On Detectable Factorizations of Cubic Graphs

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Abstract

For a connected graph G of order $n \geq 3$ and an ordered factorization $\mathcal{F} = \{G_1, G_2, \cdots, G_k\}$ of G into k spanning subgraphs G_i $(1 \leq i \leq k)$, the color code of a vertex v of G with respect to $\mathcal{F}$ is the ordered k-tuple $c(v) = (a_1, a_2, \cdots, a_k)$ where $a_i = \deg_{G_i} v$. If distinct vertices have distinct color codes, then the factorization $\mathcal{F}$ is called a detectable factorization of G; while the detection number $\det(G)$ of G is the minimum positive integer k for which G has a detectable factorization into k factors. We study detectable factorizations of cubic graphs. It is shown that there is a unique graph F for which the Petersen graph has a detectable F-factorization into three factors. Furthermore, if G is a connected cubic graph of order $\binom{k+2}{3}$ with $\det(G) = k$, then $k \equiv 2 \pmod 4$ or $k \equiv 3 \pmod 4$. We investigate the largest order of a connected cubic graph with prescribed detection number.

Key Words:detectable coloring, detectable factorization, detection number.

AMS Subject Classification: 05C15, 05C70.

1 Introduction

We refer to the book [6] for graph theory notation and terminology not described in this paper. Let G be a connected graph of order $n \geq 3$ and let $c : E(G) \to \{1, 2, \ldots, k\}$ be a coloring of the edges of G for some positive integer k (where adjacent edges may be colored the same). The *color code* of a vertex v of G with respect to a k-coloring c of the edges of G is the ordered k-tuple

$$c(v) = (a_1, a_2, \cdots, a_k) \quad \text{(or simply, } c(v) = a_1 a_2 \cdots a_k\text{)},$$

where a_i is the number of edges incident with v that are colored i for $1 \le i \le k$. Therefore,

$$\sum_{i=1}^{k} a_i = \deg_G v.$$

The coloring c is called *detectable* if distinct vertices have distinct color codes; that is, for every two vertices of G, there exists a color such that the number of incident edges with that color is different for these two vertices. The *detection number* $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-coloring . A detectable coloring of a graph G with $\det(G)$ colors is called a *minimum detectable coloring.* Since every nontrivial graph contains at least two vertices having the same degree, the vertices of a nontrivial connected graph cannot be distinguished by their degrees alone. Therefore, every connected graph of order 3 or more has detection number at least 2. The concept of detectable coloring was studied in [1, 2, 3, 4, 5], inspired by the basic problem in graph theory that concerns finding means to distinguish the vertices of a connected graph.

To illustrate these concepts, consider the graph G shown in Figure 1(a). A coloring of the edges of G is shown in Figure 1(b). For this 3-coloring c, the color codes of its vertices are $c(u) = 110$, $c(v) = 021$, $c(w) = 210$, $c(x) = 201$, $c(y) = 101$, $c(z) = 001$. Since the vertices of G have distinct color codes, c is a detectable coloring.

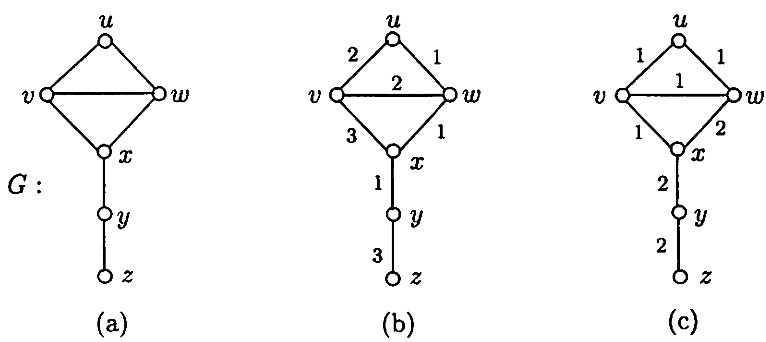


Figure 1: A detectable coloring of a graph

Figure 1(c) shows yet another detectable coloring c' of the graph G of Figure 1(a). For this coloring, $c'(u) = 20$, $c'(v) = 30$, $c'(w) = 21$, $c'(x) = 12$, $c'(y) = 02$, $c'(z) = 01$. The coloring c' uses only two colors.

Since G has a detectable 2-coloring, we can immediately conclude that $\det(G) = 2$.

Figure 2 shows minimum detectable colorings of all connected graphs of orders 3 and 4. Two of the graphs in Figure 2 illustrate a feature of detectable colorings that does not hold for standard edge colorings. The graph G_3 of Figure 2 is a subgraph of the graph G_6, while $\det(G_3) = 3$ and $\det(G_6) = 2$. Hence the fact that G is a subgraph of H does not imply in general that $\det(G) \leq \det(H)$. Furthermore, if H, F, and G are graphs with $F \leq G \leq H$, then the fact that $\det(F) = \det(H)$ does not imply in general that $\det(F) = \det(G) = \det(H)$. For example, the graph G_3 of Figure 2 is a subgraph of G_6, the graph G_6 is a subgraph of G_8, and $\det(G_3) = \det(G_8) = 3$ but $\det(G_6) = 2$.

