Sum Coloring on Certain Classes of Graphs

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Abstract

An *L(2,1) coloring* of a graph $G = (V, E)$ is a vertex coloring $f : V(G) \to \{0, 1, 2, \ldots, k\}$ such that $|f(u) - f(v)| \geq 2$ for all $uv \in E(G)$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$. The *span* $\lambda(G)$ is the smallest k for which G has an *L(2,1) coloring*. A *span* coloring is an *L(2,1) coloring* whose greatest color is $\lambda(G)$. An *L(2,1)-coloring* f is a full-coloring if $f : V(G) \to \{0, 1, 2, \ldots, \lambda(G)\}$ is onto and f is an irreducible no-hole coloring (*inh*-coloring) if $f : V(G) \to \{0, 1, 2, \ldots, k\}$ is onto for some k and there does not exist an *L(2,1)-coloring* g such that $g(u) \leq f(u)$ for all $u \in V(G)$ and $g(v) < f(v)$ for some $v \in V(G)$. The Assignment sum of f on G is the sum of all the labels assigned to the vertices of G by the *L(2,1) coloring* f. The *Sum coloring number* of G, introduced in this paper, $\sum(G)$, is the minimum assignment sum over all the possible *L(2,1) colorings* of G. f is a *Sum coloring* on G, if its assignment sum equals the *Sum coloring number*. In this paper, we investigate the *Sum coloring numbers* of certain classes of graphs. It is shown that, $\sum(P_n) = 2(n-1)$ and $\sum(C_n) = 2n$ for all n. We also give an exact value for the *Sum coloring number* of a star and conjecture a bound for the *Sum coloring number* of an arbitrary graph G, with span $\lambda(G)$.

Keywords: L(2,1) colorings; inh-coloring; Sum coloring, Sum Coloring Number; Channel assignment problems.

1 Introduction

The channel assignment problem is the problem of assigning frequencies to radio or TV transmitters subject to imposed restrictions by the distance between transmitters in such a way that communications do not interfere. We note that two transmitters may interfere with each other if they share similar frequencies and are at short distances from each other. In 1988 F. S. Roberts proposed (in a private communication with Griggs) the problem of efficiently assigning radio channels to transmitters at several locations, using non negative integers to represent channels, so that close locations receive different channels, and channels for very close locations are at least two apart. This evolved into the study of *L(2,1)-colorings* of a graph which was first studied by Griggs and Yeh [1].

More rigorously, an *L(2,1)-coloring* of a Graph $G = (V, E)$ is a vertex coloring $f : V(G) \to \{0, 1, 2, \ldots, k\}$ such that $|f(u) - f(v)| \geq 2$ for all $uv \in E(G)$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$.

The *span* $\lambda(G)$ is the smallest k for which G has an *L(2,1) coloring.* That is,

$$\lambda(G) = min\{max\, f(u) \ : \ u \in V(G),\ f \text{ an } \{L(2,1)\text{-coloring}\}.$$

A *span* coloring is an *L(2,1) coloring* whose greatest color is $\lambda(G)$.

An *L(2,1) coloring* f is a full-coloring, introduced by Fishburn and Roberts [8], if $f : V(G) \to \{0, 1, 2, \ldots, \lambda(G)\}$ is onto .

An *L(2,1) coloring* f is an irreducible no-hole coloring (*inh-coloring*) if $f : V(G) \to \{0, 1, 2, \ldots, k\}$ is onto for some k and there does not exist an *L(2,1) coloring* g such that $g(u) \leq f(u)$ for all $u \in V(G)$ and $g(v) < f(v)$ for some $v \in V(G)$. G is said to be $inh-colorable$ if there is an $inh-coloring$ on G. The *inh-coloring* concept is due to Laskar and Villalpando [9].

Suppose G is $inh-colorable$, then define the $inh-span$ $\lambda_f(G)$ of G to be the smallest k for which G has an *inh-coloring.* That is,

$$\lambda_f(G) = min\{max\, f(u) \ : \ u \in V(G), \text{f an } inh\text{-coloring}\}$$

In this papar we introduce the concept of sum-coloring and show the exact sum-coloring number of certain classes of graphs.

2 Background

The following Propositions and Theorems are due to Griggs and Yeh [1].

