On the Optimality of Belic's Lottery Designs

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Abstract

Consider a lottery scheme consisting of randomly selecting a winning t–set from a universal m–set, while a player participates in the scheme by purchasing a playing set of any number of n–sets from the universal set prior to the draw, and is awarded a prize if k or more elements in the winning t–set match those of at least one of the player's n–sets in his playing set ($1 \leq k \leq \{n, t\} \leq m$). This is called a k–prize. The player may wish to design a *smallest* playing set which guarantees the player a k–prize, no matter which winning t–set is chosen from the universal set. In this paper we consider the optimality of the 302 cardinality 7 (or less) lottery design listings in BELIC R: *Lotto Systems and Toto Systems to win Wheel Game*, [online], [cited 2003, October 31], available from: `http://www.xs4all.nl/~rbelic/`, for which $m > 20$. It is shown, by means of a computerised search technique, that 192 of these designs are optimal, whilst 78 are not, in which case we provide optimal designs. Then an additional 429 upper bounds in the tables of Belic (not necessarily of cardinality 7 or less) are improved; 126 of which are optimal. Thus, apart from the 192 designs that we show to be optimal, 204 new lottery numbers are established in this paper, and a further 304 upper bounds are improved. Finally, the optimality of 54 designs of cardinality 7 or less could not be established; however, in each of these cases a hitherto best known lower bound is provided.

Keywords: Lottery, lottery problem, lottery design.

AMS Codes: 05B05, 05B07, 05B40, 05C69, 51E10, 62K05, 62K10.

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1 Introduction

Suppose the lottery scheme $\langle m, n, t; k\rangle$ consists of randomly selecting a winning t–set w from the universal set $\mathcal{U}_m = \{1, 2, \ldots, m\}$, while a player participates in the scheme by purchasing a playing set $\mathcal{P}$ of any number of n–sets from $\mathcal{U}_m$ prior to the draw, and is awarded a prize, called a *k–prize*, if at least k elements of w match those of at least one of the player's n–sets in $\mathcal{P}$. Here we assume that $1 \leq k \leq \{n, t\} \leq m$.

Let $\Phi(\mathcal{A}, s)$ denote the set of all (unordered) s–sets from a finite set $\mathcal{A}$, so that $|\Phi(\mathcal{A}, s)| = \binom{|\mathcal{A}|}{s}$, and denote the neighbourhood set in $\langle m, n, t; k\rangle$ of any element v of $\Phi(\mathcal{U}_m, n)$ by

$$N[v] = \{\phi \in \Phi(\mathcal{U}_m, n) : \Phi(\phi, k) \cap \Phi(v, k) \neq \emptyset\}.$$

In a previous paper [3] we considered procedures for finding upper and lower bounds on solutions to the following combinatorial optimisation problem.

Definition 1 (The lottery problem) *Define a* lottery set *for* $\langle m, n, t; k\rangle$ *as a subset* $\mathcal{L}(\mathcal{U}_m, n, t; k) \subseteq \Phi(\mathcal{U}_m, n)$ *with the property that, for any element* $\phi_t \in \Phi(\mathcal{U}_m, t)$*, there exists an element* $l \in \mathcal{L}(\mathcal{U}_m, n, t; k)$ *such that* $\Phi(\phi_t, k) \cap \Phi(l, k) \neq \emptyset$*. Then the* lottery problem *is: what is the smallest possible cardinality of a lottery set* $\mathcal{L}(\mathcal{U}_m, n, t; k)$*? Denote the answer to this question by the* lottery number $L(m, n, t; k)$*. We refer to a lottery set of cardinality* $L(m, n, t; k)$ *as an* $L(m, n, t; k)$*-set for* $\langle m, n, t; k\rangle$*.* ■

Note that when $t = k$ the lottery number $L(m, n, t; k)$ reduces to the well–studied covering number $C(m, n; k)$, and that this covering number is an upper bound for the lottery number when $t \neq k$. The following growth properties of the lottery number $L(m, n, t; k)$ are due to Li [7].

Theorem 1 (Growth properties of $L(m, n, t; k)$)
(a) $L(m, n, t; k) \leq L(m + 1, n, t; k)$
(b) $L(m, n, t; k) \geq L(m, n, t + 1; k)$
(c) $L(m, n, t; k) \geq L(m + 1, n + 1, t; k)$
(d) $L(m, n, t; k) \geq L(m + 1, n + 1, t + 1; k)$
(e) $L(m, n, t; k) \geq L(m, n + 1, t; k)$
(f) $L(m, n, t; k) \leq L(m, n, t; k + 1)$
(g) $L(m, n, t; k) \leq L(m + 1, n + 1, t + 1; k + 1)$
(h) $L(m, n, t; k) \leq L(m + 1, n + 1, t; k + 1)$
(i) $L(m, n, t; k) \leq L(m + 1, n, t; k + 1)$
(j) $L(m, n, t; k) \leq L(m, n + 1, t; k + 1)$
(k) $L(m, n, t; k) \leq L(m, n, t + 1; k + 1)$
(l) $L(m, n, t; k) \leq L(m + 1, n, t + 1; k + 1)$
(m) $L(m, n, t; k) \geq L(m + 1, n, t + 1; k)$
(n) $L(m, n, t; k) \geq L(m, n + 1, t + 1; k)$ ■

Small values of lottery numbers are also known, such as the numbers $L(m,6,6;2)$ for all $6 \leq m \leq 54$, which are due to Bate & Van Rees [1]. Furthermore, listings of best known upper and lower bounds on yet undetermined values of $L(m,n,t;k)$ appear on the Internet for various combinations of the parameters m, n, t and k. For example, listings for all $1 \leq k \leq \{n,t\} \leq m \leq 20$ appear in [8], whilst listings for certain values in the ranges $m \leq 100$ and $\{n,t,k\} \leq 25$ appear in [2]. In this paper we focus on 302 small upper bounds on $L(m,n,t;k)$ not exceeding 7 in [2], which were derived via computerised heuristic search techniques, and we consider the question: "Are these bounds optimal?" We answer this question in the affirmative in 192 of these cases, and in the negative in 78 cases, and then go on to establish the true lottery number values in the latter cases. The technique that we employ in order to arrive at these answers is an exhaustive search technique, which we employed in [3] to find new, small lottery numbers and to characterise $L(m,n,t;k)$-set structures within the ranges of Li & Van Rees [8, 9]. However, in this paper we consider the optimality of bounds of the form $L(m,n,t;k) \leq \ell$ for all $\ell \leq 7$[1] within the ranges of the tables in [2], but outside the ranges considered by Li & Van Rees [8, 9] (*i.e.*, for $m > 20$), so as to avoid duplicating results.

The reader may wonder about the practical significance of small lottery numbers, since real lotteries in operation around the world [6] involve large lottery numbers, which are notoriously difficult to compute. The relevance of small lottery numbers lies in decomposition results, such as the following well-known theorem.

Theorem 2 (Upper bound decomposition)
For all $1 \leq k \leq \{n,t\} \leq m$,

$$L(m,n,t;k) \leq L(m_1,n,t_1;k) + L(m_2,n,t_2;k), \tag{1}$$

where $m_1 + m_2 = m$ *and* $t_1 + t_2 - 1 = t$.

Proof: When lottery sets for $\langle m_1,n,t_1;k\rangle$ and $\langle m_2,n,t_2;k\rangle$, from the universal sets $\mathcal{U}_{m_1} = \{1,\ldots,m_1\}$ and $\mathcal{U}_{m_2} = \{m_1+1,\ldots,m_1+m_2\}$ respectively, are conjoined, then a lottery set for $\langle m,n,t;k\rangle$ is created. This is true, because if $w \in \Phi(\mathcal{U}_{m_1} \cup \mathcal{U}_{m_2}, t)$ is the winning t-set for $\langle m,n,t;k\rangle$ and $|\mathcal{U}_{m_1} \cap w| < t_1$, then $|\mathcal{U}_{m_2} \cap w| \geq t_2$ and *vice versa*, where $t = t_1 + t_2 - 1$. ■

We use the newly established lottery numbers described above in conjunction with Theorem 2 to improve upon additional upper bounds in [2]. We also improve upper bounds by means of lottery sets that contain a disjoint n-set, by using Theorem 2 in the following way: $L(m+n,n,t+k-$

[1] We avoid covering numbers, which have been studied extensively, by considering only lottery numbers $L(m,n,t;k)$ for which $t \neq k$.

$1; k) \leq L(m, n, t; k) + L(n, n, k; k) = L(m, n, t; k) + 1$. In cases where these improved upper bounds are 7 or less, we also test for optimality. This leads to the establishment of an additional 126 new lottery numbers, and 304 improved upper bounds. There are 54 cases of upper bounds not exceeding 7 in [2], where we were not able to answer the above optimality question, in which cases we provide best known lower bounds.

2 Characterisation procedure

In [3] a search method was derived that is capable of determining lottery number lower bounds, by characterising the possible n–set overlapping structures that may be attained by lottery sets of cardinality $\ell \leq L(m, n, t; k)$ for $\langle m, n, t; k \rangle$.

This procedure was used in [3] to establish 28 new lottery numbers and to improve upon then best–known bounds for a further 29 lottery numbers. We also used the procedure in [3] to characterise all $L(m, n, t; k)$–sets for cases where $L(m, n, t; k) \leq 5$, where the parameters m, n, t and k vary within the ranges considered by Li & Van Rees [9] (*i.e.*, for $m \leq 20$). Finally, we employed the same procedure in [5] to characterise solution sets to a new incomplete version of the lottery problem, formulated in [4].

This characterisation method will be used in §3–5 to answer the optimality question posed in §1 with respect to small upper bounds in [2].

