Totally Magic Injections of Graphs

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Abstract

For a simple graph G consider an injection $\mu: V \cup E \to \mathbb{N}$. If for every vertex $x \in V$ we have $\mu(x) + \sum_{y\sim x} \mu(xy) = h$, and for every edge $xy \in E$ we have $\mu(x) + \mu(xy) + \mu(y) = k$, for some constants h and k, then μ is a totally magic injection (TMI) of G. Also, $m_t(G)$ is the smallest number in $\mathbb{N}$ such that there is a TMI $\mu: V \cup E \to \{1, 2, \ldots, m_t(G)\}$. Here we study TMIs and the number $m_t(G)$ for certain G. One theorem, the Star Theorem, is useful for eliminating many classes of well-known graphs that could have a TMI. For most n and n_j the following graphs do not have a TMI: every non-star tree, P_n, C_n, W_n, K_n, and $K_{n_1,n_2,\ldots,n_p}$. We determine $m_t(F)$ for every forest F that has a TMI, and $m_t(G)$ for every graph G with ≤ 6 vertices that has a TMI.

1 Totally Magic Injections

Let G be a simple graph with set of vertices $V(G)$ or V of size v, and set of edges $E(G)$ or E of size e. We use single letters for vertices, *eg.* x, and pairs of letters for edges, *eg.* xy. The notation $x \sim y$ indicates that x is adjacent to y. The degree of a vertex x is $deg(x)$. Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the natural numbers and let $[m] = \{1, 2, \ldots, m\}$.

Consider an injection

$$\mu: V \cup E \to \mathbb{N},$$

i.e., each vertex and edge of G is labeled with a *distinct* natural number. For a vertex $x \in V$ let its *weight* under μ be $wt(x)$:

$$wt(x) = \mu(x) + \sum_{y\sim x} \mu(xy),$$

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and for an edge $xy \in E$ let its weight under μ be $wt(xy)$:

$$wt(xy) = \mu(x) + \mu(xy) + \mu(y).$$

Then μ is a *totally magic injection* (TMI) of G if for every vertex $x \in V$ and for every edge $xy \in E$ we have

$$wt(x) = h \quad \text{and} \quad wt(xy) = k,$$

for some constants h and k.

Comparing with the definition of a totally magic labeling (TML) from [2] in which the labeling is $\lambda: V \cup E \to \{1, 2, \ldots, v + e\}$, we see that our definition of a totally magic injection is a generalization of this. We use the symbol μ for a TMI to distinguish it from the usual symbol λ for a TML.

The study of graphs with a TMI is an interesting question in its own right. Also, by weakening the definition of a TML to a TMI, we hope to shed further light on the features of a graph with a TML, since every component of a graph with a TML must have a TMI, see §2.

We call h the *vertex constant* of μ, and k the *edge constant*. When necessary we use wt_μ, h_μ,... for wt, h,... to signify μ.

Suppose that in a TMI μ of G the largest label is m, then we call μ a $[m]$-*totally magic injection*, a $[m]$-TMI. We call m the *size* of the injection. A $[m]$-TMI of G is *minimal* if G does not have a $[m']$-TMI for every $m' < m$. Such a TMI is a *minimal* TMI or a *minimal* $[m]$-TMI.

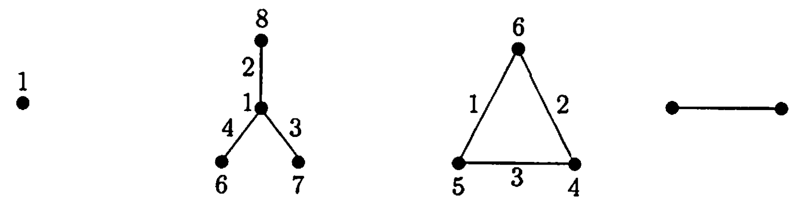
We define $m_t(G)$ to be the smallest m such that G has a $[m]$-TMI. Clearly $m_t(G) \geq v + e$ and a TML from [2] is a minimal TMI or a minimal $[v+e]$-TMI. We also define the *total deficiency* of G, $def_t(G)$, to be $m_t(G) - v - e$. If G has a TML then $m_t(G) = v + e$ and $def_t(G) = 0$.

This concludes §1. In §2 we give examples of graphs with and without a TMI. We also prove the first useful Theorems. In §3 we prove the Star Theorem which gives many 'forbidden configurations' that prevent a graph from having a TMI. Using these forbidden configurations we eliminate many classes of well-known graphs that could have a TMI. In §4 we determine $m_t(F)$ for every forest F that has a TMI. Finally, in §5, we define a TMI-survivor, and determine $m_t(G)$ for every graph G with ≤ 6 vertices that has a TMI.