Proposition 1. *([1]) Let P_n be a path on $n \geq 2$ vertices. Then,*

$$\lambda(P_n) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3, 4 \\ 4 & \text{if } n \geq 5. \end{cases}$$

Proposition 2. *([1]) Let C_n be a cycle on $n \geq 3$ vertices. Then $\lambda(C_n) = 4$.*

Theorem 1. *([1]) If T is a tree with maximum degree $\Delta \geq 1$, then*

$$\Delta + 1 \leq \lambda(T) \leq \Delta + 2.$$

Theorem 2. *([1]) Let G be a graph with maximum degree Δ. Then $\lambda(G) \leq \Delta^2 + 2\Delta$.*

Theorem 3. *([1]) The L(2,1) Problem is NP - complete.*

Conjecture 1. *([1]) For any graph G with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.*

Laskar and Villalpando [9], showed the following results.

Theorem 4. *([9]) Let T be a tree that is not a star, then there exists an inh-coloring on T.*

Theorem 5. *([9]) For any tree T with maximum degree Δ , that is not a star,*

$$\Delta + 1 \leq \lambda_f(T) \leq \Delta + 2.$$

Conjecture 2. *([9]) Let T be a tree that is not a star, then $\lambda(T)$=$\lambda_f(T)$.*

The following Theorem is due to Chang and Lu [4].

Theorem 6. *([4]) If G is a graph with maximum degree Δ and $\lambda(G) = \Delta+1$, then for any span coloring of G, a vertex of degree Δ , must be labeled 0 (or $\Delta+1$) and its neighbors must be labeled $2+i$ (or i), $i = 0, 1, \ldots, \Delta-1$.*

2.1 Background of Chromatic Sum

Schwenk and Kubicka introduced the concept of *Chromatic Sum* [11]. The chromatic sum of a graph is the smallest sum of colors among all proper colorings with natural numbers. The strength of a graph is the minimum number of colors necessary to obtain its chromatic sum. If $\eta^-(G)$ and $\eta^+(G)$ denote respectively the minimum and maximum number of colors to acheive the chromatic sum, $\Gamma(G)$ is the Grundy coloring number and $\Psi(G)$, the achromatic number of a graph, then for any graph G, the following string of inequality holds [10].

$$\chi(G) \leq \eta^-(G) \leq \eta^+(G) \leq \Gamma(G) \leq \Psi(G)$$

We extend this concept naturally to *L(2,1) coloring*.

3 Sum Coloring and Sum Coloring Number

3.1 Definitions

In this paper we introduce the *Sum coloring number* of a graph G, $\sum(G)$, which is the minimum assignment sum over all the possible *L(2,1)-colorings* of G. That is,

$$\sum(G) = min_f\{\sum_{S_{f(v)}} f(v).|S_{f(v)}| : v \in V(G) \text{ and } f \text{ is an } L(2,1)\text{-coloring}\}$$

where $S_{f(v)}$ is a set of vertices on V(G) all labeled $f(v)$. An *L(2,1) coloring* f is a *Sum coloring* on G, if its assignment sum equals the *Sum coloring number*.

Define the *inh-sum coloring number* as the minimum assignment sum over all the possible *inh-colorings* of G. That is,

$$\sum_f(G) = min_f\{\sum_{S_{f(v)}} f(v).|S_{f(v)}| : v \in V(G) \text{ and } f \text{ is an } inh-coloring)\}$$

where $S_{f(v)}$ is a set of vertices on V(G) all labeled $f(v)$.

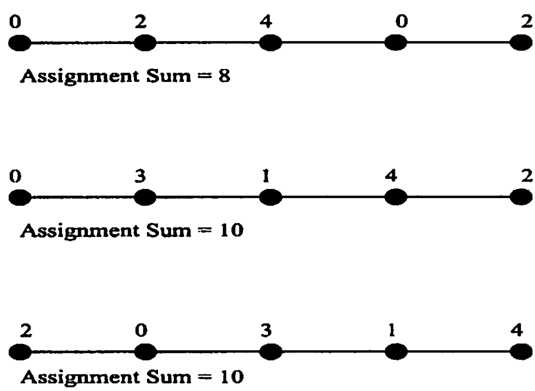

Figure 1: Example of Assignment Sums

It is easy to verify that $\sum(P_5) = 8$ but $\sum_f(P_5) = 10$. A natural question would be to ask for what n would these two sums be equal, if at all? We shall answer this question later in the paper.