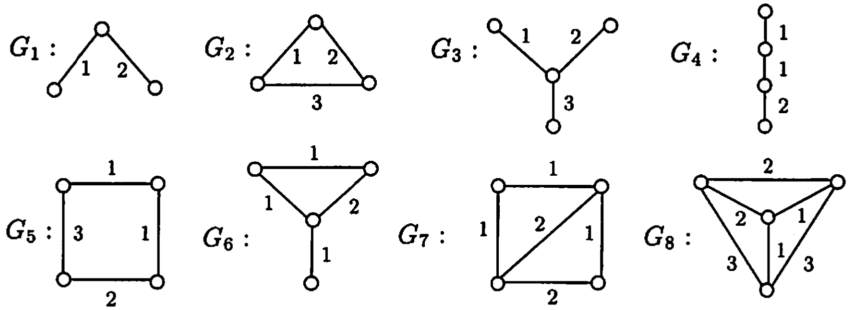


Figure 2: Minimum detectable colorings of connected graphs of small orders

As described in [5], detectable colorings can be looked at from a different point of view. For a connected graph G of order $n \geq 3$ and a factorization

$$\mathcal{F} = \{G_1, G_2, \cdots, G_k\}$$

of G into k subgraphs G_i $(1 \leq i \leq k)$, the color code of a vertex v of G with respect to $\mathcal{F}$ is the ordered k-tuple

$$c(v) = (a_1, a_2, \cdots, a_k)$$

where $a_i = \deg_{G_i} v$ and so $\sum_{i=1}^{k} \deg_{G_i} v = \deg_G v$. If distinct vertices have distinct color codes, then the factorization $\mathcal{F}$ is called a *detectable factorization* of G. A detectable factorization of G with k factors is called a *detectable k-tuple factorization*. Each ordered detectable k-tuple factorization $\mathcal{F} = \{G_1, G_2, \cdots, G_k\}$ of a graph G gives rise to a detectable k-coloring of the edges of G by assigning color i to the edges of G_i for $1 \leq i \leq k$. On the other hand, let c be a k-coloring of the edges of a connected graph G. For each integer i with $1 \leq i \leq k$, let G_i be the spanning subgraph of G whose edges are colored i. This produces a k-tuple factorization $\mathcal{F} = \{G_1,$

$G_2, \cdots, G_k\}$ of G. Then a coloring c of the edges of G is detectable if and only if for each vertex v of G, there exist two distinct factors G_s and G_t in $\mathcal{F}$ such that $\deg_{G_s} v \neq \deg_{G_t} v$. A factorization of G resulting from some detectable coloring of G is a detectable factorization of G. For example, the detectable factorization that results from the detectable 3-coloring of the graph G of Figure 1 is shown in Figure 3.

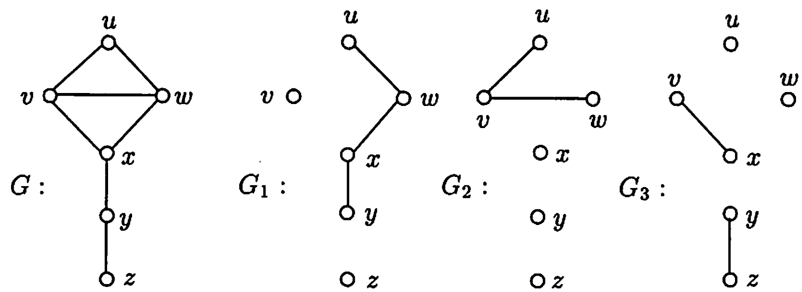


Figure 3: A detectable factorization of a graph

2 Some Known Results on Detection Numbers

If c is a coloring of the edges of a graph G and u and v are two vertices of G with $\deg_G u \neq \deg_G v$, then $c(u) \neq c(v)$. Consequently, when investigating whether a given coloring c is detectable, we need only be concerned with sets of vertices of the same degree. Therefore, it is most challenging and most interesting to find minimum detectable colorings of graphs having many vertices of the same degree. The following results were stated in [2, 5].

Theorem 2.1 *For every integer $n \geq 3$, $\det(K_n) = 3$.*

Theorem 2.2 *If G is a regular connected graph of order $n \geq 3$, then* $\det(G) \geq 3$.

Theorem 2.3 *If G is an r-regular connected graph of order $n \geq 3$, then* $\det(G) \leq (5e(r+1)!n)^{\frac{1}{r}}$.

Theorem 2.4 *Let $n \geq 3$ be an integer and let $\ell = \left\lceil \sqrt{n/2} \right\rceil$. Then*

$$\det(C_n) = \begin{cases} 2\ell & \text{if } 2\ell^2 - \ell + 1 \leq n \leq 2\ell^2 \\ 2\ell - 1 & \text{if } 2(\ell-1)^2 + 1 \leq n \leq 2\ell^2 - \ell. \end{cases}$$

Theorem 2.5 *For integers s and t with $1 \leq s \leq t$,*

$$\det(K_{s,t}) = \begin{cases} 3 & \text{if } s = t \geq 2 \\ t & \text{if } 1 = s < t \\ 2 & \text{if } t = s + 1 \\ k & \text{if } 2 \leq s < t - 1 \text{ and } k \text{ is the unique integer} \\ & \text{for which } \binom{s+k-2}{s} < t \leq \binom{s+k-1}{s} \end{cases}$$

Theorem 2.6 *A pair k, n of positive integers is realizable as the detection number and the order of some nontrivial connected graph if and only if $k = n = 3$ or $2 \leq k \leq n - 1$.*

Theorem 2.7 *Let c be a k-coloring of the edges of a graph G. There are at most $\binom{r+k-1}{r}$ different color codes for the vertices of degree r in G.*

The following result is an immediate consequence of Theorem 2.7 (see [5]).