2 Examples, First Theorems

If, for a graph G, an injection $\mu: V \cup E \to \mathbb{N}$ satisfies $wt(x) = h$ for a fixed h for all vertices $x \in V$ then μ is a *vertex-magic injection* of G. See §3.9 of [5], where it is shown that all graphs except those with $K_1 \cup K_1$ as components or a K_2 component have a vertex-magic injection. Similarly, if $wt(xy) = k$ for a fixed k for all edges $xy \in E$ then μ is an *edge-magic injection* of G. See §2.10 of [5], where it is shown that all graphs have an edge-magic injection. Thus a TMI of G is *both* a vertex-magic injection and an edge-magic injection. So, perhaps not surprisingly, graphs that have a TMI appear to be rare. In fact, (see §5), amongst the 208 graphs with ≤ 6 vertices, exactly 174 have a vertex-magic injection, all 208 have an edge-magic injection, but only 12 have a TMI.

Example 1 Four graphs. The first three have a TMI, the fourth does not.



K_1 (*i*) $K_{1,3}$ (*ii*) K_3 (*iii*) K_2 (*iv*)

(*i*) The isolated vertex K_1. Here $v = h = m = m_t(K_1) = 1$, and $e = def_t(K_1) = 0$. If we let $K_1 = \{x\}$ then we see that, for any fixed $h \in \mathbb{N}$, the injection given by: $\mu(x) = h$ is a TMI of K_1. Hence a given graph can have different TMIs.

(*ii*) The star $K_{1,3}$. Here $v = 4$, $e = 3$, $h = 10$, $k = 11$, and $m = 8$. This is the smallest m possible for a TMI of $K_{1,3}$ because $m_t(K_{1,3}) \geq v+e = 7$, but a [7]-TMI of $K_{1,3}$ would be a TML and $K_{1,3}$ does not have a TML, see Corollary 3.2 of [2]. So this [8]-TMI of $K_{1,3}$ is a minimal TMI, thus $m_t(K_{1,3}) = 8$ and $def_t(K_{1,3}) = 1$.

(*iii*) The triangle K_3. Here $v = e = 3$, $h = 9$, $k = 12$, and $m = v + e = 6$. So this [6]-TMI of K_3 is a TML, a minimal TMI, thus $m_t(K_3) = 6$ and $def_t(K_3) = 0$.

(*iv*) The isolated edge K_2. Call this edge xy. If it has a TMI μ then we must have $wt(x) = \mu(x) + \mu(xy) = wt(y) = \mu(y) + \mu(xy)$, *i.e.*, $\mu(x) = \mu(y)$. This is a contradiction because μ is an injection. So K_2 doesn't have a TMI.

The following Lemma will be useful in this paper.

Lemma 2.1 *Let G have a $[m]$-TMI with vertex constant h. Then*

(i) $h \geq m$,

(ii) $h = m$ if and only if G has an isolate.

Proof. (*i*) Let μ denote the $[m]$-TMI of G, so m has been used either as a vertex label or as an edge label in G. Suppose that m has been used as a vertex label on vertex x, so $\mu(x) = m$. Then $h = wt(x) = \mu(x) + \sum_{y \sim x} \mu(xy) = m + \sum_{y \sim x} \mu(xy) \geq m$. Now suppose that m has been used as an edge label on edge xz, so $\mu(xz) = m$. Then $h = wt(x) = \mu(x) + m + \sum_{\substack{y \sim x \\ y \neq z}} \mu(xy) > m$ since $\mu(x) > 0$, so $h \geq m$ also.
(*ii*) From (*i*) if $h = m$ then m must have been a vertex label, at vertex x say. Then $h = m$ if and only $\sum_{y \sim x} \mu(xy) = 0$, if and only if x is an isolate. ■

As usual, in the remainder of this section G is an arbitrary simple graph.

Theorem 2.2 *Let G have a TMI. Then every component of G has a TMI.*

Proof. Let G have a TMI μ, and let C be any component of G. Then clearly the 'restriction' of μ to C is a TMI of C. ■

The converse of this Theorem is not true. Consider the graph $H = K_1 \cup K_1 = \{x\} \cup \{y\}$, the union of two isolates. From Lemma 2.1(*ii*) if H has a $[m]$-TMI μ then both isolates in H must receive the label m. This is a contradiction because μ is an injection, so H doesn't have a TMI. Although this is a simple example it shows that, in general, a TMI of H_1 and a TMI of H_2 cannot necessarily be 'pieced together' to produce a TMI of $H_1 \cup H_2$. This example also shows that if G has $K_1 \cup K_1$ as components then it doesn't have a TMI. However K_1 as a component is allowed, we have:

Theorem 2.3 *Let G not have an isolate. Then $K_1 \cup G$ has a TMI if and only if G has a TMI.*

Proof. If $K_1 \cup G$ has a TMI then, from Theorem 2.2, so does G.

For the backward implication: Let μ, with constants h and k, denote the TMI of G, let us say that it is a $[m]$-TMI of G. Now G does not have an isolate so, from Lemma 2.1, $h > m$. Hence h has not been used as a label in G.