3.2 Paths

Lemma 1. *Let P_n be a path on n vertices with $u, v \in V(P_n)$, $u \neq v$. If $f(u) = 0 = f(v)$, then there exists a vertex w between u and v such that $f(w) \geq 4$. (Here we are working with the closest zeros on the labeling).*

Proof. Here we consider 2 cases, based on the distance between u and v.

case i.) Suppose $d(u, v) = 3$.

Then we have the subpath, u,v_1,v_2,v with $f(u) = 0 = f(v)$. We see that $f(v_1) = 2$ or 3 (4 or above proves the lemma) and so, $f(v_2) \geq 4$. Hence the Lemma is true for this case.

case ii.) Suppose $d(u, v) \geq 4$.

Then the subpath u-v is a path of length ≥ 5. Since $\lambda(P_n) = 4$ for $n \geq 5$, there must be a vertex in this u-v subpath with label ≥ 4.

□

Observation 1. *Let P_n be a path on n vertices. If $v \in V(P_n)$ and $f(v) = 1$ and u is a neighbor of v, then $f(u) \geq 3$.*

Theorem 7. *Let P_n be a path on n vertices. Then $\sum(P_n) = 2(n-1)$ for all positive integers n.*

Proof. The result is easily verified for P_n, $n \leq 5$. We shall prove it true for $n > 5$. Order the vertices of P_n as $v_1, v_2, \ldots, v_n$ where v_1 is the first vertex, v_n the last vertex and for every m, $m = 1, 2, \ldots, n-1$, $v_m v_{m+1} \in E(P_n)$. We develop three cases in proving this part;

case i.) $n \equiv 0(mod3)$

Consider the coloring f on $V(G)$ defined by, $f(v_1) = 0$, $f(v_2) = 3$, $f(v_3) = 1$ and

$$f(v_k) = \begin{cases} 4 & \text{if } k \equiv 1(mod3) \\ 2 & \text{if } k \equiv 2(mod3), k \geq 4 \\ 0 & \text{if } k \equiv 0(mod3) \end{cases}$$

f defined as above is an *L(2,1)-coloring* on P_n.

$$\begin{aligned} \text{Assignment Sum} &= 4 + 6(\frac{n-3}{3}) \\ &= 4 + 2(n-3) \\ &= 2n - 2 \\ &= 2(n-1). \end{aligned}$$

case ii.) $n \equiv 1(mod3)$

Redefine the coloring f on $V(G)$ at $n-3, n-2, n-1$, and n as

$$f(v_k) = \begin{cases} 3 & \text{if k=n-3} \\ 1 & \text{if k=n-2} \\ 4 & \text{if k=n-1} \\ 0 & \text{if k=n} \end{cases}$$

f defined as above is an *L(2,1)-coloring* on P_n.

$$\begin{aligned}
\text{Assignment Sum} &= 4+6(\frac{n-7}{3})+8 \\
&= 4+2(n-7)+8 \\
&= 2n-2 \\
&= 2(n-1).
\end{aligned}$$

case iii.) $n \equiv 2(mod3)$

Redefine the coloring f on $V(G)$ in case1 above at $n-4, n-3, n-2, n-1$ and n as follows,

$$f(v_k) = \begin{cases} 3 & \text{if k=n-4} \\ 1 & \text{if k=n-3} \\ 4 & \text{if k=n-2} \\ 0 & \text{if k=n-1} \\ 2 & \text{if k=n} \end{cases}$$

f defined as above is an *L(2,1)-coloring* on P_n.

$$\begin{aligned}
\text{Assignment Sum} &= 4+6(\frac{n-8}{3})+10 \\
&= 4+2(n-8)+10 \\
&= 2n-2 \\
&= 2(n-1).
\end{aligned}$$

So, in all cases we have that Assignment Sum $= 2(n-1)$. Thus $\sum(P_n) =$ minimum assignment sum $\leq 2(n-1)$.

Conversely,

a) From Observation 1, any vertex labeled 1, must have a neighbor labeled 3 or more. Pair each vertex labeled 1 with its neighbor labeled ≥ 3 and take the average of the two labels. Then average ≥ 2.

b) From Lemma 1, if $f(u) = 0 = f(v)$ (closest 0's) with $u, v \in V(P_n)$, then there is a vertex w in the uv-path such that $f(w) \geq 4$. Pair the leftmost vertex labeled 0 with a vertex w such that $f(w) \geq 4$ in the uv-path. The average of these two vertices again is ≥ 2.

c) The leftmost vertex labeled 0 without a corresponding vertex labeled 0 on the right of it on the path is not paired and is not considered in averaging. Since its value is 0, it does not affect the assignment sum.