We use the same notation to capture the overlapping structure of an $L(m, n, t; k)$–set $\mathcal{L} = \{T_1, T_2, \ldots, T_L\}$ as in [3], by defining the function

$$x^{(L)}_{(t_L t_{L-1} \cdots t_2 t_1)_2} = \left| \bigcap_{i=1}^{L} \begin{cases} T_i & \text{if } t_i = 1 \\ T_i' & \text{if } t_i = 0 \end{cases} \right|,$$

where $(t_L t_{L-1} \cdots t_2 t_1)_2$ denotes the binary representation of an integer in the range $\{0, \ldots, 2^L - 1\}$ and where T_i' denotes the complement $\mathcal{U}_m \backslash T_i$. This function induces the 2^L–integer vector

$$\vec{X}^{(L)} = \left(x^{(L)}_{(000\cdots00)_2}, x^{(L)}_{(000\cdots01)_2}, \ldots, x^{(L)}_{(111\cdots11)_2} \right),$$

which represents all the information needed to describe the n–set overlapping structure of any lottery set of minimum cardinality for $\langle m, n, t; k \rangle$. The entries of the vector $\vec{X}^{(L)}$ add up to m and may be interpreted as follows:

- there are $x^{(L)}_{(000\cdots00)_2}$ elements of $\mathcal{U}_m$ contained in no n–set of $\mathcal{L}$.
- there are $x^{(L)}_{(000\cdots01)_2}$ elements of $\mathcal{U}_m$ contained in only the n–set T_1.
- there are $x^{(L)}_{(000\cdots10)_2}$ elements of $\mathcal{U}_m$ contained in only the n–set T_2.
- there are $x^{(L)}_{(000\cdots11)_2}$ elements of $\mathcal{U}_m$ contained in both T_1 and T_2, *etc.*

Sometimes it is more convenient to write the subscripts of the $x^{(L)}$ entries in decimal form. We illustrate the above method of lottery set structure encoding by means of a simple example.

$\mathcal{U}_{26}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
T_1		×	×	×	×	×	×	×																		
T_2									×	×	×	×					×	×	×							
T_3													×	×	×	×	×	×	×							
T_4																				×	×	×	×	×	×	×

1 7 4 4 3 7

Figure 2.1: Tabular representation of the $L(26, 7, 14; 4)$–set in Example 1.

Example 1 *Consider the lottery set structure* $\vec{X}^{(4)} = (1, 7, 4, 0, 4, 0, 3, 0, 7, 0, 0, 0, 0, 0, 0, 0)$ *for* $\langle 26, 7, 14; 4\rangle$, *for which it can be shown that* $L(26, 7, 14; 4) = 4$. *An instance adhering to this lottery set structure may be found by focusing on the non-zero entries in the vector* $\vec{X}^{(4)}$: $x_0^{(4)} = 1$, $x_1^{(4)} = 7$, $x_2^{(4)} = 4$, $x_4^{(4)} = 4$, $x_6^{(4)} = 3$ *and* $x_8^{(4)} = 7$. *In binary form these are* $x^{(4)}_{(0000)_2} = 1$, $x^{(4)}_{(0001)_2} = 7$, $x^{(4)}_{(0010)_2} = 4$, $x^{(4)}_{(0100)_2} = 4$, $x^{(4)}_{(0110)_2} = 3$ *and* $x^{(4)}_{(1000)_2} = 7$, *which yield the structure of all corresponding* $L(26, 7, 14; 4)$*–sets, in terms of the number of elements from the universal set* $\mathcal{U}_{26}$ *in each term of the inclusion–exclusion counting principle. The set* $\mathcal{L} = \{\{2, 3, 4, 5, 6, 7, 8\}, \{9, 10, 11, 12, 17, 18, 19\}, \{13, 14, 15, 16, 17, 18, 19\}, \{20, 21, 22, 23, 24, 25, 26\}\}$ *emerges as an example of an* $L(26, 7, 14; 4)$*–set, which is represented in tabular form in Figure 2.1.* ■

We now summarise the method described in [3] to characterise lottery set structures of cardinality $L(m, n, t; k)$ for $\langle m, n, t; k\rangle$. One method of enumerating all $L(m, n, t; k)$–set structures for $\langle m, n, t; k\rangle$, consists of constructing a rooted tree (referred to as the *lottery tree*) of evolving overlapping structures, whose nodes resemble overlap specifications similar to that of Figure 2.2. Level i of the lottery tree contains all possible (non-isomorphic) overlapping n–set structures of cardinality i and is constructed from the nodes on level $i-1$ of the lottery tree by appending 2^{i-1} integers to (*i.e.*, doubling) each of the existing vectors $\vec{X}^{(i-1)}$. These integer appendices represent all possible (new) n–set overlappings with the existing overlappings $\{T_1, T_2, \ldots, T_{i-1}\}$ (represented by nodes on level $i-1$ of the lottery tree) when adding the i–th n–set T_i (in such a manner that $|T_i \cap T_j| < n$ for any two n–sets T_i and T_j for all $j \leq i-1$).

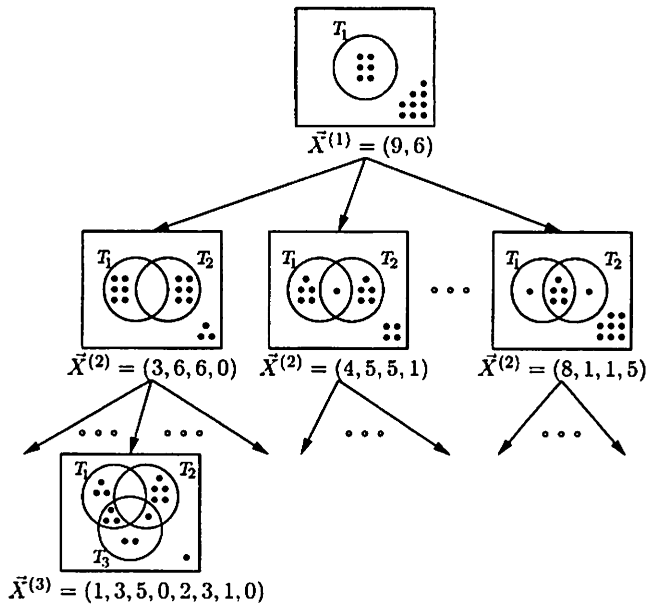

Figure 2.2: Part of the tree construction to determine all $L(15,6,6;3)$–set overlapping structures for the lottery $\langle 15,6,6;3\rangle$.

The lottery tree has $\ell + 1$ levels in total. The first level of the tree consists of the node $\vec{X}^{(1)} = (m - n, n)$ only (the root), while the nodes $\vec{X}^{(\ell)}$ on level ℓ of the tree represent potential $L(m,n,t;k)$–set structures of cardinality ℓ for $\langle m,n,t;k\rangle$. An $(\ell + 1)$–st level of nodes is added to the tree (in such a manner that $|T_{\ell+1} \cap T_j| \leq t$ for all $j \leq \ell$) in order to carry out a so-called *domination test* (*i.e.*, to test which of the nodes on level ℓ actually represent valid $L(m,n,t;k)$–sets). This domination test is achieved by testing whether *all* nodes on level $\ell+1$ overlap in at least k positions with at least one n–set of the existing ℓ n–set overlapping structure (represented by its parent node $\vec{X}^{(\ell)}$) in the tree. If this were the case, then the n–set overlapping structure represented by the parent node $\vec{X}^{(\ell)}$ would constitute a lottery set for $\langle m,n,t;k\rangle$ (and hence $L(m,n,t;k) \leq \ell$). However, if there exists at least one node on level $\ell + 1$ of the tree whose corresponding final t–set overlaps in fewer than k positions with all n–sets of the parent node overlapping structure, the parent node does not represent a lottery set. If *no* parent node represents a valid lottery set, the bound $L(m,n,t;k) > \ell$ is established.

The number of nodes on level i of the tree typically grows very rapidly as i increases, even when permutations of node structures are avoided. See

[6] for examples of the growth in the number of nodes per level and the resulting execution time required to construct the lottery tree. In the absence of pruning rules to limit the growth of the lottery tree, this complexity prohibits characterisation of most playing sets of cardinality 6 and virtually all playing sets of cardinality 7. However, with the use of a number of pruning rules, we are now able to characterise many playing sets of cardinality 6 and some playing sets of cardinality 7. However, even with these pruning rules in place, characterisation of playing sets of cardinality 8 by means of the above search technique is impossible with current computing technology. The pruning rules (implemented just after level ℓ of the tree, before the domination test) were:

(1) If $L(m-1,n,t-1;k) > L(m,n,t;k) = \ell$ and $x_0^{(\ell)} > 0$, then the structure corresponding to the vector $\vec{X}^{(\ell)}$ is *not* a lottery set, and may hence be omitted from the tree.

(2) If $\min\{x^{(\ell)}_{(100\cdots0)_2}, k-1\} + \cdots + \min\{x^{(\ell)}_{(000\cdots1)_2}, k-1\} + x_0^{(\ell)} \geq t$, then the structure corresponding to the vector $\vec{X}^{(\ell)}$ is *not* a lottery set, and may hence be omitted from the tree.

Rule (1) follows from the fact that if a specific element of $\mathcal{U}_m$ is not utilised in a lottery set of cardinality ℓ for $\langle m,n,t;k\rangle$, then $L(m-1,n,t-1;k) \leq \ell$. In rule (2) we add up the number of elements (not exceeding $k-1$ per set) that are in at most one n–set of the structure corresponding to $\vec{X}^{(\ell)}$. If there are t or more such elements, there exists a t–set having no k–intersection with any of the n–sets in the structure $\vec{X}^{(\ell)}$, and hence the structure does not represent a lottery set.

Furthermore, the following two rules were implemented just after level $\ell-1$ of the tree, and are based on the same idea as rule (2) above, in an obvious manner.

(3) If $x_0^{(\ell-1)} \geq t+n-k+1$, then all possible structures corresponding to the vector $\vec{X}^{(\ell-1)}$ are *not* a lottery set, and may hence be omitted from the tree.

(4) If $\min\{x^{(\ell-1)}_{(100\cdots0)_2}, k-1\} + \cdots + \min\{x^{(\ell-1)}_{(000\cdots1)_2}, k-1\} + x_0^{(\ell-1)} \geq n+t$, then all possible structures corresponding to the vector $\vec{X}^{(\ell-1)}$ are *not* not a lottery set, and may hence be omitted from the tree.

We do not give the full implementation details here of the characterisation procedure described above, but rather demonstrate the tree construction by means of the simple schematic representation in Figure 2.2.

