The metamorphosis of λ-fold kite systems into maximum packings of λK_n with triangles*

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1 Introduction

A *kite* is a triangle with a tail consisting of a single edge. A *λ-fold kite system* (*λ-fold triple system*) of order n is a pair (X, B), where B is a collection of kites (triangles) which partitions the edge set of λK_n (λ copies of the complete undirected graph on n vertices) with vertex set X. When $\lambda = 1$ we will simply say kite system (triple system). If we drop the quantification "partitions" we have the definition of λ-fold *partial* kite system (λ-fold *partial* triple system).

A *packing* of λK_n with triangles is a triple (X, T, L), where T is a collection of edge disjoint triangles and L is the set of edges not belonging to a triangle in T. The collection of edges L is called the *leave*. If $|T|$ is as large as possible, the packing (X, T, L) is said to be *maximum*. (So, for example, a λ-fold triple system of order n is a maximum packing of λK_n with leave the empty set.)

In what follows we will denote the m-cycle with edges $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\dots$, $\{x_{m-1}, x_m\}$, $\{x_m, x_1\}$ by any cyclic shift of $(x_1, x_2, x_3, \dots, x_m)$ or $(x_2, x_1, x_m, x_{m-1}, \dots, x_3)$ (in particular, the triangle with edges $\{a, b\}$,

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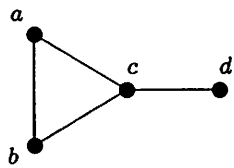
$\{b, c\}$, $\{c, a\}$ will be denoted by any cyclic shift of (a, b, c) or (b, a, c)); and the kite



by (a, b, c)-d or (b, a, c)-d. Now let (X, K) be a λ-fold kite system of order n and let $E(K) = \{\{c, d\} \mid (a, b, c)\text{-}d \in K\}$ and $T_1(K) = \{(a, b, c) \mid (a, b, c)\text{-}d \in K\}$. Then $T_1(K)$ is a λ-fold partial triple system. If the edges belonging to $E(K)$ can be arranged into a collection of triangles $T_2(K)$ with leave L, then $(X, T_1(K) \cup T_2(K), L)$ is a packing of λK_n with triangles, and is said to be a metamorphosis of (X, K). (The algorithm of going from (X, K) to $(X, T_1(K) \cup T_2(K), L)$ is also called a *metamorphosis*.) There are some fairly extensive results on similar metamorphosis problems, and the interested reader is referred to [2, 3, 5, 6, 7] for further reading. The purpose of this paper is the complete solution of the problem of constructing for each λ and for each admissible value of n a λ-fold kite system of order n having a metamorphosis into a maximum packing of λK_n with triangles with all possible leaves. It is sufficient to give a solution of this problem for $\lambda = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, since these results can be pasted together to obtain a complete solution for all other values of λ.

2 Kite systems

It is well known that the spectrum for kite systems is the set of all $n \equiv 0, 1 \pmod 8$ [3] and a maximum packing of K_n with triangles has leave [4]:

(i) a 1-factor if $n \equiv 0, 2 \pmod 6$;

(ii) a 4-cycle if $n \equiv 5 \pmod 6$;

(iii) a *tripole* (the graph consisting of $(n-4)/2$ disjoint edges and a 3-star (see Figure 1) if $n \equiv 4 \pmod 6$; and

(iv) the empty set if $n \equiv 1, 3 \pmod 6$.

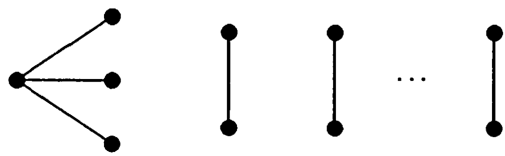


Figure 1: *tripole.*

In this section we will show that for every admissible value of n there exists a kite system of order n having a metamorphosis into a maximum packing of K_n with triangles. To begin with, we will give examples for $n = 8$, $n = 9$, $n = 16$, and $n = 17$ followed by a recursive construction for the remaining cases.

Example 2.1. **(n=8)**. Let (X, K) be the kite system with $K = \{(4,6,1)\text{-}2, (4,7,2)\text{-}3, (4,5,3)\text{-}1, (6,7,5)\text{-}1, (3,8,6)\text{-}2, (1,8,7)\text{-}3, (2,5,8)\text{-}4\}$. Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of K_8 with triangles, where $T_2(K) = \{(1,2,3)\}$ and $L = \{\{1,5\},\{2,6\},\{3,7\},\{4,8\}\}$.

Example 2.2. **(n=9)**. Let (X, K) be the kite system with $K = \{(1,2,3)\text{-}5, (4,6,5)\text{-}7, (8,9,7)\text{-}3, (4,7,1)\text{-}8, (2,5,8)\text{-}6, (3,9,6)\text{-}1, (1,5,9)\text{-}2, (6,7,2)\text{-}4, (3,8,4)\text{-}9\}$. Then $(X, T_1(K) \cup T_2(K))$ is a Steiner triple system of order 9, where $T_2(K) = \{(3,5,7),(1,6,8),(2,4,9)\}$.

Example 2.3. **(n=16)**. Let (X, K) be the kite system where $X = (\{a,b,c\} \times Z_5) \cup \{\infty\}$ and $K = \{(b_i, c_{3+i}, \infty)\text{-}a_i, (b_{4+i}, c_{3+i}, a_i)\text{-}a_{1+i}, (a_i, b_{1+i}, b_{3+i})\text{-}c_i, (c_{1+i}, c_i, a_i)\text{-}a_{2+i}, (c_{1+i}, c_{4+i}, a_{2+i})\text{-}b_{2+i}, (c_{2+i}, b_{1+i}, b_{2+i})\text{-}a_i \mid i \in Z_5\}$. From here on we will use subscript notation x_i to denote the ordered pair (x, i).) Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of K_{16} with triangles, where $T_2(K) = \{(\infty, a_0, a_1), (\infty, a_2, a_3)\} \cup \{(a_i, a_{2+i}, b_{2+i}) \mid i \in Z_5\}$ and $L = \{\{a_4,\infty\}, \{a_4,a_3\}, \{a_4,a_0\}, \{a_1,a_2\}\} \cup \{\{c_i, b_{3+i}\} \mid i \in Z_5\}$.

Example 2.4. **(n=17)**. Let (X, K) be the kite system where $X = \{x_1, x_2, x_3, x_4, x_5\} \cup Z_{12}$ and $K = \{(8,11,9)\text{-}10, (7,8,10)\text{-}0, (2,11,0)\text{-}9, (3,0,1)\text{-}10, (x_1,5,1)\text{-}2, (x_1,9,2)\text{-}4, (x_1,3,7)\text{-}6, (x_1,11,4)\text{-}1, (6,10,x_1)\text{-}x_2, (8,0,x_1)\text{-}x_3, (x_2,1,6)\text{-}9, (x_2,10,2)\text{-}3, (x_2,8,3)\text{-}5, (x_2,0,5)\text{-}2, (4,9,x_2)\text{-}x_3, (7,11,x_2)\text{-}x_4, (x_3,8,1)\text{-}11, (x_3,10,3)\text{-}4, (x_3,0,4)\text{-}6, (x_3,11,6)\text{-}3, (2,7,x_3)\text{-}x_4, (5,9,x_3)\text{-}x_5, (x_4,3,11)\text{-}10, (x_4,8,4)\text{-}5, (x_4,10,5)\text{-}7, (x_4,0,7)\text{-}4, (1,9,x_4)\text{-}x_5, (2,6,x_4)\text{-}x_1, (x_5,1,7)\text{-}9, (x_5,2,8)\text{-}5, (x_5,11,5)\text{-}6, (x_5,0,6)\text{-}8, (3,9,x_5)\text{-}x_1, (4,10,x_5)\text{-}x_2\}$. Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of K_{17} with triangles, where $T_2(K) = \{(x_1,x_3,x_5),(x_2,x_4,x_5),(0,9,10),(1,10,11),(1,2,4),(2,3,5),(3,4,6),(4,5,7),(5,6,8),(6,7,9)\}$ and $L = \{(x_1,x_2,x_3,x_4)\}$.