In conclusion then, the average on the labels on the $n-1$ vertices of P_n is ≥ 2 not counting the vertex with situation as in (c) above.

Thus, $\sum(P_n) \geq 2(n-1)$. This proves the Theorem.

□

Theorem 8. *Let P_n be a path on n vertices, $n \geq 4$. Then $\sum_f(P_n) = \sum(P_n)$ for all positive integers n except for $n = 5$.*

Proof. Clearly, $\sum_f(P_n) \geq \sum(P_n)$ since every *inh-coloring* is an *L(2,1)-coloring.*
To prove the reverse inequality, we shall construct an *inh-coloring* whose assignment sum is $2(n-1)$. Trivial for $n = 4$. For $n = 5$ Example 2 shows that $\sum_f(P_n) \neq \sum(P_n)$. Now for $n \geq 6$, order the vertices of P_n as $v_1, v_2, \ldots, v_n$ where v_1 is the first vertex, v_n the last vertex and for every m, $m = 1, 2, \ldots, n-1$, $v_m v_{m+1} \in E(P_n)$. We develop three cases in proving this part;

case i.) $n \equiv 0(mod3)$

Consider the coloring f on $V(P_n)$ defined by, $f(v_1) = 1$, $f(v_2) = 3$, $f(v_3) = 0$ and

$$f(v_k) = \begin{cases} 2 & \text{if } k \equiv 1(mod3) \\ 4 & \text{if } k \equiv 2(mod3), k \geq 6 \\ 0 & \text{if } k \equiv 0(mod3). \end{cases}$$

Then f defined as above is an *inh-coloring* on P_n.

$$\begin{aligned} \text{Assignment Sum} &= 4 + 6(\frac{n-3}{3}) \\ &= 4 + 2(n-3) \\ &= 2n - 2 \\ &= 2(n-1). \end{aligned}$$

case ii.) $n \equiv 1(mod3)$

Redefine the coloring f on $V(P_n)$ above at n as $f(v_n) = 2$. This is clearly an *inh-coloring* on P_n and

$$\begin{aligned} \text{Assignment Sum} &= 4 + 6(\frac{n-4}{3}) + 2 \\ &= 4 + 2(n-4) + 2 \\ &= 2n - 2 \\ &= 2(n-1). \end{aligned}$$

case iii.) $n \equiv 2(mod3)$

Redefine the coloring f on $V(P_n)$ in case1 above at $n-1$ and n as follows,

$$f(v_k) = \begin{cases} 3 & \text{if k=n-1} \\ 1 & \text{if k=n.} \end{cases}$$

Again f defined as above is an *inh-coloring* on P_n and

$$\begin{aligned} \text{Assignment Sum} &= 4 + 6(\frac{n-5}{3}) + 4 \\ &= 4 + 2(n-5) + 4 \\ &= 2n - 2 \\ &= 2(n-1). \end{aligned}$$

Thus $\sum_f(P_n) \leq 2(n-1) = \sum(P_n)$ and we have that $\sum_f(P_n) \leq \sum(P_n)$ and hence the equality holds.

□

3.3 Circles

Theorem 9. *Let C_n be a cycle on n vertices. Then, $\sum(C_n) = 2n$ for all positive integers $n \geq 3$.*

Proof. Order the vertices of C_n as $v_1, v_2, \ldots, v_n$ where v_1 is any vertex , $v_n v_1 \in E(C_n)$ and for every m, $m = 1, 2, \ldots, n-1$, $v_m v_{m+1} \in E(C_n)$. We develop three cases in proving this part;

case i.) $n \equiv 0(mod3)$

Consider the coloring f on $V(C_n)$ defined by,

$$f(v_k) = \begin{cases} 0 & \text{if } k \equiv 1(mod3) \\ 2 & \text{if } k \equiv 2(mod3), k \geq 3 \\ 4 & \text{if } k \equiv 0(mod3). \end{cases}$$

f defined as above is an *L(2,1)-coloring* on C_n.

$$\begin{aligned} \text{Assignment Sum} &= 6(\frac{n}{3}) \\ &= 2n. \end{aligned}$$

case ii.) $n \equiv 1(mod3)$

Redefine the coloring f on $V(C_n)$ at $n-3, n-2, n-1$, and n as

$$f(v_k) = \begin{cases} 0 & \text{if k=n-3} \\ 3 & \text{if k=n-2} \\ 1 & \text{if k=n-1} \\ 4 & \text{if k=n.} \end{cases}$$

f defined as above is an *L(2,1)-coloring* on C_n.