3 Design bounds found to be optimal

$L(21,6,7;2)=3$	$L(21,7,7;3)=3$	$L(21,7,11;4)=3^*$	$L(21,7,13;4)=3$
$L(21,8,4;2)=3^*$	$L(21,8,5;2)=3$	$L(22,7,11;4)=3^*$	$L(22,7,13;4)=3$
$L(22,8,4;2)=3^*$	$L(22,8,5;2)=3$	$L(23,7,13;4)=3$	$L(23,8,5;2)=3$
$L(24,7,13;4)=3$	$L(24,8,5;2)=3$		
$L(21,5,10;3)=4$	$L(21,6,5;2)=4^*$	$L(21,6,6;2)=4^\dagger$	$L(21,6,9;3)=4$
$L(22,6,5;2)=4^*$	$L(22,6,6;2)=4^\dagger$	$L(22,6,7;2)=4$	$L(23,6,5;2)=4^*$
$L(23,6,6;2)=4^\dagger$	$L(23,6,7;2)=4$	$L(23,10,6;3)=4$	$L(24,6,5;2)=4^*$
$L(24,6,6;2)=4^\dagger$	$L(24,6,7;2)=4$	$L(24,10,6;3)=4$	$L(25,6,6;2)=4^\dagger$
$L(25,6,7;2)=4$	$L(25,10,6;3)=4$	$L(26,6,7;2)=4$	$L(26,7,13;4)=4$
$L(26,8,5;2)=4$	$L(27,7,13;4)=4^*$	$L(27,7,15;4)=4$	$L(27,8,5;2)=4$
$L(28,7,13;4)=4^*$	$L(28,7,15;4)=4$	$L(28,8,5;2)=4$	$L(29,7,15;4)=4^*$
$L(29,7,17;4)=4$	$L(29,8,5;2)=4$	$L(30,7,15;4)=4^*$	$L(30,7,17;4)=4$
$L(30,8,5;2)=4$	$L(31,7,17;4)=4$	$L(31,8,5;2)=4$	$L(31,8,9;3)=4$
$L(32,7,17;4)=4$	$L(32,8,5;2)=4^\ddagger$	$L(32,8,9;3)=4^\ddagger$	

Table 3.1: Lotteries $\langle m,n,t;k\rangle$, $(m > 20,\ t \neq k)$ for which the optimality of the design bound $L(m,n,t;k) \leq \ell$ in [2] was established $(\ell = 3,4)$. †Due to Bate & Van Rees [1]. ‡By Theorem 1(a). *By Theorem 1(b).

The characterisation technique described in §2 was employed to verify the optimality of 192 of the 302 design bounds of the form $L(m,n,t;k) \leq \ell$, with $\ell \leq 7$, $m > 20$ and $t \neq k$, listed in [2]. In each case the search technique was used to seek lottery sets of cardinality $\ell - 1$. If no such sets were found, the optimality of the bound $L(m,n,t;k) \leq \ell$ was verified, in the sense that a new lottery number $L(m,n,t;k) = \ell$ was established. The corresponding (new) lottery numbers are listed in Tables 3.1 and 3.2 for the cases $\ell = 3,4$ and $\ell = 5,6,7$ respectively. For the cases $\ell = 3,4$ all implementations of the characterisation tree required an execution time of less than one second on an AMD 1.8GHz processor with 256Mb of memory. The execution times (in seconds) for the cases $\ell = 5,6,7$ are listed in Table 3.2.

4 Design bounds found to be suboptimal

The characterisation technique described in §2 also resulted in the improvement of a further 78 of the 302 design bounds of the form $L(m,n,t;k) \leq \ell$, with $\ell \leq 7$, $m > 20$ and $t \neq k$, listed in [2]. These improved bounds were, in turn, used in conjunction with Theorem 2 to improve additional upper bounds listed in [2], not exceeding 7. We also improved upper bounds by means of lottery sets that contain a disjoint n-set, by using Theorem 2 in the following way: $L(m+n,n,t+k-1;k) \leq L(m,n,t;k) + L(n,n,k;k) = L(m,n,t;k) + 1$. In cases where the improved upper bound did not exceed 7, the search technique was used to seek lottery sets of cardinality i, for $i = 3,4,\ldots,\ell-1$ until the first value of i was reached for which a lottery set

Lottery number	Time	Lottery number	Time	Lottery number	Time
$L(21,4,7;2) = 5^*$	-	$L(21,4,9;2) = 5$	1	$L(21,6,4;2) = 5$	1
$L(22,4,9;2) = 5$	1	$L(22,10,5;3) = 5$	14	$L(23,4,9;2) = 5$	1
$L(23,7,8;3) = 5$	4	$L(23,7,11;4) = 5$	5	$L(23,10,5;3) = 5$	16
$L(24,7,11;4) = 5^*$	-	$L(24,7,12;4) = 5$	6	$L(25,7,12;4) = 5$	6
$L(25,8,4;2) = 5$	6	$L(26,6,6;2) = 5^\dagger$	-	$L(26,8,4;2) = 5$	6
$L(26,10,3;2) = 5$	17	$L(27,6,6;2) = 5^\dagger$	-	$L(27,6,7;2) = 5$	2
$L(27,8,4;2) = 5$	6	$L(28,6,6;2) = 5^\dagger$	-	$L(28,6,7;2) = 5^*$	-
$L(28,6,8;2) = 5$	2	$L(28,8,4;2) = 5$	6	$L(29,6,6;2) = 5^\dagger$	-
$L(29,6,7;2) = 5^*$	-	$L(29,6,8;2) = 5$	2	$L(30,6,6;2) = 5^\dagger$	-
$L(30,6,7;2) = 5^*$	-	$L(30,6,8;2) = 5$	2	$L(31,6,7;2) = 5^*$	-
$L(31,6,8;2) = 5$	2	$L(34,8,6;2) = 5$	25	$L(35,8,6;2) = 5^\ddagger$	-
$L(35,8,7;2) = 5$	36	$L(36,8,6;2) = 5^\ddagger$	-	$L(36,8,7;2) = 5^\ddagger$	-
$L(37,8,6;2) = 5^\ddagger$	-	$L(37,8,7;2) = 5^\ddagger$	-	$L(38,8,6;2) = 5^\ddagger$	-
$L(38,8,7;2) = 5^\ddagger$	-	$L(39,8,7;2) = 5^\ddagger$	-	$L(40,8,7;2) = 5^\ddagger$	-
$L(45,9,6;2) = 5$	114				
$L(22,4,7;2) = 6$	1	$L(22,5,10;3) = 6$	88	$L(22,6,4;2) = 6$	137
$L(22,7,7;3) = 6$	916	$L(22,8,3;2) = 6$	4 650	$L(23,4,7;2) = 6^\ddagger$	-
$L(23,4,8;2) = 6$	2	$L(23,5,10;3) = 6^\ddagger$	-	$L(23,8,3;2) = 6^\ddagger$	-
$L(24,4,7;2) = 6^\ddagger$	-	$L(24,4,8;2) = 6^*$	-	$L(24,4,9;2) = 6$	8
$L(24,8,3;2) = 6^\ddagger$	-	$L(25,4,8;2) = 6^\ddagger$	-	$L(25,4,9;2) = 6^\ddagger$	-
$L(25,6,5;2) = 6$	149	$L(25,9,3;2) = 6$	397	$L(26,4,9;2) = 6^\ddagger$	-
$L(26,6,5;2) = 6^\ddagger$	-	$L(26,7,12;4) = 6$	19	$L(26,8,7;3) = 6$	470
$L(26,9,3;2) = 6^\ddagger$	-	$L(27,6,5;2) = 6^\ddagger$	-	$L(29,8,4;2) = 6$	17
$L(31,7,15;4) = 6$	48	$L(32,6,7;2) = 6$	142	$L(32,7,15;4) = 6^\ddagger$	-
$L(33,6,7;2) = 6^*$	-	$L(33,6,8;2) = 6$	143	$L(33,8,5;2) = 6$	5
$L(34,6,7;2) = 6^\ddagger$	-	$L(34,6,8;2) = 6^\ddagger$	-	$L(34,8,5;2) = 6^\ddagger$	-
$L(35,6,7;2) = 6^\ddagger$	-	$L(35,6,8;2) = 6^\ddagger$	-	$L(35,8,5;2) = 6^\ddagger$	-
$L(36,6,7;2) = 6^\ddagger$	-	$L(36,6,8;2) = 6^\ddagger$	-	$L(36,8,5;2) = 6^\ddagger$	-
$L(37,6,8;2) = 6^\ddagger$	-	$L(42,8,7;2) = 6$	1	$L(43,8,7;2) = 6^\ddagger$	-
$L(43,8,8;2) = 6$	1	$L(44,8,7;2) = 6^\ddagger$	-	$L(44,8,8;2) = 6^\ddagger$	-
$L(44,8,9;2) = 6$	1	$L(45,8,7;2) = 6$	-	$L(45,8,8;2) = 6^\ddagger$	-
$L(45,8,9;2) = 6^\ddagger$	-	$L(46,8,7;2) = 6^\ddagger$	-	$L(46,8,8;2) = 6^\ddagger$	-
$L(46,8,9;2) = 6^\ddagger$	-	$L(47,8,7;2) = 6^\ddagger$	-	$L(47,8,9;2) = 6^\ddagger$	-
$L(48,8,7;2) = 6^\ddagger$	-	$L(48,8,9;2) = 6^\ddagger$	-		
$L(21,3,8;2) = 7^*$	-	$L(21,3,9;2) = 7$	1	$L(21,4,6;2) = 7$	8
$L(21,5,9;3) = 7$	196	$L(22,3,9;2) = 7^\ddagger$	-	$L(22,4,6;2) = 7^\ddagger$	-
$L(23,6,4;2) = 7$	3 285	$L(24,6,4;2) = 7^\ddagger$	-	$L(26,4,8;2) = 7$	1
$L(27,4,8;2) = 7^\ddagger$	-	$L(27,4,9;2) = 7^\ddagger$	-	$L(28,4,8;2) = 7^\ddagger$	-
$L(28,4,9;2) = 7^\ddagger$	-	$L(28,6,5;2) = 7$	5 021	$L(29,4,9;2) = 7^\ddagger$	-
$L(31,6,6;2) = 7^\dagger$	-	$L(32,6,6;2) = 7^\dagger$	-	$L(33,6,6;2) = 7^\dagger$	-
$L(37,7,6;2) = 7$	44	$L(38,6,8;2) = 7$	8	$L(38,7,6;2) = 7^\ddagger$	-
$L(39,6,8;2) = 7^\ddagger$	-	$L(40,6,8;2) = 7^\ddagger$	-	$L(41,6,8;2) = 7^\ddagger$	-
$L(41,8,6;2) = 7$	495	$L(42,6,8;2) = 7^\ddagger$	-	$L(42,8,6;2) = 7^\ddagger$	-
$L(43,8,6;2) = 7^\ddagger$	-	$L(44,8,6;2) = 7^\ddagger$	-	$L(50,8,8;2) = 7$	262
$L(51,8,8;2) = 7^\ddagger$	-	$L(51,8,9;2) = 7$	262	$L(52,8,8;2) = 7^\ddagger$	-
$L(52,8,9;2) = 7^\ddagger$	-	$L(53,8,8;2) = 7^\ddagger$	-	$L(53,8,9;2) = 7^\ddagger$	-
$L(54,8,8;2) = 7^\ddagger$	-	$L(54,8,9;2) = 7^\ddagger$	-	$L(55,8,9;2) = 7^\ddagger$	-
$L(56,8,9;2) = 7^\ddagger$	-				

