circ C_{3x+2}$ is E_3-cordial.

If $m = 3p+2$, we have a E_3-cordial labeling g (say) of K_{3p+2} with the label numbers $v(0,1,2) = (p+1, p, p+1)$ and $e_g(0,1,2) = \left(\frac{3p^2+3p}{2}, \frac{3p^2+3p}{2}+1, \frac{3p^2+3p}{2}\right)$. We do not have to worry about the edges of K_{3p+2}. For this case we define two more labelings as follows:

Labeling D: Take $(a,b,c) = (x, x+1, x+1)$. The spokes are labeled differently. Leave aside one vertex of the label 1, 2 each. Now for the other spokes, assign the label i to a spoke wv if the label of the vertex v on the cycle is i. Assign the label 2 to the spoke joining w to the extra vertex with the label 1 and the label 0 to the spoke joining w to the extra vertex with the label 2. The hub gets the label 2. The label numbers are $v_D(0,1,2) = (x+1, x, x+2), e_D(0,1,2) = (2x+1, 2x+1, 2x+2)$. Clearly this is not a E_3-cordial labeling.

Labeling E: Take $(a,b,c) = (x, x+1, x+1)$. The spokes are labeled differently. Leave aside one vertex of the label 1, 2 each. Now for the other spokes, assign the label i to a spoke wv if the label of the vertex v on the cycle is i. Assign the label 0 to the spoke joining w to the extra vertices with the label 1 and the label 0. The hub gets the label 0. The label numbers are $v_E(0,1,2) = (x+1, x+1, x+1), e_E(0,1,2) = (2x+2, 2x+1, 2x+1)$. Clearly this is a E_3-cordial labeling.

Now attach a copy of W_{3x+2} with the labeling B to all but one vertices with the label 0. The hub does not change the label. For the additional vertex with the label 0, attach a copy of W_{3x+2} with the labeling E. This vertex retains its label to 0. Attach a copy of W_{3x+2} with the labeling A to each vertex with the label 1. The hub will change its label to 2. Finally, attach a copy of W_{3x+1} with the labeling C to all but one vertices with the label 2. The hub will change the label to 1. To the remaining vertex with the labeling 2 attach a copy of W_{3x+2} with the labeling D. The hub will change its label to 1.. The resulting labeling f will have vertex-label numbers $v_f(0,1,2) = p\{v_C(0,1,2)+v_B(0,1,2)+v_A(0,1,2)\}+v_E(0,1,2)+v_D(0,1,2) = (p(3x+3)+2x+2, p(3x+3)+2x+2, p(3x+3)+2x+2)$. The additional edge-label numbers are $Ae_f(0,1,2) = p(e_C(0,1,2)+e_B(0,1,2)+e_A(0,1,2))+e_D(0,1,2)+e_E(0,1,2) = (p(6x+4)+4x+3, p(6x+4)+4x+2, p(6x+4)+4x+3)$. Hence $K_{3p+2} \circ C_{3x+2}$ is E_3-cordial. •

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