Let z be the isolate in $K_1 \cup G$. Now define an injection μ_{ext} of $K_1 \cup G$ as follows: $\mu_{ext}(z) = h$, and $\mu_{ext}(x) = \mu(x)$ for every vertex $x \in V(G)$ and

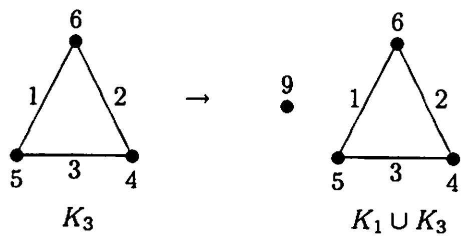
$\mu_{ext}(xy) = \mu(xy)$ for every edge $xy \in E(G)$. It is straightforward to check that μ_{ext} is a TMI of $K_1 \cup G$. ∎

Corollary 2.4 *Let G have a TMI μ and not have an isolate. Then μ can be extended to a TMI of $K_1 \cup G$, with the same vertex and edge constants.*

Proof. The TMI of $K_1 \cup G$ μ_{ext} defined above in the proof of Theorem 2.3 has the same constants as μ. ∎

Remark We will always use the notation μ_{ext} for the extension of μ from G to $K_1 \cup G$ as indicated in Corollary 2.4.

Example 2 The extension of a [6]-TMI of K_3 with $h = 9$ and $k = 12$ to a [9]-TMI of $K_1 \cup K_3$ with $h = 9$ and $k = 12$.



Recalling that K_2 doesn't have a TMI, a necessary condition for G to have a TMI is given below:

Theorem 2.5 *Let G have a TMI. Then either*

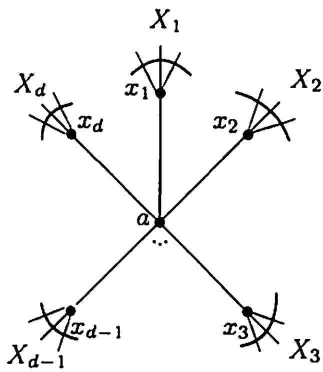
(i) G has exactly one isolate and all remaining components have ≥ 3 vertices and have a TMI, or

(ii) all components of G have ≥ 3 vertices and have a TMI. ∎

3 Star Theorem, Connected G without a TMI, Forbidden Configurations

The following Theorem has proved useful in eliminating graphs that may have a TMI. Many results produced from it appear in [2], namely Theorems 3.3, 3.5, 3.7, 3.9, and 3.11, and Corollaries 3.4, and 3.6. Thus many of the Theorems in [2] which were used to exclude the possibility of TMLs actually exclude the possibility of TMIs. We call this Theorem the 'Star Theorem'. We note that the Star Theorem could have been used to prove many of the results in [2]. We only consider connected graphs in this section.

Let G have a TMI μ with vertex and edge constants h and k. Let a be any vertex in G with neighbors $\{x_1, x_2, \ldots, x_d\}$. For every $i = 1, 2, \ldots, d$ let $X_i = \sum_{\substack{y \sim x_i \\ y \neq a}} \mu(x_i y)$, *i.e.*, let X_i be the sum of the labels of all edges incident to x_i and 'outside' the star centered at a.



Theorem 3.1 (Star Theorem) *With the above G, μ, and a we have*

$$X_1 = X_2 = \cdots = X_d = h - k + \mu(a),$$

which is constant for a fixed vertex a.

Proof. For every $i = 1, 2, \ldots, d$ we have

$$\begin{aligned} h = wt(x_i) &= \mu(x_i) + \mu(x_i a) + X_i, \\ k = wt(x_i a) &= \mu(x_i) + \mu(x_i a) + \mu(a), \end{aligned}$$

so $X_i = h - k + \mu(a)$, which is constant for a fixed vertex a. ∎

The next result is due to Brown [1].

Corollary 3.2 *Let G have a TMI μ. Let α be any edge in G then*

$$\mu(\alpha) + \sum_{\beta \sim \alpha} \mu(\beta) = 2h - k,$$

a constant, where the sum is over all edges β adjacent to α.

Proof. Here $\alpha = ab$.

Let A be the sum of the labels of all edges incident to a and outside the star centered at b, define B similarly. From above $A = h - k + \mu(b)$ and $B = h - k + \mu(a)$. So $\mu(\alpha) + \sum_{\beta \sim \alpha} \mu(\beta) = \mu(\alpha) + A + B = 2h - k$. ■

The next five Theorems are all proved using the Star Theorem, after each Theorem we give a simple Corollary.

Theorem 3.3 *Let G have a TMI. If G has a vertex of degree 1 then the component of G containing it is a star.*

Proof. Let x_1 have degree 1 and let x_1 be adjacent to a. Let a also be adjacent to the vertices $\{x_2, \dots, x_d\}$. Applying the Star Theorem at a yields $X_1 = X_2 = \cdots = X_d$. But $X_1 = 0$ because x_1 has degree 1, so $X_2 = \cdots = X_d = 0$ also, and a is the center of a star. ■

Corollary 3.4

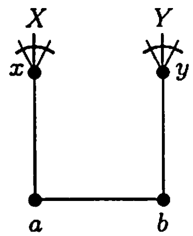
(i) Every connected graph with a vertex of degree 1 that is not a star does not have a TMI.