In order to handle the remaining cases we need the following example.

Example 2.5. **(A $K_{4,4,4}$ kite system having a metamorphosis into a packing of $K_{4,4,4}$ with triangles with leave the empty set).** Let $K_{4,4,4}$ have parts $\{x\} \times Z_4$, $\{y\} \times Z_4$, and $\{z\} \times Z_4$ and define a collection of kites $K = \{(x_2, y_1, z_0)\text{-}x_1, (y_2, z_1, x_0)\text{-}y_1, (x_1, z_2, y_0)\text{-}z_1, (y_0, z_0, x_3)\text{-}y_3, (x_3, y_1, z_1)\text{-}x_2, (x_3, y_2, z_2)\text{-}x_0, (x_0, z_0, y_3)\text{-}z_3, (y_3, z_1, x_1)\text{-}y_2, (y_3, z_2, x_2)\text{-}y_0, (x_0, y_0, z_3)\text{-}x_3, (x_1, z_3, y_1)\text{-}z_2, (x_2, z_3, y_2)\text{-}z_0\}$. Then $T_1(K) \cup T_2(K)$, where $T_2(K) = \{(x_0, y_1, z_2), (x_2, y_0, z_1), (x_1, y_2, z_0), (x_3, y_3, z_3)\}$, partitions the edge set of $K_{4,4,4}$ into triangles.

The main ingredient we will need in our recursive construction is a $\{3\}$-GDD of type 2^k or $2^{k-2}4^1$. It is well known (see [1]) that there exists a $\{3\}$-GDD of type 2^k for every $k \equiv 0, 1 \pmod 3$ and a $\{3\}$-GDD of type $2^{k-2}4^1$ for every $k \equiv 2 \pmod 3 \geq 5$.

The $8k + r$ Construction. Write $8k + r = 4(2k) + r$, where $2k \geq 6$ and $r \in \{0, 1\}$. Let S be a set of size $2k$, R be a set of size r, and $(S, \mathcal{G}, T)$ a $\{3\}$-GDD of type 2^k (for $k \equiv 0, 1 \pmod 3$) or $2^{k-2}4^1$ (for $k \equiv 2 \pmod 3$). Set $X = R \cup (S \times Z_4)$ and define a collection K of kites as follows.

(1) For every group $g \in \mathcal{G}$, let $(R \cup (g \times Z_4), K_g)$ be a kite system of order $4|g| + r$ having a metamorphosis into a maximum packing with triangles with leave L_g (see Examples 2.1, 2.2, 2.3, and 2.4); put $K_g \subseteq K$.

(2) For every $t = (x, y, z) \in T$, let $(K_{4,4,4}, K_t)$ be a kite system with parts $\{x\} \times Z_4$, $\{y\} \times Z_4$, $\{z\} \times Z_4$, having a metamorphosis into a packing of $K_{4,4,4}$ with triangles with leave the empty set (see Example 2.5); put $K_t \subseteq K$.

Then (X, K) is a kite system of order $8k + r$. The metamorphosis is the following: for each group in $\mathcal{G}$ use the metamorphosis in (1) and for each triple in T use the metamorphosis in (2). The leave is $L = \cup_{g \in \mathcal{G}} L_g$; we have the following four cases.

(a) *The case of* $r = 0$, $k \equiv 0, 1 \pmod 3$ (and so $8k + r \equiv 0, 2 \pmod 6$). Each L_g is a 1-factor of K_8 with vertex set $g \times Z_4$ and so L is a 1-factor of K_{8k+r}.

(b) *The case of* $r = 0$, $k \equiv 2 \pmod 3$ (and so $8k + r \equiv 4 \pmod 6$). Each L_g is a 1-factor of K_8 with vertex set $g \times Z_4$ with one exception, a tripole, corresponding to the group of size 4 and so L is a tripole.

(c) *The case of* $r = 1$, $k \equiv 0, 1 \pmod 3$ (and so $8k + r \equiv 1, 3 \pmod 6$). $L_g = \emptyset$ for each $g \in \mathcal{G}$ and so $L = \emptyset$.

(d) *The case of* $r = 1$, $k \equiv 2 \pmod 3$ (and so $8k + r \equiv 5 \pmod 6$). $L_g = \emptyset$ for each $g \in \mathcal{G}$ with one exception, a 4-cycle, corresponding to the group of size 4 and so L is a 4-cycle.

Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of K_{8k+r} with triangles.

Lemma 2.1. *There exists a kite system of order n having a metamorphosis into a maximum packing of K_n with triangles for all $n \equiv 0, 1 \pmod 8$.* □

3 2-fold kite systems

It is well known that the spectrum for 2-fold kite systems is the set of all $n \equiv 0, 1 \pmod 4$ [3] and a maximum packing of $2K_n$ with triangles has leave [1]:

(i) the empty set if $n \equiv 0, 1, 3, 4 \pmod 6$; and

(ii) a double edge if $n \equiv 2, 5 \pmod 6$.

In this section we will show that for every admissible value of n there exists a 2-fold kite system of order n having a metamorphosis into a maximum packing of $2K_n$ with triangles. To begin with, we will give examples for $n = 4$, $n = 5$, $n = 8$, and $n = 9$ followed by a recursive construction for the remaining cases.

Example 3.1. (n=4). Let (X, K) be the 2-fold kite system where $K = \{(3,4,1)\text{-}2,\ (1,4,2)\text{-}3,\ (2,4,3)\text{-}1\}$. Then $(X,\ T_1(K) \cup T_2(K))$ is a 2-fold triple system, where $T_2(K) = \{(1,2,3)\}$.

Example 3.2. (n=5). Let (X, K) be the 2-fold kite system where $K = \{(3,5,1)\text{-}2,\ (4,5,2)\text{-}3,\ (2,4,3)\text{-}1,\ (2,5,1)\text{-}4,\ (3,5,4)\text{-}1\}$. Then $(X,\ T_1(K) \cup T_2(K), L)$ is a maximum packing of $2K_5$, where $T_2(K) = \{(1,2,3)\}$ and $L = \{\{1,4\}, \{1,4\}\}$.

Example 3.3. (n=8). Let (X, K_1) be a kite system of order 8 having a metamorphosis into a maximum packing of K_8 with triangles with leave $L = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}\}$ (see Example 2.1), and (X, K_2) the kite system where $K_1 = \{(5,8,1)\text{-}2, (5,6,3)\text{-}1,\ (2,3,4)\text{-}1,\ (7,8,2)\text{-}5,\ (1,7,6)\text{-}2,\ (4,5,7)\text{-}3,\ (4,6,8)\text{-}3\}$. Then $(X, K_1 \cup K_2)$ is a 2-fold kite system of order 8. The metamorphosis is the following: use the metamorphosis of (X, K_1) and delete the tail from each of the kites of K_2; reassemble these edges including the edges of L into the triangles $\{(1,3,4),\ (2,5,6),\ (3,7,8)\}$ with leave $\{\{1,2\}, \{1,2\}\}$.

Example 3.4. **(n=9).** Take a kite system of order 9 having a metamorphosis into a Steiner triple system (see Example 2.2) and double each kite. The result is a 2-fold kite system of order 9 having a metamorphosis into a 2-fold triple system.

In order to handle the remaining cases we need the following example.

Example 3.5. **(A $K_{2,2,2}$ kite system having a metamorphosis into a packing of $K_{2,2,2}$ with triangles with leave the empty set).** Let $(K_{2,2,2}, K)$ be the kite system where $K = \{(x_1, y_1, z_0)\text{-}y_0,\ (y_1, z_1, x_0)\text{-}z_0,\ (x_1, z_1, y_0)\text{-}x_0\}$. Then $T_1(K) \cup T_2(K)$, where $T_2(K) = \{(x_0, y_0, z_0)\}$, partitions the edge set of $K_{2,2,2}$ into triangles with parts $\{x\} \times Z_2$, $\{y\} \times Z_2$, $\{z\} \times Z_2$.