Table 3.2: Lotteries $\langle m, n, t; k\rangle$, $(m > 20,\ t \neq k)$ for which the optimality of the design bound $L(m,n,t;k) \leq \ell$ in [2] was established $(\ell = 5, 6, 7)$. †Due to Bate & Van Rees [1]. ‡By Theorem 1(a). *By Theorem 1(b).

was found, thereby establishing the new lottery number $L(m, n, t; k) = i$; if no lottery set was found, $L(m, n, t; k) = \ell$. These (new) lottery numbers are listed in Table 4.3. In each case the structure of one valid $L(m, n, t; k)$-set is given, using the $\vec{X}$-vector notation of §2 (but omitting brackets and commas) or by giving two sets of lottery parameters for which lottery sets may be conjoined, as described by Theorem 2. The time (in seconds) required to implement the search technique described in §2 on an AMD 1.8GHz processor with 256Mb of memory, so as to verify the lower bound, are also given in Table 4.3 where applicable. (Implementations of the search technique for the upper bound were aborted the moment when minimal lottery designs were found; therefore execution times are not given for the upper bounds.)

5 Improved design upper bounds

In addition to the results reported above, a number of further upper bounds could be improved, using Theorem 2 in conjunction with the results of §3 and §4. These 304 additional bound improvements appear in Table 5.4, for which lottery sets may be constructed by conjoining smaller lottery sets corresponding to the lottery numbers on the right hand side of (1), as described in the proof of Theorem 2.

6 Inconclusive design bounds

Finally, the question of optimality of 54 design bounds in [2] with cardinality not exceeding 7 could not be resolved, due to unreasonably long execution times required to run the tree search procedure to optimality. In such cases the best lower bounds for which the procedure could, in fact, be run within a reasonable time span, are given in Table 6.5.

7 Conclusion

In this paper we considered the optimality of the 302 cardinality 7 (or less) lottery design listings in [2], for which $m > 20$ and which are not coverings ($t \neq k$). It was shown, by means of a computerised search technique, that 192 of these design bounds are optimal, as summarised in Tables 3.1–3.2. A further 204 design bounds were improved by providing alternative, optimal designs, as listed in Table 4.3. In this way, apart from the 192 bounds that we confirmed to be optimal, a total of 204 new lottery numbers were established. The optimality of 54 design bounds not exceeding 7 could not be established; however, in each of these cases a hitherto best known

$\langle m,n,t;k\rangle$	[2], L	One solution	$\langle m,n,t;k\rangle$	[2], L	One solution
$\langle 21,6,8;3\rangle$	$6,5^{b}$	$\langle 15,6,6;3\rangle+1$	$\langle 21,7,10;4\rangle$	$6,3^{a}$	07707000
$\langle 21,8,3;2\rangle$	$6,5^{b}$	$\langle 13,8,2;2\rangle+1$	$\langle 21,9,3;2\rangle$	$6,4^{a}$	0900003000303030
$\langle 22,6,8;3\rangle$	$8,6^{b}$	$\langle 16,6,6;3\rangle+1$	$\langle 22,6,9;3\rangle$	$5,4^{a}$	0640402060000000
$\langle 22,7,9;3\rangle$	$6,3^{a}$	27606010	$\langle 22,7,10;4\rangle$	$7,6^{c}$	$\langle 15,7,7;4\rangle+1$
$\langle 22,7,12;4\rangle$	$7,3^{a}$	27606010	$\langle 22,9,3;2\rangle$	$6,4^{a}$	0900004000404010
$\langle 22,9,4;2\rangle$	$7,3^{a}$	09404050	$\langle 23,6,9;3\rangle$	$6,4^{a}$	0650501060000000
$\langle 23,7,4;2\rangle$	$7,5^{b}$	$\langle 16,7,3;2\rangle+1$	$\langle 23,7,7;3\rangle$	$7,6^{\ddagger}$	$\langle 16,7,5;3\rangle+1$
$\langle 23,7,9;3\rangle$	$5,3^{a}$	27707000	$\langle 23,7,12;4\rangle$	$5,3^{a}$	27707000
$\langle 23,8,4;2\rangle$	$5,3^{a}$	08707010	$\langle 23,9,3;2\rangle$	$6,5^{b}$	$\langle 14,9,2;2\rangle+1$
$\langle 23,9,4;2\rangle$	$7,3^{a}$	09505040	$\langle 24,6,9;3\rangle$	$7,4^{a}$	0660600060000000
$\langle 24,7,4;2\rangle$	$7,5^{\ddagger}$	$\langle 17,7,3;2\rangle+1$	$\langle 24,7,8;3\rangle$	$6,5^{\ddagger}$	$\langle 17,7,6;3\rangle+1$
$\langle 24,7,9;3\rangle$	$5,4^{a}$	0730304070000000	$\langle 24,7,14;4\rangle$	$5,3^{a}$	47606010
$\langle 24,8,4;2\rangle$	$5,3^{a}$	08808000	$\langle 24,9,3;2\rangle$	$6,5^{\ddagger}$	$\langle 15,9,2;2\rangle+1$
$\langle 24,9,4;2\rangle$	$7,3^{a}$	09606030	$\langle 24,9,5;2\rangle$	$8,3^{a}$	09606030
$\langle 24,10,8;4\rangle$	$8,4^{a}$	$\langle 14,10,5;4\rangle+1$	$\langle 25,6,8;2\rangle$	$5,4^{a}$	3640402060000000
$\langle 25,6,14;4\rangle$	$11,4^{a}$	$\langle 19,6,11;4\rangle+1$	$\langle 25,7,4;2\rangle$	$7,6^{d}$	$\langle 18,7,3;2\rangle+1$
$\langle 25,7,8;3\rangle$	$8,5^{b}$	$\langle 18,7,6;3\rangle+1$	$\langle 25,7,9;3\rangle$	$5,4^{a}$	0740403070000000
$\langle 25,7,13;4\rangle$	$5,4^{a}$	0740403070000000	$\langle 25,7,14;4\rangle$	$45,3^{a}$	47707000
$\langle 25,8,5;2\rangle$	$4,3^{a}$	18808000	$\langle 25,9,4;2\rangle$	$7,3^{\ddagger}$	09707020
$\langle 25,9,5;2\rangle$	$10,3^{a}$	09707020	$\langle 25,10,8;4\rangle$	$8,4^{a}$	$\langle 15,10,5;4\rangle+1$
$\langle 26,6,8;2\rangle$	$5,4^{a}$	3650501060000000	$\langle 26,6,14;4\rangle$	$11,6^{e}$	$\langle 20,6,11;4\rangle+1$
$\langle 26,7,8;3\rangle$	$8,6^{f}$	$\langle 19,7,6;3\rangle+1$	$\langle 26,7,9;3\rangle$	$5,4^{a}$	0750502070000000
$\langle 26,7,14;4\rangle$	$5,4^{a}$	1740403070000000	$\langle 26,9,4;2\rangle$	$8,3^{\ddagger}$	09808010
$\langle 26,9,5;2\rangle$	$12,3^{\ddagger}$	09808010	$\langle 26,9,7;3\rangle$	$6,3^{a}$	09808010
$\langle 26,10,6;3\rangle$	$6,5^{g}$	$\langle 16,10,4;3\rangle+1$	$\langle 26,10,9;4\rangle$	$6,4^{a}$	$\langle 16,10,6;4\rangle+1$
$\langle 27,6,8;2\rangle$	$5,4^{a}$	3660600060000000	$\langle 27,7,9;3\rangle$	$6,4^{a}$	0760601070000000
$\langle 27,7,14;4\rangle$	$5,4^{a}$	1750502070000000	$\langle 27,9,4;2\rangle$	$8,3^{\ddagger}$	09909000
$\langle 27,9,5;2\rangle$	$14,3^{\ddagger}$	09909000	$\langle 27,9,7;3\rangle$	$6,3^{a}$	09909000
$\langle 28,6,11;3\rangle$	$14,5^{b}$	$\langle 22,6,9;3\rangle+1$	$\langle 28,7,9;3\rangle$	$7,4^{a}$	0770700070000000
$\langle 28,7,14;4\rangle$	$6,4^{a}$	1760601070000000	$\langle 28,7,17;4\rangle$	$4,3^{a}$	77707000
$\langle 28,9,4;2\rangle$	$9,5^{g}$	$\langle 19,9,3;2\rangle+1$	$\langle 28,9,5;2\rangle$	$15,3^{\ddagger}$	19909000
$\langle 29,7,14;4\rangle$	$7,4^{a}$	1770700070000000	$\langle 29,8,6;2\rangle$	$5,4^{a}$	1840404080000000
$\langle 29,9,4;2\rangle$	$10,5^{\ddagger}$	$\langle 20,9,3;2\rangle+1$	$\langle 29,9,5;2\rangle$	$17,4^{a}$	0920207090000000
$\langle 30,6,12;3\rangle$	$7,5^{b}$	$\langle 24,6,10;3\rangle+1$	$\langle 30,7,5;2\rangle$	$8,6^{b}$	$\langle 23,7,4;2\rangle+1$
$\langle 30,7,14;4\rangle$	$8,6^{h}$	$\langle 21,7,10;4\rangle$	$\langle 30,8,6;2\rangle$	$5,4^{a}$	1850503080000000
$\langle 30,8,7;2\rangle$	$5,4^{a}$	2840404080000000	$\langle 30,9,4;2\rangle$	$9,5^{\ddagger}$	$\langle 21,9,3;2\rangle+1$
$\langle 30,9,5;2\rangle$	$19,4^{\ddagger}$	0930306090000000	$\langle 31,6,12;3\rangle$	$8,5^{\ddagger}$	$\langle 25,6,10;3\rangle+1$
$\langle 31,7,5;2\rangle$	$8,6^{\ddagger}$	$\langle 24,7,4;2\rangle+1$	$\langle 31,7,14;4\rangle$	$9,6^{\ddagger}$	$\langle 21,7,10;4\rangle$
$\langle 31,8,6;2\rangle$	$5,4^{a}$	1860602080000000	$\langle 31,8,7;2\rangle$	$5,4^{a}$	2850503080000000
$\langle 31,9,4;2\rangle$	$9,5^{\ddagger}$	$\langle 22,9,3;2\rangle+1$	$\langle 31,9,5;2\rangle$	$21,4^{\ddagger}$	0940405090000000

Table 4.3: Lotteries $\langle m,n,t;k\rangle$, $(m>20,\ t\neq k)$ for which the optimality of the design upper bounds in [2] were found to be suboptimal. Where Theorem 2 was invoked, only one lottery of the decomposition is given, and a disjoint set is indicated with a "**+ 1**". The upper bounds listed in [2] and exact values of the lottery numbers appear in the column labelled "[2], L". ‡Lower bound by Theorem 1(a). Execution times (in seconds) to establish lower bounds are given by the following superscripts: ^{a}less than 1, ^{b}1, ^{c}977, ^{d}2, ^{e}130, ^{f}6, ^{g}3, ^{h}22.