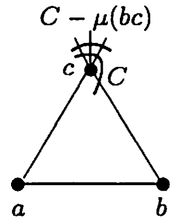
(ii) Every tree that is not a star does not have a TMI. ■

In the next section we will see that for every $n \geq 2$ the star $K_{1,n}$ does in fact have a TMI.

Theorem 3.5 *Let G have a TMI. If G has two adjacent vertices of degree 2 then the component of G containing them is a triangle.*

Proof. Let the TMI of G be μ, and let the two adjacent vertices of degree 2 be a and b and let $a \sim x$ and $b \sim y$, see (i) below:





(i) (ii)

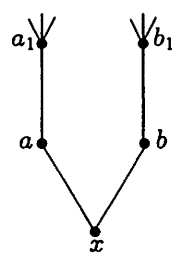
Applying the Star Theorem at a gives $X = \mu(by)$ and at b gives $Y = \mu(ax)$. Now $wt(x) = wt(y)$ gives $\mu(x) + \mu(ax) + X = \mu(y) + \mu(by) + Y$, hence $\mu(x) = \mu(y)$, and, because μ is an injection, we have $x = y$. So we have (ii) above, where we denote vertex $x = y$ by c. Now applying the Star Theorem at a again gives $C = \mu(bc)$, so $C - \mu(bc) = 0$. But $C - \mu(bc)$ is the sum of the edge labels of all edges incident to c and outside triangle abc, thus abc is a triangle component. ∎

For $n \geq 1$ let P_n denote the path on n vertices, and for $n \geq 3$ let C_n denote the cycle on n vertices.

Corollary 3.6 *For every $n \geq 4$ the path P_n and the cycle C_n does not have a TMI.* ∎

Theorem 3.7 *Suppose that G contains two vertices with a common neighbor. If these two vertices each have degree 2, or are adjacent and each have degree 3, then G does not have a TMI.*

Proof. Let the two vertices be a and b with common neighbor x. For the case in which $deg(a) = deg(b) = 2$ let $a \sim a_1$ and $b \sim b_1$.



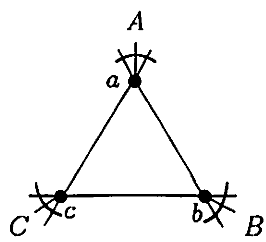
Suppose that G has a TMI μ. Applying the Star Theorem at x gives $\mu(aa_1) = \mu(bb_1)$, a contradiction. Similarly if $deg(a) = deg(b) = 3$ and $a \sim b$. ∎

For $n \geq 3$ let W_n denote the wheel with one vertex in the center and n vertices on the rim, and let F_n denote the fan obtained from W_n by removing one rim edge. By using Theorem 3.7 with a and b as rim vertices and x as the center we have:

Corollary 3.8 *For every $n \geq 3$ the wheel W_n and the fan F_n does not have a TMI.* ∎

Theorem 3.9 *Let G have a TMI, and let G contain a triangle. Then the sum of the labels of all edges outside the triangle and incident with any one vertex of the triangle is the same, whichever vertex of the triangle is chosen.*

Proof. Let the TMI of G be μ, and let the three vertices of a triangle be a, b, and c. Let A be the sum of the labels of all edges incident to a and outside the triangle, *i.e.*, except for edges ab and ac; define B and C similarly.

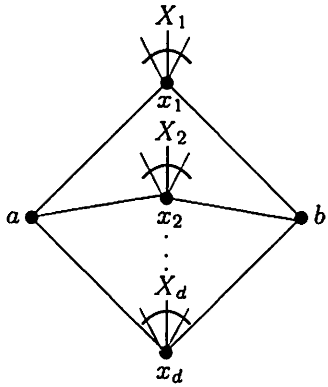


Applying the Star Theorem at a gives $B + \mu(bc) = C + \mu(bc)$, so $B = C$; similarly applying it at b gives $A = C$. Hence $A = B = C$. ∎

Corollary 3.10 *Any graph that contains a triangle with exactly one or exactly two vertices of degree 2 does not have a TMI.* ∎

Theorem 3.11 *Let G have two vertices that are each adjacent to exactly the same set of other vertices of size ≥ 2. Then G does not have a TMI. (The two vertices themselves may be adjacent or non-adjacent.)*

Proof. Consider the case when the two vertices a and b are non-adjacent, let them both be adjacent to $\{x_1, x_2, \ldots, x_d\}$ where $d \geq 2$, and to no other vertices.