Theorem 2.8 *For each detectable k-coloring of a connected graph G of order at least 3, there are at most $\binom{r+k-1}{r}$ vertices of degree r.*

This theorem implies the following result.

Theorem 2.9 *If G is a connected r-regular graph of order n having detection number k, then*

$$n \leq \binom{r+k-1}{r}.$$

The contrapositive of Theorem 2.9 gives the following.

Theorem 2.10 *Let G be a connected r-regular graph of order n. If $n > \binom{r+k-1}{r}$ for some positive integer k, then $\det(G) > k$.*

It therefore follows that the maximum order of a connected r-regular graph with detection number k is $\binom{r+k-1}{r}$. Suppose that G is a connected r-regular graph of order $\binom{r+k-1}{r}$ having detection number k. Then there exists a k-coloring of the edges of G such that for each of the $\binom{r+k-1}{r}$ possible color codes, there is exactly one vertex having that color code. Each such coloring gives rise to a k-tuple factorization $\mathcal{F}$ of G, where each of the k factors has a degree sequence containing $\binom{k-3+t}{k-2}$ terms equal to $r - t + 1$ for $t = 1, 2, \ldots, r + 1$. Note that

$$\binom{k-2}{k-2} + \binom{k-1}{k-2} + \cdots + \binom{r+k-2}{k-2} = \binom{r+k-1}{k-1} = \binom{r+k-1}{r}.$$

Since the size of G is $\frac{r}{2}\binom{r+k-1}{r}$, it follows that $k \mid \frac{r}{2}\binom{r+k-1}{r}$.

3 Cubic Graphs

In this section we turn to the main topic of this paper, namely detectable factorizations (or colorings) of cubic (3-regular) graphs. The cubic graph of smallest order is K_4 and $\det(K_4) = 3$ by Theorem 2.1. There are two cubic graphs of order 6 namely $K_{3,3}$ and $K_3 \times K_2$. We have seen that $\det(K_{3,3}) = 3$ by Theorem 2.5. A detectable 3-coloring of $K_3 \times K_2$ in Figure 4 shows that $\det(K_3 \times K_2) = 3$.

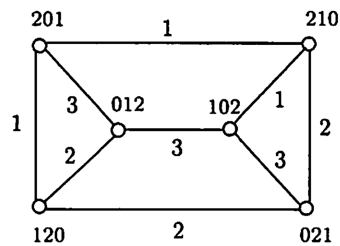


Figure 4: A minimum detectable 3-coloring $K_3 \times K_2$

Not only is the detection number of $K_3 \times K_2 = C_3 \times K_2$ equal to 3, so too is the detection number of $C_4 \times K_2 = Q_3$ equal to 3. Furthermore, $\det(C_5 \times K_2) = 3$. Detectable 3-colorings of these two cubic graphs are shown in Figure 5.

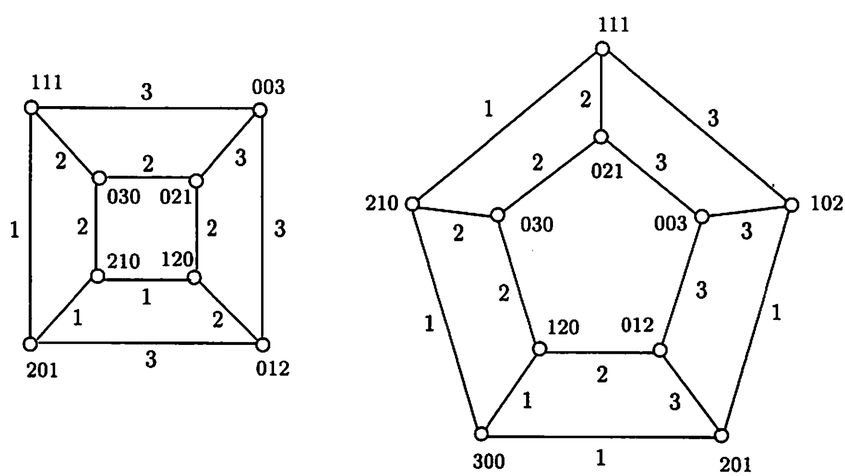


Figure 5: Detectable 3-colorings of Q_3 and $C_5 \times K_2$

Another interesting feature of the 3-coloring of the edges of $C_5 \times K_2$ shown in Figure 5 is that each factor whose edges are colored the same is isomorphic to the forest F shown in Figure 6. For graphs F and G, an *F-factorization* of G is a factorization $\mathcal{F}$ of G in which every factor in $\mathcal{F}$ is isomorphic F. Such a factorization of G is also called an *isomorphic factorization* of G. Thus the detectable 3-coloring of $C_5 \times K_2$ in Figure 5 gives rise to an isomorphic factorization of $C_5 \times K_2$ into the forest in Figure 6.