$$\begin{aligned} \text{Assignment Sum} &= 6(\frac{n-4}{3})+8 \\ &= 2(n-4)+8 \\ &= 2n. \end{aligned}$$

case iii.) $n \equiv 2(mod3)$

Redefine the coloring f on $V(C_n)$ in case1 above at $n-1$ and n as follows,

$$f(v_k) = \begin{cases} 1 & \text{if k=n-1} \\ 3 & \text{if k=n} \end{cases}$$

f defined as above is an *L(2,1)-coloring* on C_n.

$$\begin{aligned} \text{Assignment Sum} &= 6(\frac{n-2}{3})+4 \\ &= 2(n-2)+4 \\ &= 2n. \end{aligned}$$

So, in all cases we have that Assignment Sum$= 2n$. Thus $\sum(C_n) =$ minimum assignment sum $\leq 2n$.

Conversely,

a) From Observation 1, any vertex labeled 1, must have a neighbor labeled 3 or more. Pair each vertex labeled 1 with its neighbor labeled ≥ 3 and take the average of the two labels. Then average ≥ 2.

b) From Lemma 1, if $f(u) = 0 = f(v)$ (closest 0's) with $u, v \in V(P_n)$, not necessarily distinct, then there is a vertex w in the uv-path such that $f(w) \geq 4$. Pair the leftmost vertex labeled 0 with a vertex w such that $f(w) \geq 4$ in the uv-path. We observe that, there will be no leftmost vertex labeled 0 without a corresponding rightmost vertex labeled 0 as in the case of paths. The average of these two vertices again is ≥ 2.

In conclusion then, the average on the labels on the n vertices of C_n is ≥ 2..

Thus, $\sum(C_n) \geq 2n$. This proves the Theorem.

□

Theorem 10. *Let C_n be a cycle on n vertices. Then $\sum_f(C_n) = \sum(C_n)$ for all positive integers $n \geq 5$ except for $n = 6$.*

Proof. Clearly, $\sum_f(C_n) \geq \sum(C_n)$ since every *inh-coloring* is an *L(2,1)-coloring.*
To prove the reverse inequality, we shall construct an *inh-coloring* whose assignment sum is $2n$. Trivial for $n = 5$. For $n = 6$, $\sum_f(C_6) = 15 \neq \sum(C_6) = 12$. Now for $n \geq 7$, order the vertices of C_n as $v_1, v_2, \ldots, v_n$ where v_1 is any vertex, $v_n v_1 \in E(C_n)$, and for every m, $m = 1, 2, \ldots, n-1$, $v_m v_{m+1} \in E(C_n)$. We develop three cases in proving this part;

case i.) $n \equiv 0(mod3)$

Consider the coloring f on $V(P_n)$ defined by,

$$f(v_k) = \begin{cases} 0 & \text{if k=1} \\ 3 & \text{if k=2} \\ 1 & \text{if k=3} \\ 4 & \text{if k=4} \\ 0 & \text{if k=5} \\ 3 & \text{if k=6} \\ 1 & \text{if k=7} \\ 4 & \text{if k=8} \\ 2 & \text{if k=9} \end{cases}$$

and for $k > 9$,

$$f(v_k) = \begin{cases} 0 & \text{if } k \equiv 1(mod3) \\ 4 & \text{if } k \equiv 2(mod3) \\ 2 & \text{if } k \equiv 0(mod3). \end{cases}$$

Then f defined as above is an *inh-coloring* on C_n.

$$\begin{aligned} \text{Assignment Sum} &= 6(\frac{n-9}{3}) + 18 \\ &= 2(n-9) + 18 \\ &= 2n. \end{aligned}$$

case ii.) $n \equiv 1(mod3)$

Consider the coloring f on $V(C_n)$ defined as follows,

$$f(v_k) = \begin{cases} 0 & \text{if } k \equiv 1(mod3) \\ 2 & \text{if } k \equiv 2(mod3), k \geq 6 \\ 4 & \text{if } k \equiv 0(mod3) \end{cases}$$