The 4k+r Construction. Write $4k + r = 2(2k) + r$, where $2k \geq 6$ and $r \in \{0, 1\}$. Let $S = \{1, 2, \ldots, 2k\}$, R be a set of size r, $(S, \mathcal{G}, T)$ be a $\{3\}$-GDD of the type 2^k (for $k \equiv 0, 1 \pmod 3$) or $2^{k-2}4^1$ (for $k \equiv 2 \pmod 3$), with groups $g_1 = \{1, 2\}$, $g_2 = \{3, 4\}$, $\ldots$, $g_k = \{2k-1, 2k\}$ or $g_1 = \{1, 2, 3, 4\}$, $g_2 = \{5, 6\}$, $\ldots$, $g_{k-1} = \{2k-1, 2k\}$, respectively. Set $X = R \cup (S \times Z_2)$ and define a collection K of kites as follows.

(1) For every group $g \in \mathcal{G}$, let $(R \cup (g \times Z_2), K_g)$ be a 2-fold kite system of order $2|g| + r$ having a metamorphosis into a maximum packing with triangles with leave L_g (see Examples 3.1, 3.2, 3.3, and 3.4); put $K_g \subseteq K$.

(2) For every $t = (x, y, z) \in T$, let $(K_{2,2,2}, K_t)$ be the kite system in Example 3.5; put $2K_t \subseteq K$.

Then (X, K) is a 2-fold kite system of order $4k + r$. The metamorphosis is the following: for each group in $\mathcal{G}$ use the metamorphosis in (1) and for each triple in T use the metamorphosis in (2); combine the leaves in (1) and (2) to obtain the leave L as the case may be.

(a) *The case of* $r = 0$, $k \equiv 0, 1 \pmod 3$ (and so $4k + r \equiv 0, 4 \pmod 6$). $L_g = \emptyset$ for each $g \in \mathcal{G}$ and so the leave $L = \cup_{g \in \mathcal{G}} L_g$ is the empty set.

(b) *The case of* $r = 0$, $k \equiv 2 \pmod 3$ (and so $4k + r \equiv 2 \pmod 6$). Each L_g is the empty set with one exception, a double edge, corresponding to the group of size 4 and so the leave $L = \cup_{g \in \mathcal{G}} L_g$ is a double edge.

(c) *The case of* $r = 1$, $k \equiv 0, 1 \pmod 3$ (and so $4k + r \equiv 1, 5 \pmod 6$). Use the metamorphosis in (1) with leaves $\{\{1_0, 2_0\}, \{1_0, 2_0\}\}$, $\{\{3_0, 4_0\}, \{3_0, 4_0\}\}$, $\ldots$, $\{\{(2k-1)_0, (2k)_0\}, \{(2k-1)_0, (2k)_0\}\}$. These leaves plus all edges of the triples (x_0, y_0, z_0) is a copy of $2K_{2k}$. Replace these deleted edges with a maximum packing of $2K_{2k}$ with a leave L. The result is a maximum packing of $2K_{4k+1}$ with triangles with leave L.

(d) *The case of* $r = 1$, $k \equiv 2 \pmod 3$ (and so $4k + r \equiv 3 \pmod 6$). Use the metamorphosis in (1) with leaves $\emptyset$, $\{\{5_0, 6_0\}, \{5_0, 6_0\}\}$, $\{\{7_0, 8_0\}, \{7_0, 8_0\}\}$, ... ,$\{\{(2k-1)_0, (2k)_0\}, \{(2k-1)_0, (2k)_0\}\}$. These leaves plus all edges of the triples (x_0, y_0, z_0) is a copy of $2K_{2k}$ with a hole of size 4, $\{1_0, 2_0, 3_0, 4_0\}$. Replace these deleted edges with a 2-fold triple system of order $2k$ with a hole of size 4 (see [1]). The result is a maximum packing of $2K_{4k+1}$ with triangles with leave $L = \emptyset$

Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of $2K_{4k+r}$ with triangles.

Lemma 3.1. *There exists a 2-fold kite system of order n having a metamorphosis into a maximum packing of $2K_n$ with triangles for all $n \equiv 0, 1$ (mod 4).* □

4 3-fold kite systems

It is well known that the spectrum for 3-fold kite systems is the set of all $n \equiv 0, 1 \pmod 8$ [3] and a maximum packing of $3K_n$ with triangles has leave [1]:

(i) a 1-factor if $n \equiv 0 \pmod 6$;

(ii) the empty set if $n \equiv 1, 3, 5 \pmod 6$;

(iii) a graph on n vertices with $(n+4)/2$ edges and odd vertex degrees (see Figure 2) if $n \equiv 2 \pmod 6$; and

(iv) a tripole if $n \equiv 4 \pmod 6$.

In this section we will show that for every admissible value of n there exists a 3-fold kite system of order n having a metamorphosis into a maximum packing of $3K_n$ with triangles with all possible leaves. To begin with, we will settle the case $n \equiv 0 \pmod 8 \equiv 2 \pmod 6$.

Lemma 4.1. *For every $n \equiv 0 \pmod 8 \equiv 2 \pmod 6$ there exists a 3-fold kite system of order n having a metamorphosis into a maximum packing of $3K_n$ with triangles with all possible leaves.*

Proof. Take three kite systems of order n (X, K_1), (X, K_2), and (X, K_3), with $X = \{1, 2, \dots, n\}$, having a metamorphosis into a maximum packing of K_n with triangles with leaves the 1-factors L_1, L_2, and L_3, respectively (see Section 2). Then $(X, K_1 \cup K_2 \cup K_3)$ is a 3-fold kite system of order

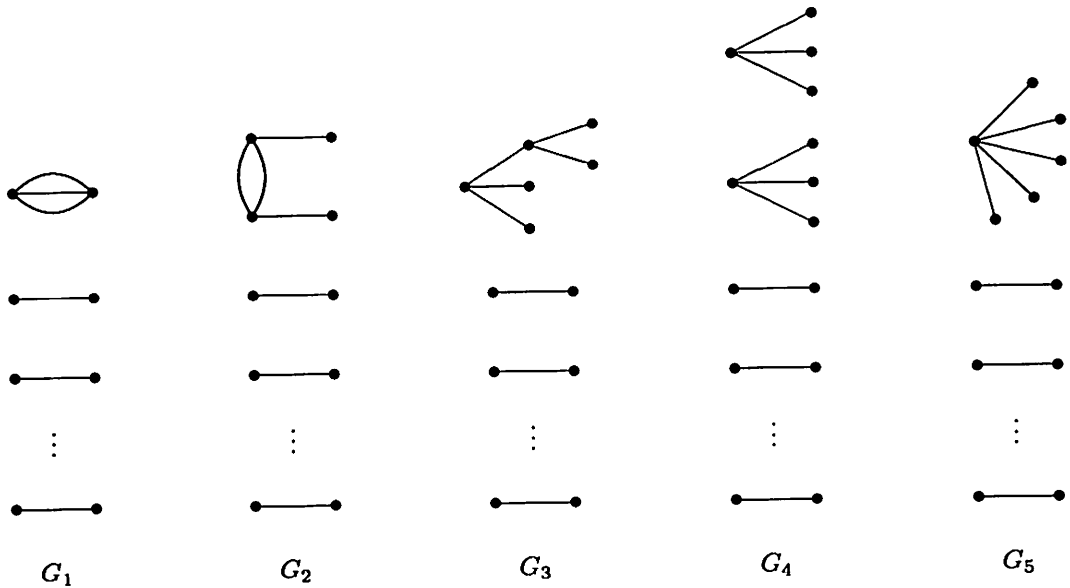


Figure 2: All possible graphs on n vertices with $(n+4)/2$ edges and odd vertex degrees