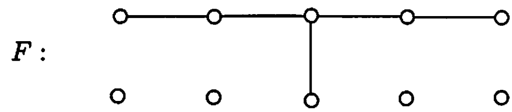


Figure 6: A factor in a detectable 3-tuple factorization of $C_5 \times K_2$

In the case of cubic graphs with detection number 3, we have the following useful observations.

Observation 3.1 *If a cubic graph G contains a detectable 3-coloring, then the order of G is at most 10.*

Observation 3.2 *If G is a connected cubic graph of order 10 with $\det(G) = 3$ and $\mathcal{F}$ is a detectable 3-tuple factorization of G, then every factor in $\mathcal{F}$ has degree sequence*

$$s : 3,2,2,1,1,1,0,0,0,0.$$

Furthermore, every factor in $\mathcal{F}$ is isomorphic to one of the graphs in Figure 7.

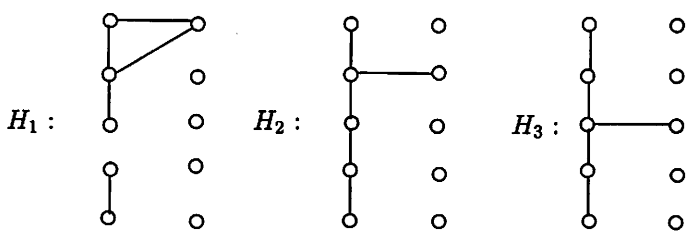
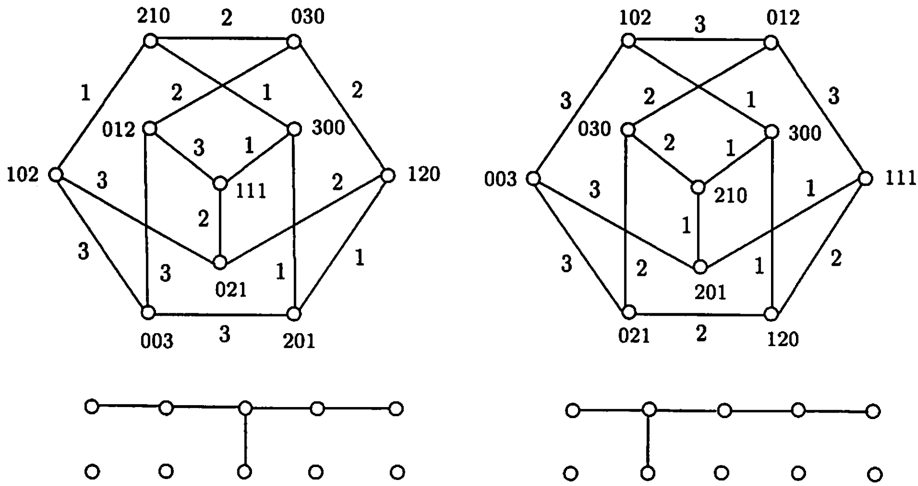


Figure 7: The possible factors in a detectable 3-tuple factorization of connected cubic graph of order 10

Undoubtedly, the best known cubic graph of order 10 is the Petersen graph P. Necessarily, $\det(P) \geq 3$. That $\det(P) = 3$ is verified in Figure 8, where two detectable 3-colorings of P are given. The first factorization of P (into G_1, G_2, and G_3) is not an isomorphic factorization; while the second one is an H_2-factorization, where H_2 is shown in Figure 7. In fact, H_2 is the only graph for which the Petersen graph has a detectable isomorphic



$G_1 = G_3 = H_3$ $G_2 = H_2$

Figure 8: Two minimum detectable 3-colorings of the Petersen graph and the resulting factorizations

factorization into three factors. In order to show that, we first present two lemmas.

Lemma 3.3 *There exist exactly two F-factorizations of the Petersen graph P, where F is the forest of Figure 9.*

Proof. Label the vertices of P as shown in Figure 9. Let $\mathcal{F}$ be an F-factorization of the Petersen graph P, where F is the forest of Figure 9. Since P is vertex-transitive, we may assume that r is the vertex of degree 3 in the first factor F_1 of $\mathcal{F}$. We consider two cases.

Case 1. *The vertices r, u, v, w, t and s are the nonisolated vertices of F_1.* Since x and y are adjacent, not both x and y can be vertices of degree 3 in factors in $\mathcal{F}$. Therefore, at least one of q and z has degree 3 in a factor of $\mathcal{F}$. Assume, without loss of generality, that z has degree 3 in the factor F_2 of $\mathcal{F}$.

We claim that y must have degree 1 in F_2 for assume, to the contrary, that y has degree 2 in F_2. Then F_2 contains either xy or uy. If $xy \in E(F_2)$, then F_3 contains a component isomorphic to K_2 with the edge uy and so $F_3 \not\cong F$, producing a contradiction. Thus F_2 contains uy. Necessarily then, x is the vertex of degree 3 in F_3, which implies that $qw \in E(F_2)$. However then, F_3 contains the path x, q, t, s and so $F_3 \not\cong F$, again a contradiction. Thus, as claimed, y has degree 1 in F_2.