For $i = 1, 2, \ldots, d$ let X_i be the sum of the labels of all edges incident to x_i and outside the neighborhood graph shown above, *i.e.*, except edges $x_i a$ and $x_i b$. Applying the Star Theorem to a gives $X_i + \mu(x_i b) = h - k + \mu(a)$, and to b gives $X_i + \mu(x_i a) = h - k + \mu(b)$. So $\mu(x_i a) - \mu(x_i b) = \mu(b) - \mu(a) \neq 0$ for every i. Now $wt(a) = wt(b)$, *i.e.*, $\mu(a) + \sum_{i=1}^{d} \mu(x_i a) = \mu(b) + \sum_{i=1}^{d} \mu(x_i b)$. So $\sum_{i=1}^{d} (\mu(x_i a) - \mu(x_i b)) = \mu(b) - \mu(a)$, *i.e.*, $d\,(\mu(b) - \mu(a)) = \mu(b) - \mu(a)$, a contradiction because $d \geq 2$. The proof is similar if $a \sim b$. ■

For $n \geq 1$ let K_n denote the complete graph with n vertices, and for $p \geq 2$ let $K_{n_1, n_2, \ldots, n_p}$ denote the complete p-partite graph with $n_j \geq 1$ vertices in part j for every $j = 1, 2, \ldots, p$.

Corollary 3.12

(i) For every $n \geq 4$ the complete graph K_n does not have a TMI.

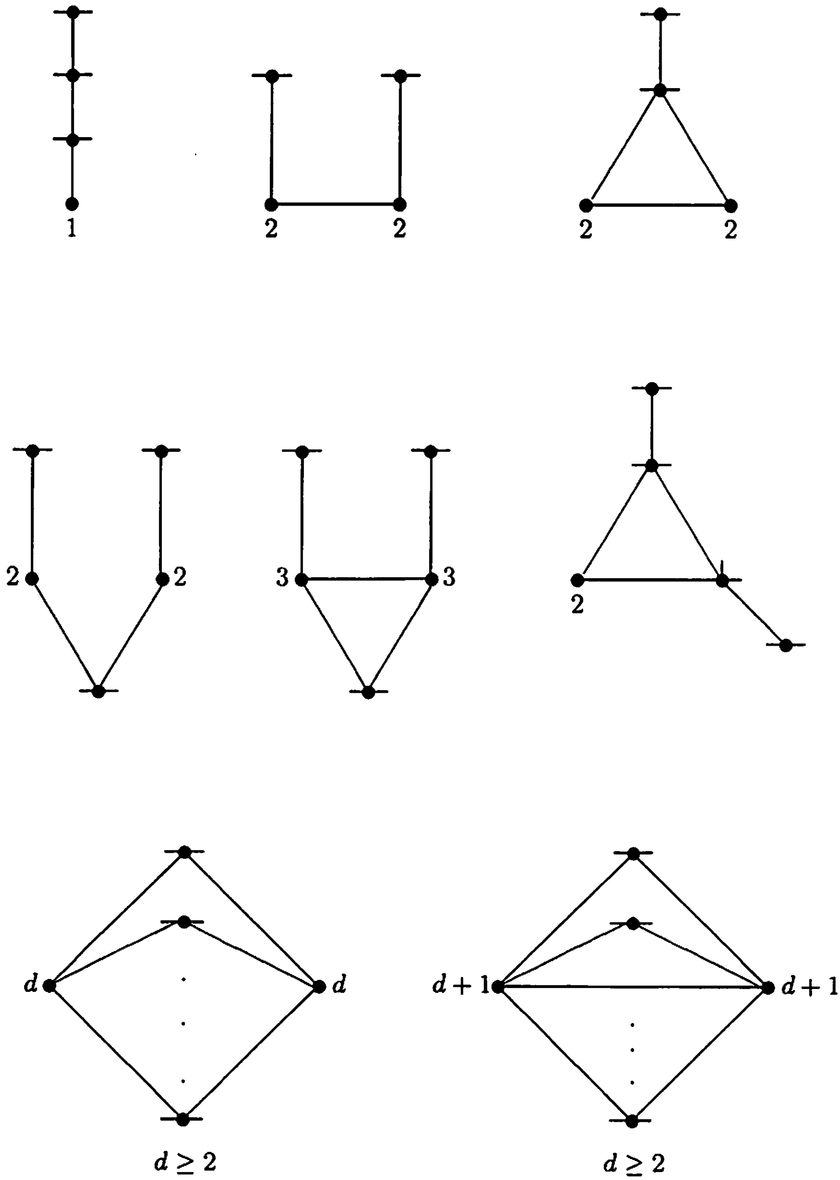
(ii) $p = 2$. For every pair $\{n_1, n_2\}$ with both $n_1 \geq 2$ and $n_2 \geq 2$ the complete bipartite graph K_{n_1, n_2} does not have a TMI.

(iii) $p \geq 3$. For every p-set $\{n_1, n_2, \ldots, n_p\}$ with at least one $n_j \geq 2$ the complete p-partite graph $K_{n_1, n_2, \ldots, n_p}$ does not have a TMI. ■

If a graph G has one of the configurations mentioned in Theorems 3.3, 3.5, 3.7, 3.9, or 3.11 then it does not have a TMI. Thus the configurations given in these Theorems are 'forbidden configurations', they forbid G from having a TMI. Below we have shown all forbidden configurations which come from these Theorems. Here a number i next to a vertex indicates a vertex of degree exactly i, and the two short lines attached to a vertex indicate that there may or may not be further edges incident to it.

Forbidden Configurations

If a graph contains any of these configurations then it does not have a TMI.