n. The metamorphosis is the following: use the metamorphoses of (X, K_1), (X, K_2), and (X, K_3) with $L_1 \supseteq F_1 = \{\{1,3\},\{4,6\},\{7,9\},\dots,\{n-4,n-2\}\}$, $L_2 \supseteq F_2 = \{\{1,2\},\{4,5\},\{7,8\},\dots,\{n-4,n-3\}\}$, and $L_3 \supseteq F_3 = \{\{3,2\},\{6,5\},\{9,8\},\dots,\{n-2,n-3\}\}$ and combine the edges of F_1, F_2, and F_3 to obtain $(n-2)/3$ triangles, $(1,2,3)$, $(4,5,6)$, and $(n-4,n-3,n-2)$. Rearrange the remaining edges of L_1, L_2, and L_3 as the case may be:
(1) Let $\{n,n-1\} \in L_1$, $\{n,n-1\} \in L_2$, and $\{n,n-1\} \in L_3$. The result is a maximum packing of $3K_n$ with triangles with leave a 3-times repeated edge and $(n-2)/2$ disjoint edges.
(2) Let $\{\{2,n-1\},\{5,n\}\} \subseteq L_1$, $\{\{3,n-1\},\{6,n\}\} \subseteq L_2$, and $\{\{1,n-1\},\{4,n\}\} \subseteq L_3$; combine these edges into the two 3-stars, $\{\{n-1,1\},\{n-1,2\},\{n-1,3\}\}$ and $\{\{n,4\},\{n,5\},\{n,6\}\}$. The result is a maximum packing of $3K_n$ with triangles with leave two 3-stars and $(n-8)/2$ disjoint edges.
(3) Without loss of generality suppose $(4,5,n-1) \in T_2(K_1)$. Let $\{\{2,n-1\},\{5,n\}\} \subseteq L_1$, $\{3,n-1\} \in L_2$, and $\{\{1,n-1\},\{4,n\}\} \subseteq L_3$; rearrange these edges along with the edges of $(4,5,n-1)$ into the triangle $(4,5,n)$ and the 5-star $\{\{n-1,1\},\{n-1,2\},\{n-1,3\},\{n-1,4\},\{n-1,5\}\}$. The result is a maximum packing of $3K_n$ with triangles with leave one 5-star and $(n-6)/2$ disjoint edges.
(4) Let $\{\{2,n-1\},\{5,n\}\} \subseteq L_1$, $\{\{3,n-1\},\{6,n\}\} \subseteq L_2$, and $\{n,n-1\} \in L_3$. The result is a maximum packing of $3K_n$ with triangles with leave $\{n-1,2\}$, $\{n-1,3\}$, $\{n-1,n\}$, $\{n,5\}$, $\{n,6\}$ and $(n-6)/2$ disjoint edges.
(5) Let $\{n,n-1\} \in L_1$, $\{n,n-1\} \in L_2$, and $\{\{1,n-1\},\{4,n\}\} \subseteq L_3$. The result is a maximum packing of $3K_n$ with triangles with leave $\{n-1,n\}$, $\{n-1,n\}$, $\{4,n\}$, $\{1,n-1\}$ and $(n-4)/2$ disjoint edges. □

Lemma 4.2. *For every $n \equiv 0,1 \pmod 8$ there exists a 3-fold kite system of order n having a metamorphosis into a maximum packing of $3K_n$ with triangles with all possible leaves.*

Proof. If $n \equiv 0 \pmod 8 \equiv 2 \pmod 6$ the statement follows from Lemma 4.1. Let $n \equiv 0,1 \pmod 8$, $n \not\equiv 2 \pmod 6$.
Let (1) (X, K_1) be a kite system of order n having a metamorphosis into a maximum packing of K_n with triangles with leave L_1 and (2) (X, K_2) a 2-fold kite system of order n having a metamorphosis into a maximum packing of $2K_n$ with triangles with leave L_2. Then $(X, K_1 \cup K_2)$ is 3-fold kite system of order n. The metamorphosis is the following: use the metamorphosis in (1) and (2) and combine the leaves L_1 and L_2 to obtain the leave L as the case may be.

(a) *The case of $n \equiv 0 \pmod 6$.* L_1 is a 1-factor and $L_2 = \emptyset$. The leave is $L = L_1 \cup L_2$, a 1-factor.

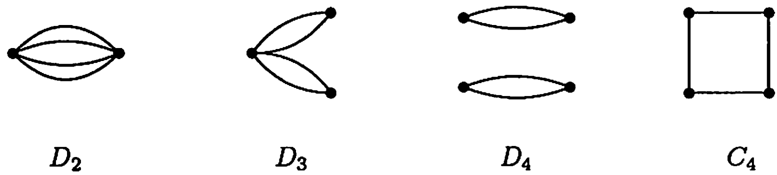


Figure 3: all possible graphs with 4 edges and even vertex degrees.

(b) *The case of* $n \equiv 1, 3 \pmod 6$. $L_1 = L_2 = \emptyset$ and so the leave $L = L_1 \cup L_2$ is the empty set.

(d) *The case of* $n \equiv 4 \pmod 6$. L_1 is a tripole and $L_2 = \emptyset$. The leave is $L = L_1 \cup L_2$, a tripole.

(e) *The case of* $n \equiv 5 \pmod 6$. Use the metamorphosis in (1) with leave the 4-cycle $L_1 = (a, b, c, d)$ and use the metamorphosis in (2) with leave $L_2 = \{\{a, c\}, \{a, c\}\}$. Reassemble the edges belonging to L_1 and L_2 into the 2 triangles (a, b, c) and (a, c, d). The result is a maximum packing of $3K_n$ with leave $L = \emptyset$.

Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of $3K_n$ with triangles. □

5 4-fold kite systems

It is well known that the spectrum for 4-fold kite systems is the set of all integers $n \geq 4$ [3] and a maximum packing of $4K_n$ with triangles has leave [1]:

(i) the empty set if $n \equiv 0, 1, 3, 4 \pmod 6$; and

(ii) a graph with 4 edges and even vertex degrees (see Figure 3) if $n \equiv 2, 5 \pmod 6$.

In this section we will show that for every $n \geq 4$ there exists a 4-fold kite system of order n having a metamorphosis into a maximum packing of $4K_n$ with triangles with all possible leaves.
To begin with, we will handle the case $n \equiv 0, 1 \pmod 4$.

Example 5.1. (n=5, with leave a 4-cycle). Let (Z_5, K) be the kite system where $K = \{(2 + i, 3 + i, i)\text{-}(1 + i),\ (4 + i, 3 + i, i)\text{-}(2 + i) \mid i \in Z_5\}$. $E(K)$ is a copy of K_5. Replace these deleted edges with a maximum packing of K_5 with triangles with leave a 4-cycle (see Section 2). The result is a maximum packing of $4K_5$ with triangles with leave a 4-cycle.

Example 5.2. (n=8, with leave a 4-cycle). Let (X, K) be the 2-fold kite system of order 8 of Example 3.3 and (X, K') be a 2-fold kite system of order 8 having a metamorphosis into a maximum packing with triangles with leave $\{\{2,3\},\{2,3\}\}$. Then $(X, K \cup K')$ is a 4-fold kite system of order 8. The metamorphosis is the following: use the metamorphoses of (X, K) and (X, K'). Since the triangle $(1,3,4) \in T_2(K)$, its edges can be used in a rearrangement and so combine them with the leaves $\{\{1,2\},\{1,2\}\}$ and $\{\{2,3\},\{2,3\}\}$ and reassemble these deleted edges into the triangle $(1,2,3)$ and the 4-cycle $(1,2,3,4)$.