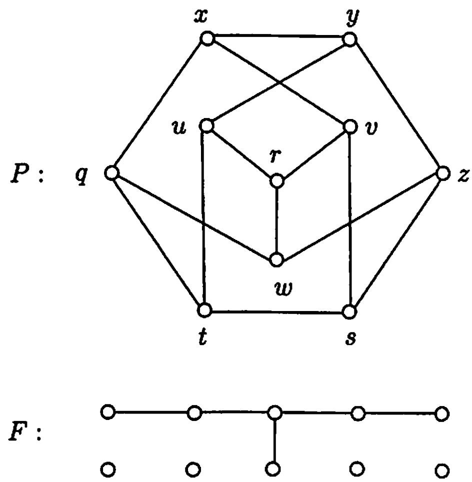


Figure 9: The Petersen graph P and a factor of P

Since y has degree 1 in F_2, the vertices w and s have degree 2 in F_2. However then, the factor F_3 is isomorphic to F, resulting in the F-factorization shown in Figure 10(a).

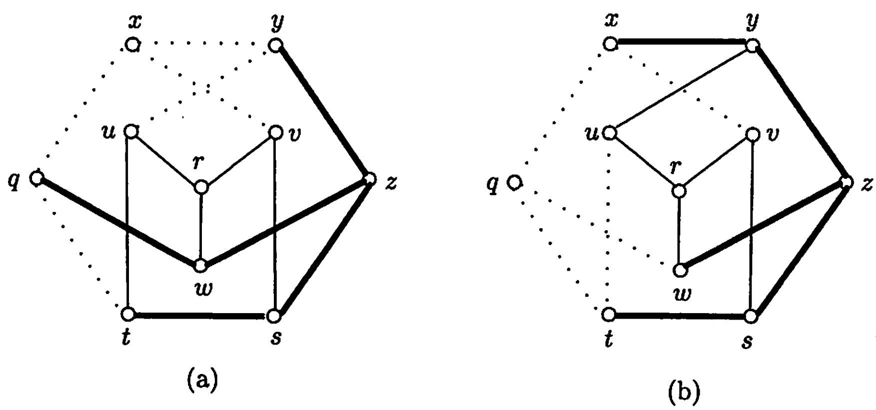


Figure 10: Two F-factorizations of the Petersen graph

Case 2. The vertices r, u, v, w, y and s are the nonisolated vertices of F_1. Observe first that if either x or t is a vertex of degree 3 in a factor in $\mathcal{F}$, then q has degree 2 in that factor and q cannot be the vertex of degree 3 in a factor of $\mathcal{F}$. Furthermore, this says that not both x and t can be the vertex of degree 3 in a factor in $\mathcal{F}$. However, if q is a vertex of degree 3 in a factor in $\mathcal{F}$, then x and t have degree 1 or 2 in that factor, implying that neither x not t is a vertex of degree 3 in a factor in $\mathcal{F}$. Consequently, z must be a vertex of degree 3 in a factor, say F_2, in $\mathcal{F}$. Since $\deg_{F_3} v = 1$ and $\deg_{F_3} y \leq 1$, it follows that x cannot be the vertex of degree 3 in F_3. Furthermore, since $\deg_{F_3} u = 1$ and $\deg_{F_3} s \leq 1$, it follows that t cannot be the vertex of degree 3 in F_3. This implies that q is the vertex of degree 3 in F_3. Thus the F-factorization $\mathcal{F}$ is uniquely determined (see Figure 10(b)).

Since the vertex w is adjacent to the three vertices of degree 3 in the F-factorization shown in Figure 10(b) and there is no such vertex for the F-factorization shown in Figure 10(a), these two factorizations are distinct. ∎

We are now prepared to show that the Petersen graph P has a unique detectable isomorphic factorization into three factors.

Theorem 3.4 *The only graph F for which the Petersen graph has a detectable F-factorization into three factors is when F is isomorphic to the graph H_2 of Figure 7.*

Proof. By Observation 3.2, the only graphs F for which the Petersen graph P could have a detectable F-factorization into three factors are the three graphs of Figure 7. Since P is triangle-free, P cannot have an H_1-factorization. We have seen in Figure 8 that there is a detectable H_2-factorization of P. By Lemma 3.3, there are exactly two distinct H_3-factorizations $\mathcal{F}_1$ and $\mathcal{F}_2$ of the Petersen graph, where $\mathcal{F}_1$ is the F-factorization described in Figure 10(a) and $\mathcal{F}_2$ is the F-factorization described in in Figure 10(b). In $\mathcal{F}_1$, u and v have the same color codes; while in $\mathcal{F}_2$, y and s have the same color codes. Therefore, neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ is detectable and so the graph H_2 of Figure 7 is the only graph F for which P has a detectable F-factorization into three factors. ∎

The graphs G and H in Figure 11 are also cubic graphs of order 10. The detection numbers of both G and H are also 3, as is shown in Figure 11. The resulting detectable 3-tuple factorization of G is an H_3-factorization, where H_3 is the forest in Figure 7; while the resulting detectable 3-tuple factorization of H is an H_1-factorization, where H_1 is the graph in Figure 7.