4 Forests

Here we determine $m_t(F)$ and $def_t(F)$ for every forest F that has a TMI.

First consider the star $F = K_{1,n}$ for $n \geq 2$. From Corollary 3.2 of [2] the only star with a TML is $K_{1,2}$, hence $m_t(K_{1,2}) = 5$.

Theorem 4.1 *For $n \geq 3$ we have* $m_t(K_{1,n}) = \binom{n+2}{2} - 2.$

Proof. Let the center of $K_{1,n}$ be c, let the outer vertices be $\{x_1, x_2, \ldots, x_n\}$, and let $K_{1,n}$ have TMI μ. Without loss of generality let the n edge labels satisfy:

$$1 \leq \mu(cx_1) < \mu(cx_2) < \cdots < \mu(cx_n). \tag{1}$$

For any fixed $j = 1, 2, \ldots, n$ we have $wt(c) = wt(x_j)$, so

$$\mu(c) + \sum_{i=1}^{n} \mu(cx_i) = \mu(x_j) + \mu(cx_j),$$

and so

$$\mu(x_j) = \mu(c) + \sum_{\substack{i=1 \\ i \neq j}}^{n} \mu(cx_i). \tag{2}$$

Now, from (1) and (2), for $j = 1, 2, \ldots, n$ clearly $\mu(x_j)$ is largest when $j = 1$. That is, $\mu(x_1) > \mu(x_j)$ for every $j = 2, \ldots, n$. Also from (2) we have $\mu(x_1) > \mu(c)$, hence $\mu(x_1)$ is the largest vertex label. But again from (2) we have $\mu(x_1) > \mu(cx_n)$, so, from (1) again, $\mu(x_1)$ is larger than every edge label. Thus $\mu(x_1) = m$ is the largest label.

Now we minimize $m = \mu(x_1) = \mu(c) + \sum_{i=2}^{n} \mu(cx_i)$. We first determine where to place the label 1.

From (1), for each $i = 2, \ldots, n$, we have $\mu(cx_i) \geq 2$, *i.e.*, 1 is not an edge label for any edge cx_i for $i = 2, \ldots, n$. Hence, to minimize m, we first let $\mu(c) = 1$. Now if $\mu(cx_2) = 2$ then $\mu(cx_1) = 1$, a contradiction. So we finish the minimization of m by letting $\mu(cx_2) = 3$, $\mu(cx_3) = 4$, ..., *i.e.*, by letting $\mu(cx_i) = i + 1$ for each $i = 2, \ldots, n$. Hence the minimum m is $m = 1 + 3 + \cdots + (n+1) = \binom{n+2}{2} - 2$.

We now let $\mu(cx_1) = 2$ and $\mu(x_i) = \binom{n+2}{2} - (i+1)$ for each $i = 2, \ldots, n$, so all vertices and edges have been labeled.

The labels $\{1, 2, \ldots, n+1, \binom{n+2}{2} - (n+1), \binom{n+2}{2} - n, \ldots, \binom{n+2}{2} - 2\}$ are all distinct provided that $\binom{n+2}{2} - (n+1) > n+1$, *i.e.*, provided that $n \geq 3$. It is straightforward to check that this labeling is a TMI of $K_{1,n}$, (with $h = \binom{n+2}{2}$ and $k = \binom{n+2}{2} + 1$), and is therefore a minimal TMI. ∎

Corollary 4.2 *For $n \geq 3$ we have $m_t(K_1 \cup K_{1,n}) = \binom{n+2}{2}$.*

Proof. The above minimal TMI μ of $K_{1,n}$ has $h = \binom{n+2}{2}$. Now apply Corollary 2.4 with $G = K_{1,n}$ to obtain the TMI μ_{ext} of $K_1 \cup K_{1,n}$ with $m_{\mu_{ext}} = \binom{n+2}{2}$.

Now let μ_1 be an arbitrary TMI of $K_1 \cup K_{1,n}$. Then, from Lemma 2.1(*ii*), since $K_1 \cup K_{1,n}$ has an isolate, we have $m_{\mu_1} = h_{\mu_1}$. Let c denote the center of $K_{1,n}$ then $h_{\mu_1} = wt_{\mu_1}(c) \geq \sum_{i=1}^{n+1} = \binom{n+2}{2}$. So $m_{\mu_1} \geq \binom{n+2}{2}$, and hence μ_{ext} above is a minimal TMI of $K_1 \cup K_{1,n}$. ∎

Theorem 4.3 *The only forests F that have a TMI are K_1, or $K_{1,n}$ for $n \geq 2$, or $K_1 \cup K_{1,n}$ for $n \geq 2$. Furthermore we have ($n \geq 3$):*

$$m_t(F) = \begin{cases} 1,\ 5,\ 6 & \text{if } F = K_1, K_{1,2}, K_1 \cup K_{1,2}, \\ \binom{n+2}{2} - 2 & \text{if } F = K_{1,n}, \\ \binom{n+2}{2} & \text{if } F = K_1 \cup K_{1,n}. \end{cases}$$

Proof. Let the forest F have a TMI. Every component of F has a vertex of degree 1, and so, from Theorem 3.3, every component of F is a star, and F is a union of stars. Hence, from Theorem 2.5, either (*i*) $F = K_1 \cup K_{1,n_1} \cup K_{1,n_2} \cup \cdots$ where each $n_i \geq 2$, or (*ii*) $F = K_{1,n_1} \cup K_{1,n_2} \cup \cdots$ where each $n_i \geq 2$.