Lemma 5.1. *There exists a 4-fold kite system of every order* $n \equiv 0, 1 \pmod 4$ *having a metamorphosis into a maximum packing of* $4K_n$ *with triangles with all possible leaves.* □

Proof. As was pointed out in Section 3, the spectrum for 2-fold kite systems having metamorphoses into maximum packings of $2K_n$ with triangles is the set of all $n \equiv 0, 1 \pmod 4$. So let (X, K) and (X, K') be two such systems of order n. Then $(X, K \cup K')$ is a 4-fold kite system of order n. Clearly, if $(X, T_1(K) \cup T_2(K), L)$ and $(X, T_1(K') \cup T_2(K'), L')$ are metamorphoses of (X, K) and (X, K'), respectively, into maximum packings of $2K_n$ with triangles, then $(X, T_1(K \cup K') \cup T_2(K \cup K'), L \cup L')$, is a metamorphosis of $(X, K \cup K')$ into a maximum packing of $4K_n$ with triangles, where $T_2(K \cup K') = T_2(K) \cup T_2(K')$. If $n \equiv 0, 1, 3, 4 \pmod 6$, $L = L' = \emptyset$ and so $L \cup L' = \emptyset$. If $n \equiv 2, 5 \pmod 6$, L and L' are isomorphic to $2K_2$ (a double edge) and there are three possibilities: $L = L'$; L and L' have one point in common; L and L' have no point in common. It follows that $L \cup L'$ is isomorphic to D_2, D_3, or D_4, respectively (see Figure 3). There are four possible leaves for a 4-fold kite system of order $n \equiv 2, 5 \pmod 6$; the fourth is a 4-cycle. The following is a solution for a 4-cycle. For $n = 5$ and $n = 8$ see Examples 5.1 and 5.2. For $n \equiv 2, 5 \pmod 6 \geq 17$ use the $4k + r$ Construction (see Section 3) to obtain (X, K) and (X, K') with the metamorphoses $(X, T_1(K) \cup T_2(K), L)$ and $(X, T_1(K') \cup T_2(K'), L')$, respectively, where $L = \{\{1_0, 2_0\}, \{1_0, 2_0\}\}$, $L' = \{\{1_0, 3_0\}, \{1_0, 3_0\}\}$, and $(2_0, 3_0, z_0) \in T_2(K)$. The edges of $(2_0, 3_0, z_0))$ can be used in a rearrangement; so combine them with the leaves L and L' and reassemble these deleted edges into the triangle $(1_0, 2_0, 3_0)$ and the 4-cycle $(1_0, 2_0, z_0, 3_0)$. □
We will now handle the case $n \equiv 2, 3 \pmod 4$. The following examples take care of the cases $n = 6$, $n = 7$, $n = 10$, and $n = 11$.

Example 5.3. (n=6). Let $(\{\infty\} \cup Z_5, K)$ be the 4-fold kite system where $K = \{(i, 2+i, \infty)\text{-}(i+1),\ (1+i, 2+i, i)\text{-}\infty,\ (2+i, 4+i, i)\text{-}(i+1) \mid i \in Z_5\}$. Then $(X, T_1(K) \cup T_2(K))$ is a 4-fold triple system of order 6, where $T_2(K) = \{(i, 1+i, \infty) \mid i \in Z_5\}$.

Example 5.4. **(n=7)**. Let (Z_7, K) be the 4-fold kite system where $K = \{(i, 1+i, 3+i)\text{-}(i+6),\ (i, 1+i, 3+i)\text{-}(i+5)\ (2+i, 3+i, 5+i)\text{-}(i+6) \mid i \in Z_7\}$. Then $(X, T_1(K) \cup T_2(K))$ is a 4-fold triple system of order 7, where $T_2(K) = \{(i, 2+i, 3+i) \mid i \in Z_7\}$.

Example 5.5. **(n=10)**. Let $(\{\infty\} \cup Z_9, K)$ be the 4-fold kite system where $K = \{(i, 4+i, \infty)\text{-}(1+i),\ (4+i, 6+i, i)\text{-}\infty,\ (3+i, 5+i, 1+i)\text{-}i,\ (4+i, 1+i, i)\text{-}(3+i),\ (2+i, 1+i, i)\text{-}(3+i),\ \mid i \in Z_9\}$. Then $(X, T_1(K) \cup T_2(K))$ is a 4-fold triple system of order 10, where $T_2(K) = \{(i, 1+i, \infty) \mid i \in Z_5\} \cup (2\{(0,3,6), (1,4,7), (2,5,8)\})$.

Example 5.6. **(n=11, with all possible leaves)**. Let $S = \{x_0, x_1, x_2, x_3, x_4\}$ and $X = S \cup Z_5 \cup \{\infty\}$. Define a collection K of kites as follows.

(1) Let (S, K') be a 4-fold kite system of order 5 having a metamorphosis into a maximum packing with triangles (see Example 5.1 and Lemma 5.1); put $K' \subseteq K$.

(2) For every $i \in Z_5$ set $K_i = \{(i-1, i+1, x_i)\text{-}\infty, (i-2, i+2, x_i)\text{-}(i+1), (x_i, i, \infty)\text{-}(i+1), (\infty, i, x_i)\text{-}(i+2), (i-1, i+1, x_i)\text{-}i, (x_i, i-2, i+2)\text{-}i, (\infty, i, x_i)\text{-}(i-2), (i-2, i+2, x_i)\text{-}(i-1), (x_i, i+1, i-1)\text{-}(i-2)\}$ with arithmetic modulo 5; put $K_i \subseteq K$.

Then (X, K) is a 4-fold kite system of order 11. The metamorphosis is the following: use the metamorphosis in (1) with leave L; for every $i \in Z_5$, delete the tail from each kite in K_i and reassemble them into the triangles $(\infty, i+1, x_i)$, $(i, i+2, x_i)$, and $(i-1, i-2, x_i)$. The result is a maximum packing of $4K_{11}$ with triangles with leave L.

In order to handle the remaining cases we need the following examples. In what follows a *hole* is a set of vertices H with all edges between any two vertices in H removed.

Example 5.7. **(A 4-fold kite system of order 6 with a hole of size 2 having a metamorphosis into a packing of $4(K_6 \backslash K_2)$ with triangles with leave a double edge)**. Let $K = \{(1,2,a)\text{-}3,\ (2,3,a)\text{-}1,\ (0,3,a)\text{-}2,\ (a,1,0)\text{-}b,\ (a,1,2)\text{-}3,\ (a,0,3)\text{-}b,\ (b,0,1)\text{-}3,\ (1,3,b)\text{-}2,\ (b,3,2)\text{-}0,\ (b,2,0)\text{-}a,\ (b,3,1)\text{-}2,\ (b,2,0)\text{-}3,\ (0,1,2)\text{-}3,\ (0,3,1)\text{-}b\}$. Then (X, K) is a 4-fold kite system of order 6 with hole $\{a, b\}$ and $(X, T_1(K) \cup T_2(K), L)$ is a packing of $4(K_6 \backslash K_2)$ with triangles, where $T_2(K) = \{(a,1,2), (a,0,3), (b,0,2), (b,1,3)\}$ and $L = \{\{2,3\}, \{2,3\}\}$.

Example 5.8. **(A 4-fold kite system of order 7 with a hole of size 3 having a metamorphosis into a packing of $4(K_7 \backslash K_3)$ with triangles with leave the empty set)**. Let $K = \{(a,4,3)\text{-}1,\ (a,4,2)\text{-}b,\ (a,3,2)\text{-}c,\ (a,1,2)\text{-}4,\ (a,3,1)\text{-}b,\ (a,4,1)\text{-}c,\ (b,4,3)\text{-}a,\ (b,2,4)\text{-}1,\ (b,2,3)\text{-}c,\ (b,2,1)\text{-}a,$

$(b,1,3)$-2, $(b,1,4)$-c, $(c,3,4)$-a, $(c,2,4)$-b, $(c,2,3)$-4, $(c,1,2)$-a, $(c,1,3)$-b, $(c,4,1)$-2$\}$. Then (X,K) is a 4-fold kite system of order 7 with hole $\{a,b,c\}$ and $T_1(K) \cup T_2(K)$ is a decomposition of $4(K_7 \setminus K_3)$ into triangles, where $T_2(K) = \{(a,3,4),(a,1,2),(b,2,4),(c,2,3),(b,1,3),(c,1,4)\}$.