⟨m, n, t; k⟩	[2], L	One solution	⟨m, n, t; k⟩	[2], L	One solution
⟨32, 6, 8; 2⟩	$6,5^{b}$	⟨26, 6, 7; 2⟩ + 1	⟨32, 6, 12; 3⟩	$9,7^{c}$	⟨23, 6, 9; 3⟩
⟨32, 7, 5; 2⟩	$8,7^{d}$	⟨25, 7, 4; 2⟩ + 1	⟨32, 8, 6; 2⟩	$5,4^{a}$	1870701080000000
⟨32, 8, 7; 2⟩	$5,4^{a}$	2860602080000000	⟨32, 9, 4; 2⟩	$9,6^{e}$	⟨23, 9, 3; 2⟩ + 1
⟨32, 9, 5; 2⟩	$10,4^{a}$	0950504090000000	⟨33, 6, 12; 3⟩	$10,7^{\ddagger}$	⟨24, 6, 9; 3⟩
⟨33, 7, 15; 4⟩	$8,7^{b}$	⟨21, 7, 10; 4⟩	⟨33, 7, 16; 4⟩	$11,5^{b}$	⟨26, 7, 13; 4⟩ + 1
⟨33, 7, 17; 4⟩	$6,5^{b}$	⟨26, 7, 14; 4⟩ + 1	⟨33, 8, 6; 2⟩	$5,4^{a}$	1880800080000000
⟨33, 8, 7; 2⟩	$5,4^{a}$	2870701080000000	⟨33, 9, 4; 2⟩	$9,6^{\ddagger}$	⟨24, 9, 3; 2⟩ + 1
⟨33, 9, 5; 2⟩	$9,4^{a}$	0960603090000000	⟨34, 7, 16; 4⟩	$11,5^{\ddagger}$	⟨21, 7, 10; 4⟩
⟨34, 7, 17; 4⟩	$6,5^{\ddagger}$	⟨27, 7, 14; 4⟩ + 1	⟨34, 8, 7; 2⟩	$5,4^{a}$	2880800080000000
⟨34, 9, 5; 2⟩	$10,4^{\ddagger}$	0970702090000000	⟨34, 9, 6; 2⟩	$11,4^{a}$	0970702090000000
⟨34, 10, 8; 3⟩	$11,5^{f}$	⟨24, 10, 6; 3⟩ + 1	⟨35, 7, 16; 4⟩	$11,5^{\ddagger}$	⟨21, 7, 10; 4⟩
⟨35, 7, 17; 4⟩	$8,5^{a}$	⟨28, 7, 14; 4⟩ + 1	⟨35, 9, 5; 2⟩	$10,4^{\ddagger}$	⟨26, 9, 4; 2⟩ + 1
⟨35, 9, 6; 2⟩	$11,4^{\ddagger}$	⟨26, 9, 5; 2⟩ + 1	⟨35, 10, 4; 2⟩	$7,5^{f}$	⟨25, 10, 3; 2⟩ + 1
⟨35, 10, 8; 3⟩	$11,5^{\ddagger}$	⟨25, 10, 6; 3⟩ + 1	⟨36, 7, 17; 4⟩	$8,5^{a}$	⟨29, 7, 14; 4⟩ + 1
⟨36, 9, 5; 2⟩	$10,4^{\ddagger}$	⟨27, 9, 4; 2⟩ + 1	⟨36, 9, 6; 2⟩	$11,4^{\ddagger}$	⟨27, 9, 5; 2⟩ + 1
⟨36, 10, 4; 2⟩	$7,6^{\ddagger}$	⟨26, 10, 3; 2⟩ + 1	⟨36, 10, 8; 3⟩	$11,6^{g}$	⟨26, 10, 6; 3⟩ + 1
⟨37, 8, 5; 2⟩	$8,7^{h}$	⟨21, 8, 3; 2⟩	⟨37, 9, 5; 2⟩	$10,6^{i}$	⟨24, 9, 4; 2⟩
⟨37, 9, 6; 2⟩	$11,4^{\ddagger}$	⟨28, 9, 5; 2⟩ + 1	⟨38, 9, 5; 2⟩	$10,6^{\ddagger}$	⟨25, 9, 4; 2⟩
⟨38, 9, 6; 2⟩	$12,5^{j}$	⟨29, 9, 5; 2⟩ + 1	⟨39, 8, 6; 2⟩	$7,5^{b}$	⟨31, 8, 5; 2⟩ + 1
⟨39, 8, 8; 2⟩	$6,5^{b}$	⟨31, 8, 7; 2⟩ + 1	⟨39, 9, 5; 2⟩	$11,6^{\ddagger}$	⟨30, 9, 4; 2⟩ + 1
⟨39, 9, 6; 2⟩	$12,5^{\ddagger}$	⟨30, 9, 5; 2⟩ + 1	⟨40, 8, 6; 2⟩	$7,5^{\ddagger}$	⟨32, 8, 5; 2⟩ + 1
⟨40, 8, 8; 2⟩	$6,5^{\ddagger}$	⟨32, 8, 7; 2⟩ + 1	⟨40, 8, 9; 2⟩	$6,5^{b}$	⟨32, 8, 8; 2⟩ + 1
⟨40, 9, 5; 2⟩	$11,6^{\ddagger}$	⟨31, 9, 4; 2⟩ + 1	⟨40, 9, 6; 2⟩	$13,5^{\ddagger}$	⟨31, 9, 5; 2⟩ + 1
⟨41, 8, 7; 2⟩	$6,5^{b}$	⟨33, 8, 6; 2⟩ + 1	⟨41, 8, 8; 2⟩	$6,5^{\ddagger}$	⟨31, 8, 7; 2⟩ + 1
⟨41, 8, 9; 2⟩	$6,5^{\ddagger}$	⟨31, 8, 8; 2⟩ + 1	⟨41, 9, 6; 2⟩	$14,5^{\ddagger}$	⟨32, 9, 5; 2⟩ + 1
⟨42, 8, 8; 2⟩	$6,5^{\ddagger}$	⟨34, 8, 7; 2⟩ + 1	⟨42, 8, 9; 2⟩	$8,5^{\ddagger}$	⟨34, 8, 8; 2⟩ + 1
⟨42, 9, 6; 2⟩	$13,5^{\ddagger}$	⟨33, 9, 5; 2⟩ + 1	⟨43, 8, 9; 2⟩	$6,5^{\ddagger}$	⟨35, 8, 8; 2⟩ + 1
⟨43, 9, 6; 2⟩	$13,5^{\ddagger}$	⟨34, 9, 5; 2⟩ + 1	⟨44, 9, 6; 2⟩	$13,5^{\ddagger}$	⟨35, 9, 5; 2⟩ + 1
⟨44, 9, 7; 2⟩	$14,5^{j}$	⟨35, 9, 6; 2⟩ + 1	⟨45, 9, 7; 2⟩	$14,5^{\ddagger}$	⟨36, 9, 6; 2⟩ + 1
⟨45, 10, 10; 3⟩	$15,6^{k}$	⟨35, 10, 8; 3⟩ + 1	⟨46, 9, 6; 2⟩	$13,7^{l}$	⟨25, 9, 4; 2⟩
⟨46, 9, 7; 2⟩	$14,5^{\ddagger}$	⟨37, 9, 6; 2⟩ + 1	⟨46, 9, 8; 2⟩	$15,5^{\ddagger}$	⟨37, 9, 7; 2⟩ + 1
⟨46, 10, 11; 3⟩	$14,5^{m}$	⟨36, 10, 9; 3⟩ + 1	⟨47, 8, 8; 2⟩	$7,6^{\ddagger}$	⟨39, 8, 7; 2⟩ + 1
⟨47, 9, 6; 2⟩	$13,7^{\ddagger}$	⟨25, 9, 4; 2⟩	⟨47, 9, 7; 2⟩	$14,6^{b}$	⟨25, 9, 4; 2⟩
⟨47, 9, 8; 2⟩	$15,5^{\ddagger}$	⟨38, 9, 7; 2⟩ + 1	⟨47, 10, 11; 3⟩	$14,5^{\ddagger}$	⟨37, 10, 9; 3⟩ + 1
⟨48, 8, 8; 2⟩	$7,6^{\ddagger}$	⟨40, 8, 7; 2⟩ + 1	⟨48, 9, 6; 2⟩	$13,7^{\ddagger}$	⟨39, 9, 5; 2⟩ + 1
⟨48, 9, 7; 2⟩	$14,6^{\ddagger}$	⟨25, 9, 4; 2⟩	⟨48, 9, 8; 2⟩	$15,6^{b}$	⟨39, 9, 7; 2⟩ + 1
⟨48, 9, 9; 2⟩	$16,5^{j}$	⟨39, 9, 8; 2⟩ + 1	⟨48, 10, 11; 3⟩	$14,5^{\ddagger}$	⟨38, 10, 9; 3⟩ + 1
⟨49, 8, 8; 2⟩	$7,6^{\ddagger}$	⟨41, 8, 7; 2⟩ + 1	⟨49, 8, 9; 2⟩	$7,6^{\ddagger}$	⟨41, 8, 8; 2⟩ + 1
⟨49, 9, 6; 2⟩	$13,7^{\ddagger}$	⟨40, 9, 5; 2⟩ + 1	⟨49, 9, 7; 2⟩	$14,6^{\ddagger}$	⟨25, 9, 4; 2⟩
⟨49, 9, 8; 2⟩	$15,6^{\ddagger}$	⟨25, 9, 4; 2⟩	⟨49, 9, 9; 2⟩	$18,6^{\ddagger}$	⟨25, 9, 5; 2⟩

Table 4.3: (continued) Lotteries $\langle m, n, t; k\rangle$, $(m > 20,\ t \neq k)$ for which the optimality of the design upper bounds in [2] were found to be suboptimal. Where Theorem 2 was invoked, only one lottery of the decomposition is given, and a disjoint set is indicated with a " **+ 1**". The upper bounds listed in [2] and exact values of the lottery numbers appear in the column labelled "[2], L". ‡Lower bound by Theorem 1(a). Execution times (in seconds) to establish lower bounds are given by the following superscripts: ^{a}less than 1, ^{b}1, ^{c}20, ^{d}770, ^{e}104, ^{f}3, ^{g}700, ^{h}5 668, ^{i}13, ^{j}2, ^{k}8, ^{l}3 201, ^{m}4.