Now suppose that F contains at least two stars K_{1,n_i}, where each $n_i \geq 2$. Then it is straightforward to show that the labels on the centers of each of these stars is $k - h$, a contradiction. Hence F contains at most one star $K_{1,n}$, where $n \geq 2$.

Hence either (*i*) $F = K_1$ or $F = K_1 \cup K_{1,n}$ for $n \geq 2$, or (*ii*) $F = K_{1,n}$ for $n \geq 2$. Now the three forests K_1, $K_{1,2}$, and $K_1 \cup K_{1,2}$ each have a TML. Hence $m_t(K_1) = 1$, $m_t(K_{1,2}) = 5$, and $m_t(K_1 \cup K_{1,2}) = 6$. This fact and the previous two results complete the proof. ∎

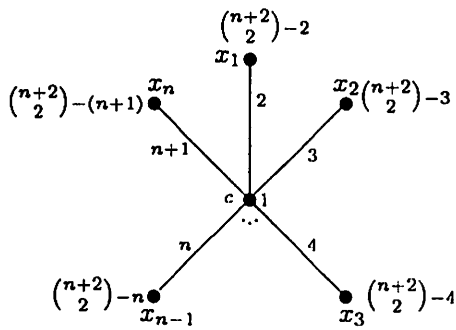
Finally, using $def_t(F) = m_t(F) - v - e$, we have:

Theorem 4.4 *Let F be a forest that has a TMI. Then the total deficiency of F is given by ($n \geq 3$):*

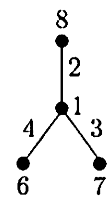
$$def_t(F) = \begin{cases} 0,\ 0,\ 0 & \text{if } F = K_1, K_{1,2}, K_1 \cup K_{1,2}, \\ \dfrac{n^2 - n - 4}{2} & \text{if } F = K_{1,n}, \\ \dfrac{n^2 - n - 2}{2} & \text{if } F = K_1 \cup K_{1,n}. \end{cases}$$

∎

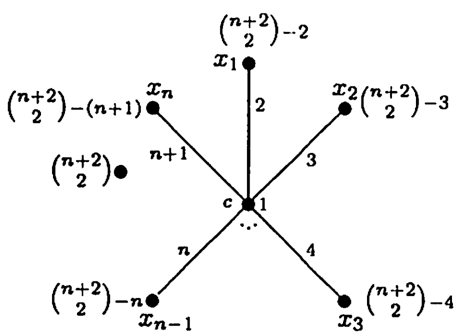
Example 3 The minimal $\left[\binom{n+2}{2}-2\right]$-TMI of $K_{1,n}$ together with the $n=3$ case, giving a minimal [8]-TMI of $K_{1,3}$, is shown first below. Then the minimal $\left[\binom{n+2}{2}\right]$-TMI of $K_1 \cup K_{1,n}$ together with the $n=3$ case, giving a minimal [10]-TMI of $K_1 \cup K_{1,3}$, is shown.



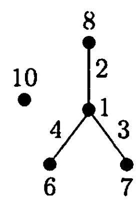
$K_{1,n}$



$K_{1,3}$



$K_1 \cup K_{1,n}$



$K_1 \cup K_{1,3}$

5 TMI-Survivor, Graphs with ≤ 6 vertices that have a TMI

In this section we define a TMI-survivor. Then we find all graphs G with ≤ 6 vertices that have a TMI, and, for each of them, the numbers $m_t(G)$ and $def_t(G)$.

Because of Theorem 2.2 when searching for graphs with a TMI we start searching amongst connected graphs.

Borrowing the idea of a survivor from §6 of [2] we define a TMI-*survivor* to be a simple graph ($\neq K_2$) that is connected and that does not contain any of the forbidden configurations given at the end of §3. (Note that a TMI-survivor need not have a TMI, however the smallest that does not have a TMI has 7 vertices.)

A list of the 143 connected graphs with ≤ 6 vertices is given on pp.7–9 of Steinbach [4]. Of these there are exactly six TMI-survivors: K_1, $K_{1,2}$, K_3, $K_{1,3}$, $K_{1,4}$, and $K_{1,5}$, and all six have a TMI.

We then used these TMI-survivors as components to find a further six disconnected graphs that have a TMI: $K_1 \cup K_{1,2}$, $K_1 \cup K_3$, $K_1 \cup K_{1,3}$, $K_1 \cup K_{1,4}$, $K_{1,2} \cup K_3$, and $K_3 \cup K_3$.