The 4k + s Construction. Write $4k + s = 2(2k) + s$, where $2k \geq 6$ and $s \in \{2,3\}$. Let $S = \{1,2,\ldots,2k\}$, R a set of size s, and $(S,\mathcal{G},T)$ a $\{3\}$-GDD of the type 2^k (for $k \equiv 0,1 \pmod 3$) or $2^{k-2}4^1$ (for $k \equiv 2 \pmod 3$), with groups $g_1 = \{1,2\}$, $g_2 = \{3,4\}$, ... , $g_k = \{2k-1,2k\}$ or $g_1 = \{1,2,3,4\}$, $g_2 = \{5,6\}$, ... , $g_{k-1} = \{2k-1,2k\}$, respectively. Set $X = R \cup (S \times Z_2)$ and define a collection K of kites as follows.

(1) Let $(R \cup (g_1 \times Z_2), K_{g_1})$ be a 4-fold kite system of order $2|g_1| + s$ having a metamorphosis into a maximum packing with triangles with leave L_{g_1} (see Examples 5.3, 5.4, 5.5, and 5.6); put $K_{g_1} \subseteq K$.

(2) For every group $g \in \mathcal{G} \setminus \{g_1\}$, let $(R \cup (g \times Z_2), K_g)$ be a 4-fold kite system of order $2|g| + s$ with a hole of size s having a metamorphosis into a packing of $4(K_{2|g|+s} \setminus K_s)$ with triangles with leave L_g (see Examples 5.7 and 5.8); put $K_g \subseteq K$.

(3) For every $t = (x,y,z) \in T$, let (S_t, K_t) be the $K_{2,2,2}$ kite system of Example 3.5; put $4K_t \subseteq K$.

Then (X,K) is a 4-fold kite system of order $4k + s$. The metamorphosis is the following: for the group g_1 use the metamorphosis in (1); for each group in $\mathcal{G} \setminus \{g_1\}$ use the metamorphosis in (2); for each triple in T use the metamorphosis in (3); and combine the leaves in (1), (2), and (3) to obtain the leave L as the case may be.

(a) *The case of* $s = 2$, $k \equiv 0,1 \pmod 3$ (and so $4k + s \equiv 2,0 \pmod 6$). $L_{g_1} = \emptyset$ (see Example 5.3); use the metamorphosis in (2) with leaves $\{\{3_0,4_0\},\{3_0,4_0\}\}$, $\{\{5_0,6_0\},\{5_0,6_0\}\}$, ... ,$\{\{(2k-1)_0,(2k)_0\},\{(2k-1)_0,(2k)_0\}\}$. These leaves plus all edges of $2(x_0,y_0,z_0)$ for each $(x,y,z) \in T$ is a copy of $2K_{2k}$ with a hole of size 2, $\{1_0,2_0\}$; replace these deleted edges with a packing of $2(K_{2k} \setminus K_2)$ with triangles with leave L. More precisely, if $k \equiv 0 \pmod 3$, take a decomposition $(S \times \{0\}, T')$ of $2K_{2k}$ into triangles (for the existence see Section 3) such that $\{\{1_0,2_0,z_0\},\{1_0,2_0,z'_0\}\} \in T'$, with $z_0 \neq z'_0$, and delete the double edge $\{\{1_0,2_0\},\{1_0,2_0\}\}$; the leave is a 4-cycle $L = (1_0,z_0,2_0,z'_0)$. If $k \equiv 1 \pmod 3$, take a decomposition of $2K_{2k}$ into triangles with leave $\{\{1_0,2_0\},\{1_0,2_0\}\}$ (see Section 3) and delete the double edge $\{\{1_0,2_0\},\{1_0,2_0\}\}$; the result is a decomposition of $2(K_{2k} \setminus K_2)$ into triangles with leave $L = \emptyset$.

(b) *The case of* $s = 2$, $k \equiv 2 \pmod 3$ (and so $4k + s \equiv 4 \pmod 6$). $L_{g_1} = \emptyset$ (see Example 5.5); use the metamorphosis in (2) with leaves $\{\{5_0, 6_0\}, \{5_0, 6_0\}\}, \{\{7_0, 8_0\}, \{7_0, 8_0\}\}, \dots, \{\{(2k-1)_0, (2k)_0\}, \{(2k-1)_0, (2k)_0\}\}$. These leaves plus all edges of $2(x_0, y_0, z_0)$ for each $(x, y, z) \in T$ is a copy of $2K_{2k}$ with a hole of size 4, $\{1_0, 2_0, 3_0, 4_0\}$. Replace these deleted edges with a packing of $2(K_{2k} \setminus K_4)$ with triangles with leave $L = \emptyset$ (see [1]).

(c) *The case of* $s = 3$, $k \equiv 0, 1 \pmod 3$ (and so $4k + s \equiv 3, 1 \pmod 6$). $L_{g_1} = \emptyset$ (see Example 5.4) and $L_g = \emptyset$ for each group in $\mathcal{G} \setminus \{g_1\}$ (see Example 5.8) and so $L = \emptyset$.

(d) *The case of* $s = 3$, $k \equiv 2 \pmod 3$ (and so $4k + s \equiv 5 \pmod 6$). $L_g = \emptyset$ for each group in $\mathcal{G} \setminus \{g_1\}$ (see Example 5.8) and so $L = L_{g_1}$ (C_4 or D_i, $i \in \{2, 3, 4\}$; see Example 5.6).

Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of $4K_{4k+s}$ with triangles.

Remark. Note that in the above construction in the case when $s = 2$ and $k \equiv 0 \pmod 3$, we constructed a 4-fold kite system having a metamorphosis into a maximum packing with triangles with leave a 4-cycle. In this case there are other three possible leaves. In order to give a solution for the remaining leaves, we need the following example.

Example 5.9. (n=14, with all possible leaves). For a 4-cycle see previous remark. In order to obtain a solution for the remaining possible leaves we need the following partial kite systems.

(a) *A partial kite system of order 14 with leave a 1-factor, having a metamorphosis into a partial triple system.* Let (X_1, K_1) be the partial kite system where $K_1 = \{(9,11,1)\text{-}2, (10,14,1)\text{-}4\}, (12,13,1)\text{-}3, (8,11,2)\text{-}3, (10,12,2)\text{-}5, (13,14,2)\text{-}4, (8,14,3)\text{-}4, (9,12,3)\text{-}6, (11,13,3)\text{-}5, (8,9,4)\text{-}5, (10,13,4)\text{-}7, (12,14,4)\text{-}6, (8,13,5)\text{-}6, (9,10,5)\text{-}1, (11,14,5)\text{-}7, (8,12,6)\text{-}7, (9,14,6)\text{-}2, (10,11,6)\text{-}1, (8,10,7)\text{-}1, (9,13,7)\text{-}3, (11,12,7)\text{-}2\}$. K_1 is a decomposition of $K_{14} \setminus L_1$ into kites, where $L_1 = \{\{1,8\}, \{2,9\}, \{3,10\}, \{4,11\}, \{5,12\}, \{6,13\}, \{7,14\}\}$. Then $(X, T_1(K_1) \cup T_2(K_1))$ is a decomposition of $K_{14} \setminus L_1$ into triangles, where $T_2(K_1) = \{(1,2,4), (2,3,5), (3,4,6), (4,5,7), (5,6,1), (6,7,2), (7,1,3)\}$.

(b) *A partial kite system* (Z_{13}, K_2) *with leave two triangles with one vertex in common, having a metamorphosis into a triple system.* Let $K_2 = \{(4+i, 1+i, i)\text{-}(6+i), (10+i, 7+i, 6+i)\text{-}(8+i), (7+i, 2+i, i)\text{-}(8+i) \mid 0 \le i \le 5\}$. K_2 is a decomposition of $K_{13} \setminus L_2$ with vertex set Z_{13} into kites, where $L_2 = \{(6,1,12),(0,3,12)\}$. Then $(X, T_1(K_2) \cup T_2(K_2))$ is a decomposition of $K_{13} \setminus L_2$ into triangles, where $T_2(K_2) = \{(i, 6+i, 8+i) \mid 0 \le i \le 5\}$.