Another cubic graph G of order 10 is shown in Figure 12. That $\det(G) = 3$ is shown by the factorization $\mathcal{F}'$ of G, where $\mathcal{F}' = \{F_1, F_2, F_3\}$. There is no detectable isomorphic factorization of G into three factors, however.

Proposition 3.5 *There is no detectable isomorphic factorization of the connected cubic graph G of Figure 12 into three factors.*

Proof. Suppose that there is such a detectable F-factorization $\mathcal{F}$. If F is a forest, then either $F \cong H_2$ or $F \cong H_3$ of Figure 7. Then the bridge e of

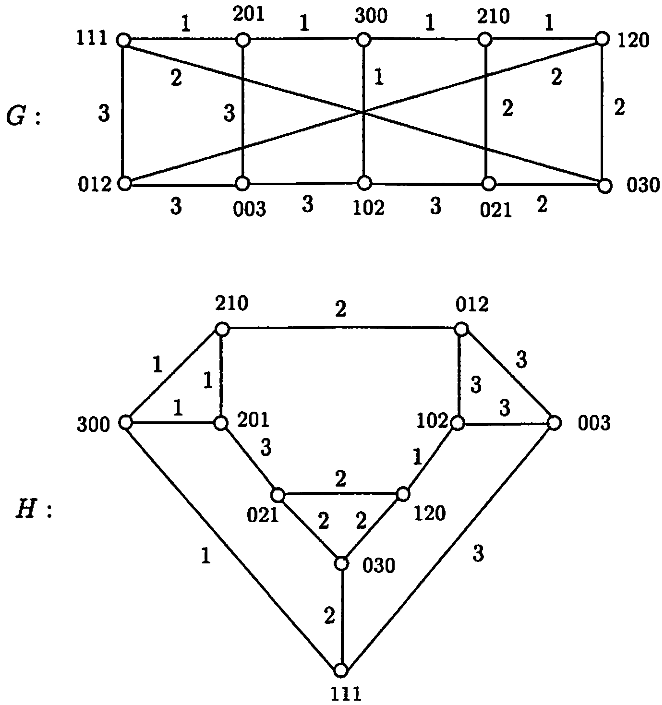


Figure 11: Detectable 3-colorings of two cubic graphs of order 10

G belongs to one factor. Since the nontrivial component T of F has order 6, T is not a subgraph of $G - e$. If $F \cong H_1$ of Figure 7, then the bridge e of G must be the component of some factor in $\mathcal{F}$ that is isomorphic to K_2. However, if G_1 denotes the component of order 4 and size 4 in F, then one of the components of $G - e$ must contain two edge-disjoint copies of G_1. However, each such component has size 7 and so this is impossible. ∎

By Proposition 3.5, the connected cubic graph G of Figure 12 has no detectable isomorphic factorization into three factors. On the other hand, every factor F of G is isomorphic to one of the graphs in Figure 7. Thus exactly two factors in every detectable 3-tuple factorization of the graph G in Figure 12 are isomorphic. This is not the case for the connected cubic graph H of Figure 13. The detectable 3-coloring of the the graph H shown Figure 13 results in three factors, no two of which are are isomorphic. Necessarily, these three factors are the three graphs in Figure 7. Such a factorization is called a *irregular factorization* of a graph. Therefore, the graph H of Figure 13 has an irregular 3-tuple detectable factorization;

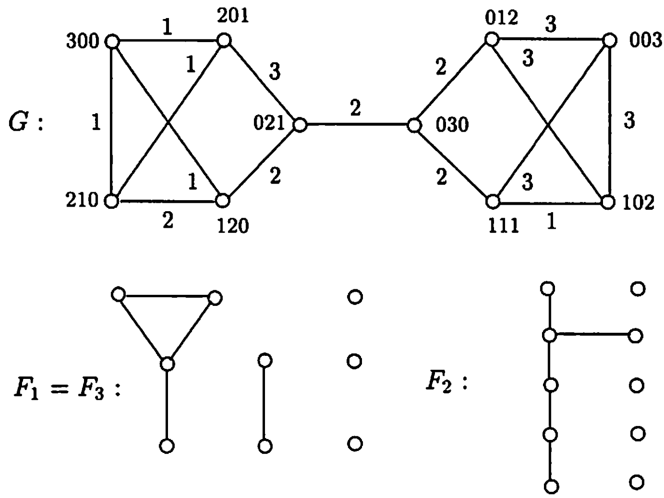


Figure 12: Another cubic bipartite graph of order 10
having detection number 3

while the Petersen graph and the graph G of Figure 12 do not have such a factorization.