$\langle m,n,t;k\rangle$	[2], L	One solution	$\langle m,n,t;k\rangle$	[2], L	One solution
$\langle 49,10,11;3\rangle$	$14,5^{\ddagger}$	$\langle 39,10,9;3\rangle+1$	$\langle 50,8,9;2\rangle$	$7,6^{\ddagger}$	$\langle 42,8,8;2\rangle+1$
$\langle 50,9,7;2\rangle$	$14,6^{\ddagger}$	$\langle 25,9,4;2\rangle$	$\langle 50,9,8;2\rangle$	$16,6^{\ddagger}$	$\langle 25,9,4;2\rangle$
$\langle 50,9,9;2\rangle$	$20,6^{\ddagger}$	$\langle 25,9,5;2\rangle$	$\langle 50,10,11;3\rangle$	$14,5^{\ddagger}$	$\langle 40,10,9;3\rangle+1$
$\langle 51,9,7;2\rangle$	$15,6^{\ddagger}$	$\langle 27,9,4;2\rangle$	$\langle 51,9,8;2\rangle$	$16,6^{\ddagger}$	$\langle 27,9,4;2\rangle$
$\langle 51,9,9;2\rangle$	$22,6^{\ddagger}$	$\langle 27,9,5;2\rangle$	$\langle 52,9,7;2\rangle$	$15,6^{\ddagger}$	$\langle 27,9,4;2\rangle$
$\langle 52,9,8;2\rangle$	$17,6^{\ddagger}$	$\langle 27,9,4;2\rangle$	$\langle 52,9,9;2\rangle$	$23,6^{\ddagger}$	$\langle 27,9,5;2\rangle$
$\langle 53,9,7;2\rangle$	$16,6^{\ddagger}$	$\langle 44,9,6;2\rangle+1$	$\langle 53,9,8;2\rangle$	$18,6^{\ddagger}$	$\langle 44,9,7;2\rangle+1$
$\langle 53,9,9;2\rangle$	$25,6^{\ddagger}$	$\langle 44,9,8;2\rangle+1$	$\langle 54,9,7;2\rangle$	$16,6^{\ddagger}$	$\langle 45,9,6;2\rangle+1$
$\langle 54,9,8;2\rangle$	$17,6^{\ddagger}$	$\langle 45,9,7;2\rangle+1$	$\langle 54,9,9;2\rangle$	$27,6^{\ddagger}$	$\langle 45,9,8;2\rangle+1$
$\langle 55,8,8;2\rangle$	$9,7^{\ddagger}$	$\langle 32,8,5;2\rangle$	$\langle 55,9,8;2\rangle$	$17,6^{\ddagger}$	$\langle 46,9,7;2\rangle+1$
$\langle 55,9,9;2\rangle$	$29,6^{\ddagger}$	$\langle 46,9,8;2\rangle+1$	$\langle 56,8,8;2\rangle$	$9,7^{\ddagger}$	$\langle 48,8,7;2\rangle+1$
$\langle 56,9,8;2\rangle$	$17,7^{a}$	$\langle 34,9,5;2\rangle$	$\langle 56,9,9;2\rangle$	$18,6^{\ddagger}$	$\langle 34,9,6;2\rangle$
$\langle 57,8,9;2\rangle$	$8,7^{\ddagger}$	$\langle 33,8,6;2\rangle$	$\langle 57,9,8;2\rangle$	$17,7^{\ddagger}$	$\langle 35,9,5;2\rangle$
$\langle 57,9,9;2\rangle$	$18,7^{b}$	$\langle 35,9,6;2\rangle$	$\langle 57,10,13;3\rangle$	$18,6^{c}$	$\langle 47,10,11;3\rangle+1$
$\langle 58,9,8;2\rangle$	$17,7^{\ddagger}$	$\langle 36,9,5;2\rangle$	$\langle 58,9,9;2\rangle$	$18,7^{\ddagger}$	$\langle 36,9,6;2\rangle$
$\langle 58,10,13;3\rangle$	$18,6^{\ddagger}$	$\langle 48,10,11;3\rangle+1$	$\langle 59,9,8;2\rangle$	$17,7^{\ddagger}$	$\langle 50,9,7;2\rangle+1$
$\langle 59,9,9;2\rangle$	$18,7^{\ddagger}$	$\langle 37,9,6;2\rangle$	$\langle 59,10,13;3\rangle$	$18,6^{\ddagger}$	$\langle 49,10,11;3\rangle+1$
$\langle 60,9,8;2\rangle$	$17,7^{\ddagger}$	$\langle 51,9,7;2\rangle+1$	$\langle 60,9,9;2\rangle$	$18,7^{\ddagger}$	$\langle 37,9,6;2\rangle$
$\langle 60,10,13;3\rangle$	$18,6^{\ddagger}$	$\langle 50,10,11;3\rangle+1$	$\langle 61,9,8;2\rangle$	$17,7^{\ddagger}$	$\langle 52,9,7;2\rangle+1$
$\langle 61,9,9;2\rangle$	$18,7^{\ddagger}$	$\langle 37,9,6;2\rangle$	$\langle 62,9,8;2\rangle$	$17,7^{\ddagger}$	$\langle 53,9,7;2\rangle+1$
$\langle 62,9,9;2\rangle$	$18,7^{\ddagger}$	$\langle 37,9,6;2\rangle$	$\langle 63,9,8;2\rangle$	$17,7^{\ddagger}$	$\langle 54,9,7;2\rangle+1$
$\langle 63,9,9;2\rangle$	$19,7^{\ddagger}$	$\langle 54,9,8;2\rangle+1$	$\langle 64,9,9;2\rangle$	$19,7^{\ddagger}$	$\langle 55,9,8;2\rangle+1$

Table 4.3: (continued) Lotteries $\langle m,n,t;k\rangle$, ($m>20$, $t\neq k$) for which the optimality of the design upper bounds in [2] were found to be suboptimal. Where Theorem 2 was invoked, only one lottery of the decomposition is given, and a disjoint set is indicated with a " + 1". The upper bounds listed in [2] and exact values of the lottery numbers appear in the column labelled "[2], L". ‡Lower bound by Theorem 1(a). Execution times (in seconds) to establish lower bounds are given by the following superscripts: ^{a}1 466, ^{b}1 486, ^{c}1.

lower bound was provided in Table 6.5. The bounds in Table 6.5 seem tantalisingly close, and certainly present an opportunity for further work.

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$\langle m,n,t;k\rangle$	Old	m,t	New	$\langle m,n,t;k\rangle$	Old	m,t	New
$\langle 21,4,9;3\rangle$	16	17,7	15	$\langle 21,6,7;3\rangle$	9	15,5	8
$\langle 21,6,10;4\rangle$	12	15,7	11	$\langle 21,7,6;3\rangle$	8	14,4	7
$\langle 21,8,8;4\rangle$	8	13,5	7	$\langle 22,7,11;5\rangle$	31	15,7	25
$\langle 22,10,7;4\rangle$	10	12,4	6	$\langle 23,6,8;3\rangle$	8	17,6	7
$\langle 23,7,10;4\rangle$	8	16,7	7	$\langle 23,10,7;4\rangle$	10	13,4	8
$\langle 24,6,8;3\rangle$	9	18,6	8	$\langle 25,6,9;3\rangle$	8	19,7	7
$\langle 25,9,6;3\rangle$	7	16,4	6	$\langle 26,6,9;3\rangle$	9	20,7	7
$\langle 26,6,13;4\rangle$	15	20,10	8	$\langle 26,10,8;4\rangle$	8	16,5	6
$\langle 27,6,9;3\rangle$	10	21,7	9	$\langle 27,6,13;4\rangle$	15	21,10	12
$\langle 27,6,14;4\rangle$	11	21,11	9	$\langle 27,8,6;3\rangle$	20	19,4	15
$\langle 27,9,6;3\rangle$	8	18,4	7	$\langle 28,6,14;4\rangle$	13	22,11	10
$\langle 28,6,15;4\rangle$	13	–	10^*	$\langle 28,7,8;3\rangle$	9	21,6	8
$\langle 28,8,6;3\rangle$	24	20,4	15	$\langle 29,6,13;4\rangle$	18	23,10	17
$\langle 29,6,14;4\rangle$	14	23,11	12	$\langle 29,6,15;4\rangle$	13	–	12^*
$\langle 30,6,14;4\rangle$	16	24,11	13	$\langle 30,7,13;4\rangle$	9	23,10	8
$\langle 32,7,14;4\rangle$	10	11,5	8	$\langle 32,8,7;3\rangle$	14	24,5	13
$\langle 33,10,6;3\rangle$	15	23,4	12	$\langle 33,10,7;3\rangle$	12	23,5	6
$\langle 34,6,12;3\rangle$	10	10,4	8	$\langle 34,9,4;2\rangle$	9	13,2	7
$\langle 34,10,6;3\rangle$	16	24,4	13	$\langle 34,10,7;3\rangle$	12	24,5	7
$\langle 35,6,12;3\rangle$	11	11,4	9	$\langle 35,7,5;2\rangle$	10	11,2	9
$\langle 35,7,9;3\rangle$	14	28,7	13	$\langle 35,9,4;2\rangle$	9	13,2	7
$\langle 35,10,6;3\rangle$	16	25,4	15	$\langle 35,10,7;3\rangle$	12	25,5	9
$\langle 36,6,12;3\rangle$	13	12,4	10	$\langle 36,7,9;3\rangle$	16	29,7	15
$\langle 36,7,16;4\rangle$	13	15,7	8	$\langle 36,9,4;2\rangle$	9	15,2	8
$\langle 36,10,7;3\rangle$	12	26,5	9	$\langle 37,7,16;4\rangle$	13	16,7	9
$\langle 37,7,17;4\rangle$	10	16,8	7	$\langle 37,9,4;2\rangle$	9	15,2	8
$\langle 37,10,7;3\rangle$	12	27,5	10	$\langle 37,10,8;3\rangle$	11	27,6	8
$\langle 38,7,16;4\rangle$	15	17,7	12	$\langle 38,7,17;4\rangle$	10	17,8	7
$\langle 38,9,9;3\rangle$	10	12,3	7	$\langle 38,10,7;3\rangle$	12	28,5	11
$\langle 38,10,8;3\rangle$	11	28,6	8	$\langle 39,7,16;4\rangle$	15	32,13	14
$\langle 39,7,17;4\rangle$	12	32,14	9	$\langle 39,9,9;3\rangle$	10	13,3	9
$\langle 39,10,8;3\rangle$	11	13,3	9	$\langle 40,6,13;3\rangle$	16	16,5	12
$\langle 40,7,16;4\rangle$	17	33,13	16	$\langle 40,10,8;3\rangle$	11	14,3	10
$\langle 41,9,5;2\rangle$	12	32,4	7	$\langle 42,9,5;2\rangle$	12	33,4	7
$\langle 43,9,5;2\rangle$	12	22,3	8	$\langle 44,9,5;2\rangle$	12	22,3	8
$\langle 44,10,8;3\rangle$	15	34,6	14	$\langle 45,7,7;2\rangle$	12	38,6	8
$\langle 45,9,5;2\rangle$	12	24,3	9	$\langle 46,7,7;2\rangle$	12	39,6	9
$\langle 46,9,5;2\rangle$	12	24,3	9	$\langle 46,10,8;3\rangle$	18	36,6	17
$\langle 46,10,10;3\rangle$	15	36,8	7	$\langle 47,7,7;2\rangle$	12	40,6	10
$\langle 47,9,5;2\rangle$	12	26,3	10	$\langle 47,10,8;3\rangle$	19	37,6	17
$\langle 47,10,9;3\rangle$	16	37,7	11	$\langle 47,10,10;3\rangle$	15	37,8	9
$\langle 48,7,7;2\rangle$	12	41,6	10	$\langle 48,9,5;2\rangle$	12	26,3	10
$\langle 48,10,8;3\rangle$	20	38,6	18	$\langle 48,10,9;3\rangle$	16	38,7	12
$\langle 48,10,10;3\rangle$	15	38,8	9	$\langle 49,5,6;2\rangle$	29	44,5	28
$\langle 49,7,7;2\rangle$	12	42,6	11	$\langle 49,9,5;2\rangle$	12	28,3	11