Thus, in total, there are exactly 12 graphs with ≤ 6 vertices that have a TMI. Of these 12 graphs eight are forests and have been considered in Theorems 4.3 and 4.4. The remaining four graphs G, and their numbers $m_t(G)$ and $def_t(G)$, are given in the Table below.

G	$m_t(G)$	$def_t(G)$	Comments
K_3	6	0	Example 1(iii)
$K_1 \cup K_3$	9	2	See Remark 1 below
$K_{1,2} \cup K_3$	13	2	See Remark 3 below
$K_3 \cup K_3$	14	2	See Remark 2 below

Remarks

1. $K_1 \cup K_3$.

Example 2 gives a [9]-TMI of $K_1 \cup K_3$. From Theorem 4 of [2] we know that $K_1 \cup K_3$ doesn't have a TML, *i.e.*, doesn't have a [7]-TMI. Let z denote the K_1 component in $K_1 \cup K_3$. From Lemma 2.1(ii) if $\{z\} \cup K_3$ has a [8]-TMI then $h = m = \mu(z) = 8$. So the remaining six labels for the K_3 component must come from the set $\{1, 2, 3, 4, 5, 6, 7\}$. But clearly using 6 or 7 as a label on the K_3 component with $h = 8$ gives a contradiction. Hence $K_1 \cup K_3$ doesn't have a [8]-TMI. So $m_t(K_1 \cup K_3) = 9$ and $def_t(K_1 \cup K_3) = 2$.

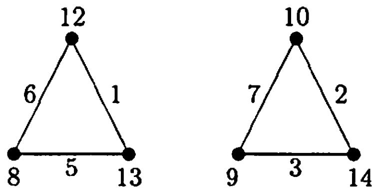
2. $K_3 \cup K_3$.

We have proved the following Theorem in [3]:

Theorem 5.1 *Let $s \geq 2$ be even.*

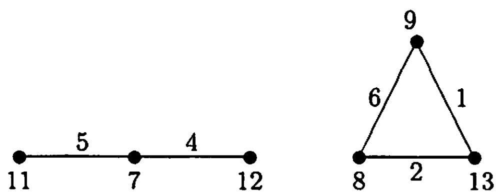
Then $m_t(sK_3) = 6s + 2$ and $def_t(sK_3) = 2$. ∎

A minimal [14]-TMI of $K_3 \cup K_3$ (with $h = 19$ and $k = 26$) is:



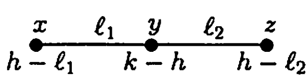
3. $K_{1,2} \cup K_3$.

Consider the above [14]-TMI of $K_3 \cup K_3$. Subtract 1 from each label, and remove the edge with label 0. This gives the [13]-TMI of $K_{1,2} \cup K_3$ (with $h = 16$ and $k = 23$) below.



To show that this is a minimal TMI we must show that $K_{1,2} \cup K_3$ doesn't have an [11]- or [12]-TMI:

Consider the 'generic' labeling of $K_{1,2} = \{x, y, z\}$ under TMI μ with constants h and k below:



That is: $\mu(x) = h - \ell_1$, $\mu(y) = k - h$, $\mu(z) = h - \ell_2$, $\mu(xy) = \ell_1$, and $\mu(yz) = \ell_2$. Now, from Corollary 3.2, we have $\ell_1 + \ell_2 = 2h - k$. So $\mu(x) + \mu(z) = k$. Thus if we include edge xz with $\mu(xz) = 0$, then $wt(xz) = k$. Then we can add 1 to all labels to arrive at a TMI of K_3. Clearly if there was an [11]- or [12]-TMI of $K_{1,2} \cup K_3$ then applying the above process to the $K_{1,2}$ component (and adding 1 to each label in the K_3 component) would produce a [12]- or [13]-TMI of $K_3 \cup K_3$, a contradiction from Remark 2. Hence the above [13]-TMI of $K_{1,2} \cup K_3$ is a minimal TMI. We have $m_t(K_{1,2} \cup K_3) = 13$ and $def_t(K_{1,2} \cup K_3) = 2$.

A summary of the situation for vertex-magic injections/edge-magic injections/TMIs for graphs with ≤ 6 vertices is given below:
(i) amongst the 143 connected graphs with ≤ 6 vertices exactly 142 (all except K_2) have a vertex-magic injection, all 143 have an edge-magic injection, but only 6 have a TMI,
(ii) amongst the 208 graphs with ≤ 6 vertices exactly 174 have a vertex-magic injection, all 208 have an edge-magic injection, but only 12 have a TMI.

References

[1] D.Brown. Personal communication, (2001).

[2] G.Exoo, A.C.H.Ling, J.P.McSorley, N.C.K.Phillips, W.D.Wallis. *Totally Magic Graphs*, Discrete Mathematics, 254, 103–113, (2002).

[3] J.P.McSorley, *Solutions of some Research Problems in Totally Magic Labelings and Totally Magic Injections of Graphs*, in preparation.

[4] P.Steinbach. *Field guide to Simple Graphs*, **1**, Design Lab, Albuquerque, (1999).

[5] W.D.Wallis. *Magic Graphs*, Birkhäuser, (2001).