for $k \neq n-3, n-2, n-1, n$ and

$$f(v_k) = \begin{cases} 0 & \text{if k=n-3} \\ 3 & \text{if k=n-2} \\ 1 & \text{if k=n-1} \\ 4 & \text{if k=n.} \end{cases}$$

$$\begin{aligned} \text{Assignment Sum} &= 6(\frac{n-4}{3})+8 \\ &= 2(n-4)+8 \\ &= 2n \end{aligned}$$

case iii.) $n \equiv 2(mod3)$

Consider the coloring f on $V(C_n)$ defined as follows,

$$f(v_k) = \begin{cases} 0 & \text{if } k \equiv 0(mod3) \\ 2 & \text{if } k \equiv 1(mod3), k \geq 7 \\ 4 & \text{if } k \equiv 2(mod3) \end{cases}$$

for $k \neq$ n-1, n and

$$f(v_k) = \begin{cases} 1 & \text{if k=n-1} \\ 3 & \text{if k=n.} \end{cases}$$

$$\begin{aligned} \text{Assignment Sum} &= 6(\frac{n-2}{3})+4 \\ &= 2(n-2)+4 \\ &= 2n \end{aligned}$$

Thus $\sum_f(C_n) \leq 2n = \sum(C_n)$ and we have that $\sum_f(C_n) \leq \sum(C_n)$ and hence equality holds.

□

3.4 Stars

Lemma 2. *Let T be a star with maximum degree Δ. Then*

$$\sum(T) = \frac{\Delta^2 + \Delta + 2}{2}$$

Proof. By Theorem 6, the following labeling in Figure 2, will give the *sum coloring* for a star, T. Adding the labels gives the required result.

□

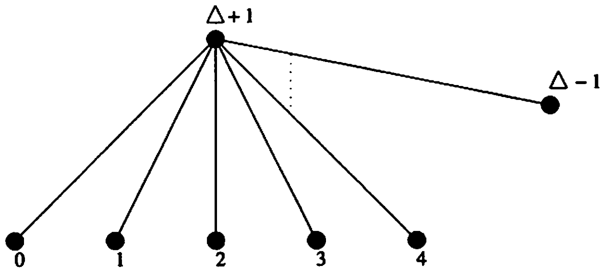

Figure 2: Sum Coloring for a Star

3.5 Arbitrary Graph G

From the examples given so far, it is tempting to think that all *sum colorings* will be span coloring. We make this clear with the following observation.

Observation 2. *A sum coloring is not necessarily a span coloring. The example in Figure 3 illustrates this fact.*

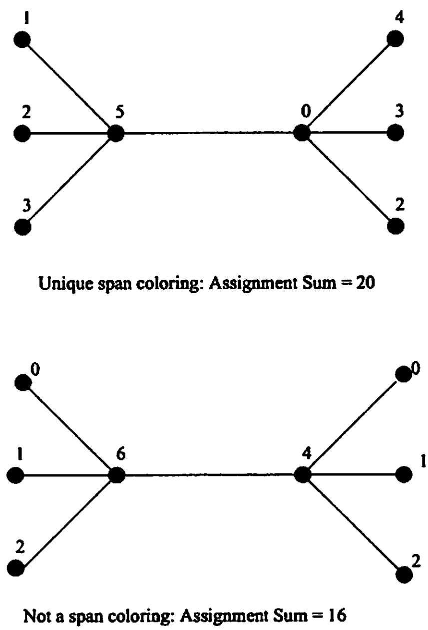

Figure 3: Example of sum coloring which is not a span coloring

Lemma 3. *Let K_n be a complete graph on n vertices. Then,*

$$\sum(K_n) = \frac{n\lambda(K_n)}{2}$$

Proof. It is obvious to see that $\lambda(K_n) = 2(n-1)$ and $\sum(K_n) = \sum_{i=1}^{n} 2(n-i)$. The proof follows directly from these equations. □

Corollary 1. *Let G be a graph on n vertices. Then*

$$\sum(G) \leq \frac{n\lambda(K_n)}{2}$$

Proof. The proof is straight forward since $G \subseteq K_n$. □

Conjecture 3. *If G is a graph on n vertices with span $\lambda(G)$, then*

$$\sum(G) \leq \frac{n\lambda(G)}{2}$$

4 Open Problems

- Find the bounds for the Sum Coloring number for several other classes of graphs.
- Find a bound relating the Sum Coloring Number and the Chromatic Sum of a graph G.
- Complexity issues regarding the Sum Coloring Number

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