Table 5.4: Lotteries $\langle m,n,t;k\rangle$, ($m > 20$, $t \neq k$) for which upper bounds were improved using Theorem 2. The values of m and t of only one component of the decomposition are given. No conclusion could be reached about the optimality of these improved upper bounds. The upper bounds listed in [2] appear in the column labelled "Old", whilst our improved upper bounds appear in the column labelled "New". *By Theorem 1(e).

$\langle m,n,t;k\rangle$	Old	m,t	New	$\langle m,n,t;k\rangle$	Old	m,t	New
$\langle 49,10,5;2\rangle$	12	14,2	8	$\langle 49,10,9;3\rangle$	16	39,7	13
$\langle 49,10,10;3\rangle$	15	23,5	10	$\langle 50,9,5;2\rangle$	12	28,3	11
$\langle 50,9,6;2\rangle$	13	41,5	8	$\langle 50,10,5;2\rangle$	9	15,2	8
$\langle 50,10,9;3\rangle$	16	40,7	14	$\langle 50,10,10;3\rangle$	15	24,5	11
$\langle 51,9,6;2\rangle$	13	42,5	8	$\langle 51,10,5;2\rangle$	10	16,2	9
$\langle 51,10,9;3\rangle$	16	41,7	15	$\langle 51,10,10;3\rangle$	15	41,8	12
$\langle 51,10,11;3\rangle$	14	25,6	9	$\langle 52,7,7;2\rangle$	14	28,4	13
$\langle 52,9,6;2\rangle$	14	43,5	9	$\langle 52,10,10;3\rangle$	15	26,5	13
$\langle 52,10,11;3\rangle$	14	26,6	10	$\langle 53,7,8;2\rangle$	15	46,7	10
$\langle 53,8,7;2\rangle$	10	32,5	9	$\langle 53,9,6;2\rangle$	14	44,5	9
$\langle 53,10,10;3\rangle$	15	27,5	14	$\langle 53,10,11;3\rangle$	14	27,6	12
$\langle 54,5,7;2\rangle$	30	49,6	29	$\langle 54,7,8;2\rangle$	15	47,7	11
$\langle 54,9,6;2\rangle$	15	45,5	10	$\langle 54,10,11;3\rangle$	14	28,6	12
$\langle 55,7,8;2\rangle$	15	48,7	11	$\langle 55,9,6;2\rangle$	15	46,5	10
$\langle 55,9,7;2\rangle$	16	10,2	8	$\langle 55,10,5;2\rangle$	11	45,4	10
$\langle 55,10,11;3\rangle$	14	29,6	13	$\langle 56,7,8;2\rangle$	15	49,7	12
$\langle 56,9,6;2\rangle$	15	47,5	11	$\langle 56,9,7;2\rangle$	16	11,2	8
$\langle 56,10,12;3\rangle$	19	46,10	8	$\langle 57,7,8;2\rangle$	15	34,5	13
$\langle 57,9,6;2\rangle$	15	48,5	11	$\langle 57,9,7;2\rangle$	16	12,2	8
$\langle 57,10,11;3\rangle$	15	47,9	12	$\langle 57,10,12;3\rangle$	19	47,10	10
$\langle 58,7,8;2\rangle$	15	34,5	13	$\langle 58,9,6;2\rangle$	15	37,4	12
$\langle 58,9,7;2\rangle$	16	13,2	8	$\langle 58,10,6;2\rangle$	13	48,5	9
$\langle 58,10,11;3\rangle$	16	48,9	13	$\langle 58,10,12;3\rangle$	19	48,10	10
$\langle 59,5,8;2\rangle$	31	54,7	30	$\langle 59,7,8;2\rangle$	15	52,7	14
$\langle 59,9,6;2\rangle$	15	50,5	12	$\langle 59,9,7;2\rangle$	16	14,2	9
$\langle 59,10,6;2\rangle$	11	49,5	9	$\langle 59,10,11;3\rangle$	18	49,9	14
$\langle 59,10,12;3\rangle$	19	49,10	11	$\langle 60,9,6;2\rangle$	15	39,4	13
$\langle 60,9,7;2\rangle$	16	15,2	9	$\langle 60,10,6;2\rangle$	11	50,5	9
$\langle 60,10,10;3\rangle$	23	50,8	21	$\langle 60,10,11;3\rangle$	19	50,9	15
$\langle 60,10,12;3\rangle$	19	50,10	12	$\langle 61,5,8;2\rangle$	33	56,7	32
$\langle 61,9,6;2\rangle$	15	39,4	13	$\langle 61,9,7;2\rangle$	16	16,2	10
$\langle 61,10,6;2\rangle$	11	26,3	10	$\langle 61,10,10;3\rangle$	24	51,8	22
$\langle 61,10,11;3\rangle$	20	51,9	16	$\langle 61,10,12;3\rangle$	19	51,10	13
$\langle 61,10,13;3\rangle$	18	51,11	10	$\langle 62,9,6;2\rangle$	15	41,4	14
$\langle 62,9,7;2\rangle$	16	53,6	10	$\langle 62,10,10;3\rangle$	24	52,8	23
$\langle 62,10,11;3\rangle$	20	52,9	17	$\langle 62,10,12;3\rangle$	19	52,10	14
$\langle 62,10,13;3\rangle$	18	52,11	11	$\langle 63,9,6;2\rangle$	15	41,4	14
$\langle 63,9,7;2\rangle$	16	18,2	11	$\langle 63,10,11;3\rangle$	20	53,9	18
$\langle 63,10,12;3\rangle$	19	53,10	15	$\langle 63,10,13;3\rangle$	18	53,11	13
$\langle 64,5,9;2\rangle$	32	59,8	31	$\langle 64,9,7;2\rangle$	16	55,6	11
$\langle 64,9,8;2\rangle$	18	55,7	9	$\langle 64,10,6;2\rangle$	12	54,5	11
$\langle 64,10,11;3\rangle$	20	54,9	19	$\langle 64,10,12;3\rangle$	19	54,10	16
$\langle 64,10,13;3\rangle$	18	54,11	13	$\langle 65,9,7;2\rangle$	17	20,2	12
$\langle 65,9,8;2\rangle$	18	20,3	9	$\langle 65,9,9;2\rangle$	20	43,6	8
$\langle 65,10,6;2\rangle$	12	55,5	11	$\langle 65,10,12;3\rangle$	19	39,7	17

Table 5.4: (continued) Lotteries $\langle m,n,t;k\rangle$, ($m > 20$, $t \neq k$) for which upper bounds were improved using Theorem 2. The values of m and t of only one component of the decomposition are given. No conclusion could be reached about the optimality of these improved upper bounds. The upper bounds listed in [2] appear in the column labelled "Old", whilst our improved upper bounds appear in the column labelled "New".