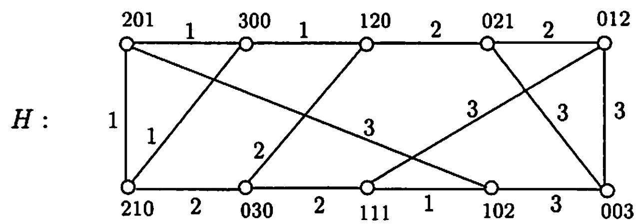
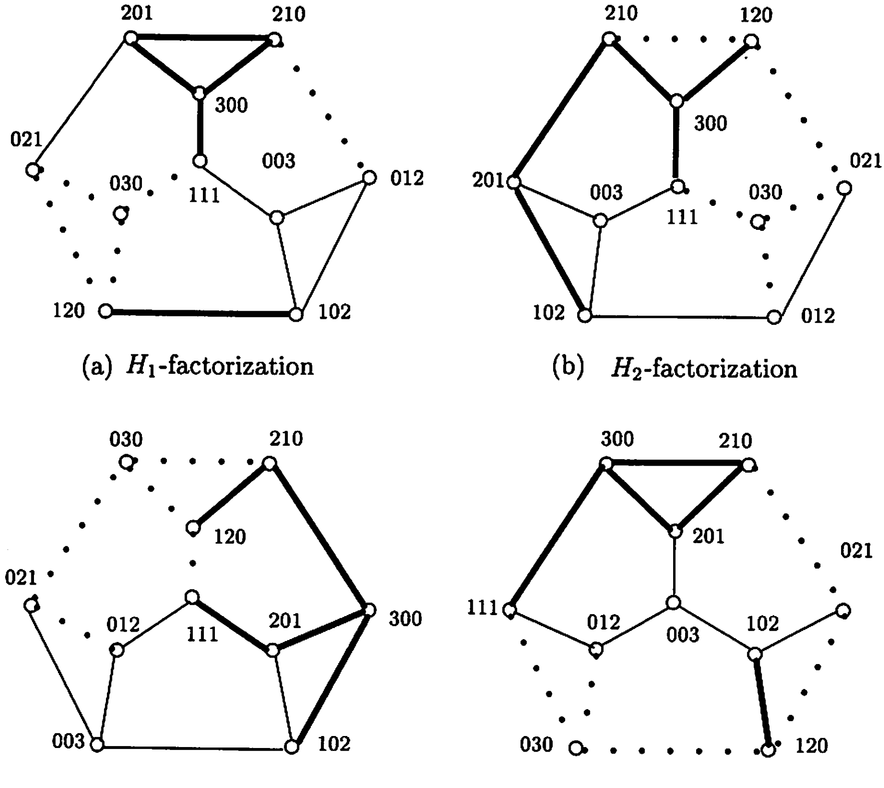


Figure 13: A cubic graph of order 10
with an irregular detectable 3-tuple factorization

We have now seen that some cubic graphs of order 10 do not have a detectable 3-tuple H_i-factorization for some graph H_i $(1 \leq i \leq 3)$ in Figure 7 and some cubic graphs of order 10 do not have a detectable irregular 3-tuple factorization. For example, the Petersen graph has neither a detectable 3-tuple H_3-factorization nor a detectable irregular 3-tuple factorization since P is triangle -free. On the other hand, Figure 14 shows four detectable 3-tuple factorizations $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$ of a cubic graph of order 10, where $\mathcal{F}_i$ is an H_i- factorization for $1 \leq i \leq 3$, while $\mathcal{F}_4$ is an irregular factorization with factors H_1, H_2, and H_3 and where the bold edges are colored 1, the dashed edges are colored 2, and the remaining edges are colored 3.



(a) H_1-factorization (b) H_2-factorization (c) H_3-factorization (d) irregular factorization

Figure 14: Four detectable 3-tuple factorizations of a cubic graph of order 10

We have seen that if G is a connected graph of order n with $\det(G) = k$, then G contains at most $\binom{r+k-1}{r}$ vertices of degree r. Therefore, in the case of cubic graphs, we have the following observation.

Observation 3.6 *If G is a connected cubic graph of order n with* $\det(G) = k$*, then*

$$n \leq \binom{k+2}{3}.$$

Not all connected cubic graphs with detection number k can have order $\binom{k+2}{3}$, however.

Theorem 3.7 *If G is a connected cubic graph of order $\binom{k+2}{3}$ with* $\det(G) = k$*, then*

$$k \equiv 2 \pmod 4 \text{ or } k \equiv 3 \pmod 4.$$

Proof. Assume, to the contrary, that there exists a connected cubic graph of order $n = \binom{k+2}{3}$ with $\det(G) = k$ such that $k \equiv 1 \pmod 4$ or $k \equiv 0 \pmod 4$. We consider these two cases.

Case 1. $k \equiv 1 \pmod 4$. Then $k = 4q + 1$ for some integer q. Observe that the order of G is

$$\begin{aligned} n &= \binom{4q+3}{3} = \frac{(4q+3)(4q+2)(4q+1)}{6} \\ &= \frac{(4q+3)(2q+1)(4q+1)}{3}, \end{aligned}$$

which is odd. This is impossible.

Case 2. $k \equiv 0 \pmod 4$. Then $k = 4q$ for some integer q. Then the order of G is

$$n = \binom{4q+2}{3} = \frac{(4q+2)(4q+1)(4q)}{6}$$

and the size of G is

$$m = \frac{3}{2}\binom{4q+2}{3} = \frac{(2q+1)(4q+1)(4q)}{2}.$$

Then G has a detectable k-tuple factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$. The size of each factor F_i $(1 \le i \le k)$ is therefore,

$$\frac{m}{k} = \frac{(2q+1)(4q+1)}{2},$$

which is not an integer, producing a contradiction. ∎

We have mentioned that the greatest possible order of a connected cubic graph with detection number 3 is 10. Furthermore, we have given several examples of such graphs. We summarize this below.