Set $X = \{\infty\} \cup Z_{13}$ and define a collection K of kites as follows.

(1) Let $(\{\infty\} \cup Z_{13}, K_1')$ be a partial kite system of order 14 with leave $L_1' = \{\{2i, 1+2i\} \mid 0 \le i \le 5\} \cup \{12, \infty\}$, having a metamorphosis into a partial triple system (see (a)); put $K_1' \subseteq K$.

(2) Let (Z_{13}, K_2') be a partial kite system of order 13 with leave $L_2 = \{(0,2,12),(1,3,12)\}$, having a metamorphosis into a triple system (see (b)); put $K_1' \subseteq K$.

(3) Put $K_3' = \{(4+2i, 5+2i, \infty)\text{-}i \mid 0 \le i \le 3\} \cup \{(2,12,0)\text{-}1, (1,12,3)\text{-}2\} \subseteq K$.

Then (X, K) is a 2-fold partial kite system of order 14 having a metamorphosis into a packing of $2K_{14}$ with triangles with leave $\{\{12,\infty\},\{12,\infty\}\}$. The metamorphosis is the following: use the metamorphoses in (1) and (2); delete the tail from each kite in K_3' and reasseble them into the triangles $(0,1,\infty)$, $(2,3,\infty)$. In order to obtain a 4-fold kite system of order 14 with metamorphosis, let (X, K') be a 2-fold partial kite system of order 14 having a metamorphosis into a packing of $2K_{14}$ with triangles with leave a double edge $\{\{a,b\},\{a,b\}\}$. Then $(X, K \cup K')$ is a 4-fold kite system of order 14 having a metamorphosis into a maximum packing of $4K_{14}$ with triangles with leave $L = \{\{12,\infty\},\{12, \infty\},\{a,b\},\{a,b\}\}$. L is isomorphic to D_2, D_3, D_4, if $|\{12,\infty\} \cap \{a,b\}| = 2$, $|\{12,\infty\} \cap \{a,b\}| = 1$, and $|\{12,\infty\} \cap \{a,b\}| = 0$, respectively.

With Example 5.9 in hand we can give a construction which completes the case $n = 4k+2$ for $k \equiv 0 \pmod 3$, $k \ge 3$. The main ingredient we will need in this construction is a $\{3\}$-GDD of the type $2^{3(k-1)}6^1$. Since such a $\{3\}$-GDD exists if and only if $k \ge 3$ (see [1]), we will give a direct construction for $n = 26$. **The 12k + 2 Construction.** Let $S = \{1, 2, \dots, 6k\}$, $k \ge 3$, R a set of size 2, and $(S, \mathcal{G}, T)$ a $\{3\}$-GDD of the type $6^1 2^{3(k-1)}$, with groups $g_1, g_2, \dots, g_{3(k-1)}$, $g_1 = \{1_0, 2_0, 3_0, 4_0, 5_0, 6_0\}$. Set $X = R \cup (S \times Z_2)$ and define a collection K of kites as follows.

(1) Let $(R \cup (g_1 \times Z_2), K_{g_1})$ be a 4-fold kite system of order 14 having a metamorphosis into a maximum packing with triangles with leave L (see Example 5.9); put $K_{g_1} \subseteq K$.

(2) The same as (2) in the $4k+s$ Construction for $s = 2$.

(3) The same as (3) in the $4k+s$ Construction.

Then (X, K) is a 4-fold kite system of order $12k+2$. The metamorphosis is the following: for the group g_1 use the metamorphosis in (1): for each group in $\mathcal{G} \setminus \{g_1\}$ use the metamorphosis in (2) with leaves $\{\{7_0, 8_0\}, \{7_0, 8_0\}\}$, $\ldots$,$\{\{(6k-1)_0, (6k)_0\}, \{(6k-1)_0, (6k)_0\}\}$; for each triple in T use the metamorphosis in (3). The leaves in (2) plus all edges of $2(x_0, y_0, z_0)$ for each $(x, y, z) \in T$ is a copy of $2K_{6k}$ with a hole of size 6, $\{1_0, 2_0, 3_0, 4_0, 5_0, 6_0\}$. Replace these edges with a packing of $2(K_{6k} \setminus K_6)$ with triangles with leave the empty set (see [1]). Then $(X, T_1(K) \cup T_2(K), L)$ is a maximum packing of $4K_{12k+2}$ with triangles with all possible leaves. In order to handle the

case $n = 26$, we need the following example.

Example 5.10. (A 4-fold kite system of order 6 with two holes of size 2, H_1, H_2, having a metamorphosis into a packing of $4(\mathbf{K_6} \setminus (\mathbf{H_1} \cup \mathbf{H_2}))$ with triangles with $\mathbf{D_i}$, $\mathbf{i} \in \{2, 3, 4\}$, as leave). Let $S_\alpha = t_\alpha \times Z_2$, where $t_\alpha = (\alpha, y, z)$. Set $K = \{(\alpha_0, y_0, z_0)\text{-}\alpha_1$, $(y_0, z_1, \alpha_1)\text{-}y_1$, $(\alpha_1, z_1, y_1)\text{-}z_0\}$ and $K' = \{(\alpha_0, y_0, z_0)\text{-}\alpha_1$, $(y_0, z_1, \alpha_1)\text{-}\alpha_0$, $(\alpha_1, z_1, y_1)\text{-}\alpha_0$, $(y_0, z_0, \alpha_1)\text{-}\alpha_0$, $(y_0, z_1, \alpha_0)\text{-}z_0$, $(\alpha_0, z_1, y_1)\text{-}\alpha_1\}$ and define three collections of kites with vertices in S_α as follows.
(1) $K_1 = 2K \cup \{(\alpha_1, y_0, z_0)\text{-}\alpha_0, (y_0, z_1, \alpha_0)\text{-}y_1, (\alpha_0, z_1, y_1)\text{-}z_0, (y_0, z_0, \alpha_1)\text{-}\alpha_0, (y_0, z_1, \alpha_0)\text{-}\alpha_1, (y_1, z_1, \alpha_0)\text{-}\alpha_1, (y_1, z_0, \alpha_0)\text{-}\alpha_1\}$.
(2) $K_2 = K' \cup \{(y_0, z_0, \alpha_0)\text{-}\alpha_1, (y_0, z_1, \alpha_1)\text{-}\alpha_0, (\alpha_1, z_1, y_1)\text{-}z_0, (y_0, \alpha_1, z_0)\text{-}y_1, (y_0, z_1, \alpha_0)\text{-}y_1, (\alpha_0, z_1, y_1)\text{-}z_0, (\alpha_1, y_1, z_0)\text{-}\alpha_0\}$.
(3) $K_3 = K' \cup \{(y_0, z_0, \alpha_0)\text{-}\alpha_1, (y_0, \alpha_1, z_1)\text{-}y_1, (z_0, y_1, \alpha_1)\text{-}z_1, (y_0, \alpha_1, z_0)\text{-}y_1, (y_0, \alpha_0, z_1)\text{-}y_1, (z_0, y_1, \alpha_0)\text{-}z_1, (\alpha_1, \alpha_0, y_1)\text{-}z_0\}$.
For every $i = 1, 2, 3$, K_i is a partition of $4(K_6 \setminus (H_1 \cup H_2))$, where K_6 is the complete graph with vertex set S_α, $H_1 = \{y_0, y_1\}$, and $H_2 = \{z_0, z_1\}$.
For $i \in \{1, 2, 3\}$, delete the tail from each of the kites of K_i and reassemble them into a collection of triangles with leave L_i as the case may be. Note that the tails of the kites in K and K' can be reassembled into $T = \{(\alpha_1, y_1, z_0)\}$ and $T' = \{(\alpha_1, \alpha_0, y_1), (\alpha_1, \alpha_0, z_0)\}$, respectively.
(1) (S_α, K_1) admits the metamorphosis $(S_\alpha, T_1(K_1) \cup T_2(K_1), L_1)$, where $T_2(K_1) = 2T \cup \{(\alpha_0, y_1, z_0)\}$ and $L_1 = 4\{\{\alpha_0, \alpha_1\}\}$.
(2) (S_α, K_2) admits the metamorphosis $(S_\alpha, T_1(K_2) \cup T_2(K_2), L_2)$, where $T_2(K_2) = T' \cup \{(\alpha_0, y_1, z_0)\}$ and $L_2 = 2\{\{\alpha_0, \alpha_1\}, \{z_0, y_1\}\}$.
(2) (S_α, K_3) admits the metamorphosis $(S_\alpha, T_1(K_3) \cup T_2(K_3), L_3)$, where $T_2(K_3) = T' \cup \{(\alpha_0, \alpha_1, z_1)\}$ and $L_3 = 2\{\{y_1, z_0\}, \{y_1, z_1\}\}$.

$\mathbf{n = 26}$, with all possible leaves. For leave a 4-cycle see the $4k+s$ Construction. Let (S, T) be a STS(13) having an almost parallel class C which partitions $S \setminus \{\alpha\}$. Set $X = S \times Z_2$ and define a collection K of kites as follows.

λ (mod 12)	order n
0,4,8	any $n \geq 4$
1,3,5,7,9,11	$0, 1 \pmod 8$
2,6,10	$0,1 \pmod 4$

Table 1: *necessary and sufficient conditions for the existence of a λ-fold kite system.*

(1) Fix a triple $t_\alpha = (\alpha, y, z) \in T$. Let (S_α, K_i) be one of the three 4-fold partial kite systems of Example 5.10 with leave L_i, $i \in \{1, 2, 3\}$; put $K_i \subseteq K$.

(2) For every $t \in T \setminus C$, $t \neq t_\alpha$, let $(t \times Z_2, K_t)$ be a $K_{2,2,2}$ kite system of order 6 having a metamorphosis into a packing of $K_{2,2,2}$ with triangles with leave the empty set (see Example 3.5); put $4K_t \subseteq K$.

(3) For every $t \in C$, let $(t \times Z_2, K_t)$ be a 4-fold kite system of order 6 having a metamorphosis into a packing of $4K_6$ with triangles with leave the empty set (see Example 5.3); put $K_t \subseteq K$.

Then (X, K) is a 4-fold kite system of order 26. Use the metamorphoses in (1), (2), and (3). The result is maximum packing of $4K_{26}$ with triangles with leave L_i, $i \in \{1, 2, 3\}$.

Lemma 5.2. *There exists a 4-fold kite system of every order $n \geq 4$ having a metamorphosis into a maximum packing of $4K_n$ with triangles with all possible leaves.* □

6 λ-fold kite systems, $5 \leq \lambda \leq 12$

Table 1 gives the necessary and sufficient conditions for the existence of a λ-fold kite system (see [3]), while Table 2 shows the leaves of maximum packings of λK_n with triangles (see [1]).

In order to give a solution for $5 \leq \lambda \leq 11$ we need the following definition. Two collections of graphs H_1 and H_2 are said to be *balanced* provided they contain *exactly* the same edges.

Let F_n be a 1-factor of K_n containing the edges $\{a, d\}, \{b, c\}$, $T_n = \{\{a, b\}, \{a, c\}, \{a, d\}\} \cup (F_n \setminus \{\{a, d\}, \{b, c\}\})$, $D_3 = \{\{a, b\}, \{a, b\}, \{a, c\}, \{a, c\}\}$, and $E = \{\{b, c\}, \{b, c\}\}$. The following collections of graphs are balanced: $A_1 = D_3 \cup F_n$ and $A_2 = \{(a, b, c)\} \cup T_n$; $B_1 = E \cup D_3$ and $B_2 = \{(a, b, c), (a, b, c)\}$; $C_1 = D_3 \cup 2E$ and $C_2 = \{(a, b, c), (a, b, c)\} \cup E$.

		$n \pmod 6$					
		0	1	2	3	4	5
$\lambda = 1$		1-factor	$\emptyset$	1-factor	$\emptyset$	tripole	4-cycle
$\lambda > 1$ (mod 6)	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	1	1-factor	$\emptyset$	1-factor	$\emptyset$	tripole	D
	2	$\emptyset$	$\emptyset$	double edge	$\emptyset$	$\emptyset$	double edge
	3	1-factor	$\emptyset$	G	$\emptyset$	tripole	$\emptyset$
	4	$\emptyset$	$\emptyset$	D	$\emptyset$	$\emptyset$	D
	5	1-factor	$\emptyset$	tripole	$\emptyset$	tripole	double edge

Table 2: *leaves of maximum packings of λK_n with triangles.* (G is a graph on n vertices with $(n+4)/2$ edges and odd vertex degrees; D is a graph with 4 edges and even vertex degrees.)

To obtain solutions for $5 \leq \lambda \leq 11$ it is sufficient to combine a λ_1-fold kite system having a metamorphosis with leave L_1 and a λ_2-fold kite system having a metamorphosis with leave L_2 (for suitable values of λ_1 and λ_2) and replace L_1 and L_2 with a balanced collection of graphs where it is necessary (see Table 3).

To obtain solutions for $\lambda = 12$ combine three copies of a 4-fold kite system with leaves $\{\{a,b\},\{a,b\},\{a,b\},\{a,b\}\}$, $\{\{a,c\},\{a,c\},\{a,c\},\{a,c\}\}$, and $\{\{b,c\},\{b,c\}, \{b,c\},\{b,c\}\}$, respectively, and replace these leaves with $\{(a,b,c),(a,b,c),(a, b,c),(a,b,c)\}$.

7 Concluding remarks

Let $\lambda \equiv 1 \pmod 6$, $\lambda \geq 13$. Write $\lambda = 6k+7$ and combine k copies of a 6-fold kite system having a metamorphosis with a 7-fold kite system having a metamorphosis. For any value of $\lambda = 12k+h$ where $0 \leq h \leq 11$, $h \neq 1,7$, combine k copies of a 12-fold kite system having a metamorphosis with a h-fold kite system having a metamorphosis.

Theorem 7.1. *For every λ and for every admissible value of n there exists a λ-fold kite system of order n having a metamorphosis into a maximum packing of λK_n with triangles with all possible leaves.* □

		λ_1	λ_2	L_1	L_2	$L_1 \cup L_2$	leave
$\lambda = 5$	$n \equiv 0 \pmod 6$	1	4	F_n	$\emptyset$	F_n	F_n
	$n \equiv 1,3 \pmod 6$	1	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	$n \equiv 4 \pmod 6$	1	4	T_n	$\emptyset$	T_n	T_n
	$n \equiv 2 \pmod 6$	1	4	F_n	D_3	$A_1 \to A_2$	T_n
	$n \equiv 5 \pmod 6$	2	3	E	$\emptyset$	E	E
$\lambda = 6$	$n \equiv 0,1,3,4 \pmod 6$	2	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	$n \equiv 2,5 \pmod 6$	2	4	E	D_3	$B_1 \to B_2$	$\emptyset$
$\lambda = 7$	$n \equiv 5 \pmod 6$	3	4	$\emptyset$	D	D	D
	$n \equiv 0,1,2,3,4 \pmod 6$	1	6	L	$\emptyset$	L	L
$\lambda = 8$	$n \equiv 0,1,3,4 \pmod 6$	4	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
	$n \equiv 2,5 \pmod 6$	4	4	D_3	$2E$	$C_1 \to C_2$	E
$\lambda = 9$		3	6	L	$\emptyset$	L	L
$\lambda = 10$		4	6	L	$\emptyset$	L	L
$\lambda = 11$		5	6	L	$\emptyset$	L	L

Table 3: $\lambda = 5, 6, 7, 8, 9, 10, 11$.

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