Theorem 3.8 *The largest order of a connected cubic graph with detection number* 3 *is* 10.

In fact, we know of no connected cubic graph of order 10 that has detection number different from 3, which leads to the following problem.

Problem 3.9 *Is the detection number of every cubic graph of order 10 equal to 3?*

In fact, there is a more general question.

Problem 3.10 *Do there exist connected cubic graphs of the same order having distinct detection numbers?*

By Observation 3.6 , if G is a connected cubic graph of order n with detection number 4, then $n \leq 20$. By Theorem 3.7, however, there is no a connected cubic graph of order 20 having detection number 4.

Theorem 3.11 *The largest order of a connected cubic graph with detection number* 4 *is* 18.

Proof. It suffices to give an example of a connected cubic graph of order 18 with detection number 4. Let $G \cong C_9 \times K_2$. Since $\det(G) \geq 4$, we need only show that there is a detectable 4-coloring of G. One such coloring is shown in Figure 15. ∎

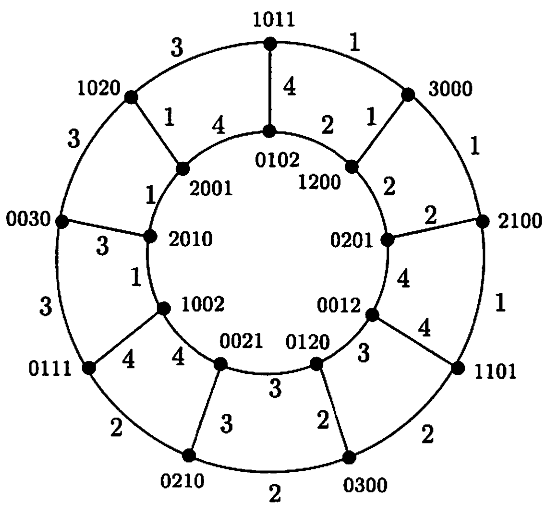


Figure 15: A detectable 4-coloring of $C_9 \times K_2$

We now turn to the problem of finding the largest order of a connected cubic graph with detection number 5. By Observation 3.6 and Theorem 3.7, the largest order cannot exceed 34.

Theorem 3.12 *The largest order of a connected cubic graph with detection number* 5 *is* 32.

Proof. In a detectable 5-coloring of a connected cubic graph of order n, exactly n of the following 35 color codes must occur:

30000	21000	10020	11100
03000	20100	01020	11010
00300	20010	00120	11001
00030	20001	00021	10101
00003	12000	10002	10110
	02100	01002	10011
	02010	00102	01110
	02001	00012	01101
	10200		01011
	01200		00111
	00210		
	00201		

At most 34 of these can be used. Assume, to the contrary, that exactly 34 of these are used in a detectable 5-coloring of a connected cubic graph G. Hence there is one color code that is not used. Since every color code contains at least two 0s, we may assume, without loss of generality, that the color code that is not used has 0 in its first coordinate. In the resulting detectable factorization $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\}$ of G, the degree sequence of F_1 is

$$s : 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$$

followed by 19 0s. However, this says that F_1 contains an odd number of odd vertices, which is impossible. Consequently, the maximum order of a connected cubic graph of with detection number 5 is at most 32.

It remains to show that there exists a connected cubic graph of order 32 with detection number 5. Let $G \cong C_{16} \times K_2$. Since $32 > 18$, it follows that $\det(G) \geq 5$. Therefore, we need only show that there is a detectable 5-coloring of G. One such coloring is shown in Figure 16. Therefore, $\det(G) = 5$. ■

We closed with a final question.

Problem 3.13 *For each integer $k \geq 6$, what is the largest integer $f(k)$ for which there exists a connected cubic graph of order $f(k)$ with detection number k?*

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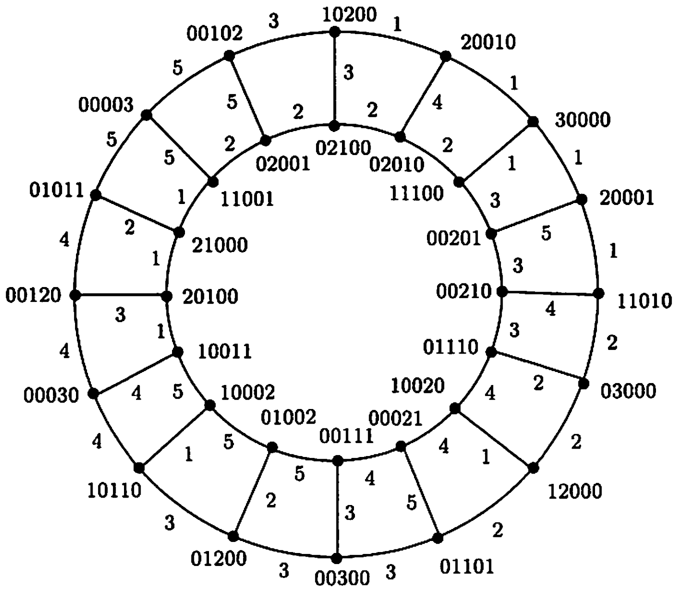

Figure 16: A detectable 5-coloring of $C_{16} \times K_2$

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