$\langle m,n,t;k\rangle$	Old	m,t	New	$\langle m,n,t;k\rangle$	Old	m,t	New
$\langle 65,10,13;3\rangle$	18	39, 8	14	$\langle 66,5,9;2\rangle$	34	61, 8	33
$\langle 66,9,7;2\rangle$	17	21, 2	12	$\langle 66,9,8;2\rangle$	19	45, 6	9
$\langle 66,9,9;2\rangle$	20	44, 6	8	$\langle 66,10,6;2\rangle$	13	56, 5	12
$\langle 66,10,12;3\rangle$	19	40, 7	18	$\langle 66,10,13;3\rangle$	18	40, 8	15
$\langle 67,5,9;2\rangle$	35	62, 8	34	$\langle 67,9,7;2\rangle$	18	58, 6	13
$\langle 67,9,8;2\rangle$	19	58, 7	9	$\langle 67,9,9;2\rangle$	20	58, 8	8
$\langle 67,10,13;3\rangle$	18	57, 11	13	$\langle 67,10,14;3\rangle$	18	57, 12	11
$\langle 68,9,7;2\rangle$	18	59, 6	13	$\langle 68,9,8;2\rangle$	19	45, 6	10
$\langle 68,9,9;2\rangle$	20	45, 6	8	$\langle 68,10,7;2\rangle$	16	58, 6	10
$\langle 68,10,13;3\rangle$	18	58, 11	14	$\langle 68,10,14;3\rangle$	18	58, 12	11
$\langle 68,10,15;3\rangle$	22	58, 13	7	$\langle 69,8,9;2\rangle$	12	45, 6	11
$\langle 69,9,7;2\rangle$	18	60, 6	14	$\langle 69,9,8;2\rangle$	19	45, 6	10
$\langle 69,9,9;2\rangle$	20	45, 6	8	$\langle 69,10,7;2\rangle$	16	59, 6	10
$\langle 69,10,13;3\rangle$	18	59, 11	15	$\langle 69,10,14;3\rangle$	18	59, 12	12
$\langle 69,10,15;3\rangle$	22	59, 13	7	$\langle 70,9,7;2\rangle$	18	61, 6	14
$\langle 70,9,8;2\rangle$	19	25, 3	11	$\langle 70,9,9;2\rangle$	20	45, 6	8
$\langle 70,10,6;2\rangle$	14	60, 5	13	$\langle 70,10,7;2\rangle$	14	35, 4	10
$\langle 70,10,13;3\rangle$	19	60, 11	16	$\langle 70,10,14;3\rangle$	18	60, 12	13
$\langle 70,10,15;3\rangle$	22	60, 13	7	$\langle 71,9,7;2\rangle$	18	50, 5	15
$\langle 71,9,8;2\rangle$	19	26, 3	11	$\langle 71,9,9;2\rangle$	20	26, 4	8
$\langle 71,10,7;2\rangle$	14	61, 6	11	$\langle 71,10,13;3\rangle$	20	61, 11	17
$\langle 71,10,14;3\rangle$	18	61, 12	14	$\langle 71,10,15;3\rangle$	22	61, 13	11
$\langle 72,9,7;2\rangle$	18	63, 6	15	$\langle 72,9,8;2\rangle$	19	27, 3	12
$\langle 72,9,9;2\rangle$	20	27, 4	8	$\langle 72,10,12;3\rangle$	25	62, 10	24
$\langle 72,10,13;3\rangle$	21	62, 11	18	$\langle 72,10,14;3\rangle$	18	62, 12	15
$\langle 72,10,15;3\rangle$	22	62, 13	12	$\langle 73,9,7;2\rangle$	18	52, 5	16
$\langle 73,9,8;2\rangle$	19	28, 3	12	$\langle 73,9,9;2\rangle$	20	28, 4	10
$\langle 73,10,7;2\rangle$	14	38, 4	12	$\langle 73,10,12;3\rangle$	26	63, 10	25
$\langle 73,10,13;3\rangle$	23	63, 11	19	$\langle 73,10,14;3\rangle$	19	63, 12	16
$\langle 73,10,15;3\rangle$	22	63, 13	14	$\langle 74,9,7;2\rangle$	18	52, 5	16
$\langle 74,9,8;2\rangle$	19	29, 3	13	$\langle 74,9,9;2\rangle$	20	29, 4	10
$\langle 74,10,7;2\rangle$	13	39, 4	12	$\langle 74,10,12;3\rangle$	28	64, 10	26
$\langle 74,10,13;3\rangle$	24	64, 11	20	$\langle 74,10,14;3\rangle$	21	64, 12	17
$\langle 74,10,15;3\rangle$	22	64, 13	14	$\langle 75,9,7;2\rangle$	18	54, 5	17
$\langle 75,9,8;2\rangle$	19	30, 3	13	$\langle 75,9,9;2\rangle$	20	30, 4	10
$\langle 75,10,7;2\rangle$	14	65, 6	12	$\langle 75,10,12;3\rangle$	28	65, 10	27
$\langle 75,10,13;3\rangle$	24	65, 11	21	$\langle 75,10,14;3\rangle$	22	65, 12	18
$\langle 75,10,15;3\rangle$	22	49, 10	15	$\langle 76,9,7;2\rangle$	18	54, 5	17
$\langle 76,9,8;2\rangle$	19	31, 3	14	$\langle 76,9,9;2\rangle$	20	31, 4	10
$\langle 76,10,7;2\rangle$	14	41, 4	13	$\langle 76,10,13;3\rangle$	24	66, 11	22
$\langle 76,10,14;3\rangle$	23	66, 12	19	$\langle 76,10,15;3\rangle$	22	50, 10	16
$\langle 77,9,8;2\rangle$	19	68, 7	14	$\langle 77,9,9;2\rangle$	21	68, 8	11
$\langle 77,10,13;3\rangle$	24	67, 11	23	$\langle 77,10,14;3\rangle$	23	67, 12	20
$\langle 77,10,15;3\rangle$	22	67, 13	14	$\langle 78,9,8;2\rangle$	20	33, 3	15
$\langle 78,9,9;2\rangle$	21	69, 8	11	$\langle 78,10,14;3\rangle$	23	52, 9	21

Table 5.4: (continued) Lotteries $\langle m,n,t;k\rangle$, ($m > 20$, $t \neq k$) for which upper bounds were improved using Theorem 2. The values of m and t of only one component of the decomposition are given. No conclusion could be reached about the optimality of these improved upper bounds. The upper bounds listed in [2] appear in the column labelled "Old", whilst our improved upper bounds appear in the column labelled "New".

$\langle m,n,t;k\rangle$	Old	m,t	New	$\langle m,n,t;k\rangle$	Old	m,t	New
$\langle 78,10,15;3\rangle$	22	68, 13	15	$\langle 78,10,16;3\rangle$	22	68, 14	12
$\langle 79,9,8;2\rangle$	20	34, 3	15	$\langle 79,9,9;2\rangle$	22	34, 4	12
$\langle 79,10,7;2\rangle$	15	44, 4	14	$\langle 79,10,14;3\rangle$	23	53, 9	22
$\langle 79,10,15;3\rangle$	22	69, 13	16	$\langle 79,10,16;3\rangle$	22	69, 14	13
$\langle 80,9,8;2\rangle$	21	35, 3	16	$\langle 80,9,9;2\rangle$	22	35, 4	12
$\langle 80,10,7;2\rangle$	15	45, 4	14	$\langle 80,10,15;3\rangle$	22	70, 13	17
$\langle 80,10,16;3\rangle$	22	70, 14	14	$\langle 81,9,8;2\rangle$	21	36, 3	16
$\langle 81,9,9;2\rangle$	22	36, 4	13	$\langle 81,10,7;2\rangle$	16	46, 4	15
$\langle 82,9,8;2\rangle$	21	37, 3	17	$\langle 82,9,9;2\rangle$	22	37, 4	13
$\langle 83,9,8;2\rangle$	21	74, 7	17	$\langle 83,9,9;2\rangle$	22	38, 4	14
$\langle 84,9,8;2\rangle$	21	39, 3	18	$\langle 84,9,9;2\rangle$	22	39, 4	14
$\langle 85,9,8;2\rangle$	21	76, 7	18	$\langle 85,9,9;2\rangle$	22	40, 4	15
$\langle 85,10,7;2\rangle$	17	75, 6	16	$\langle 86,9,8;2\rangle$	21	65, 6	19
$\langle 86,9,9;2\rangle$	22	41, 4	15	$\langle 87,9,8;2\rangle$	21	65, 6	19
$\langle 87,9,9;2\rangle$	22	42, 4	16	$\langle 88,9,8;2\rangle$	21	67, 6	20
$\langle 88,9,9;2\rangle$	22	43, 4	16	$\langle 89,9,8;2\rangle$	21	67, 6	20
$\langle 89,9,9;2\rangle$	22	44, 4	17	$\langle 90,9,9;2\rangle$	22	45, 4	17

Table 5.4: (continued) Lotteries $\langle m,n,t;k\rangle$, $(m > 20,\ t \neq k)$ for which upper bounds were improved using Theorem 2. The values of m and t of only one component of the decomposition are given. No conclusion could be reached about the optimality of these improved upper bounds. The upper bounds listed in [2] appear in the column labelled "Old", whilst our improved upper bounds appear in the column labelled "New".

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$6^{\dagger\ddagger} \le L(21,7,3;2) \le 7$	$6^{\dagger} \le L(21,7,6;3) \le 7^{*}$	$5^{\bullet} \le L(21,8,8;4) \le 7^{*}$
$5^{\bullet} \le L(22,10,7;4) \le 6^{*}$	$6^{\dagger\ddagger} \le L(23,6,8;3) \le 7^{*}$	$6^{\dagger\ddagger} \le L(23,7,10;4) \le 7^{*}$
$5^{\bullet} \le L(24,8,9;4) \le 6$	$5^{\bullet} \le L(24,10,5;3) \le 6$	$6^{\dagger} \le L(25,6,9;3) \le 7^{*}$
$6^{\dagger} \le L(25,7,11;4) \le 7$	$6^{\dagger\ddagger} \le L(25,8,3;2) \le 7$	$5^{\bullet} \le L(25,8,7;3) \le 6$
$5^{\bullet} \le L(25,9,6;3) \le 6^{*}$	$6^{\dagger\ddagger} \le L(26,6,9;3) \le 7^{*}$	$6^{\dagger} \le L(26,7,4;2) \le 7$
$5^{\bullet} \le L(26,9,6;3) \le 7$	$5^{\bullet} \le L(26,10,8;4) \le 6^{*}$	$6^{\dagger\ddagger} \le L(27,7,4;2) \le 7$
$6^{\dagger\ddagger} \le L(27,7,12;4) \le 7$	$6^{\dagger\ddagger} \le L(27,8,7;3) \le 7$	$6^{\dagger\ddagger} \le L(27,9,3;2) \le 7$
$5^{\bullet\ddagger} \le L(27,9,6;3) \le 7^{*}$	$5^{\bullet} \le L(27,10,6;3) \le 7$	$5^{\bullet} \le L(27,10,9;4) \le 6$
$6^{\dagger\ddagger} \le L(28,9,3;2) \le 7$	$5^{\bullet} \le L(28,9,7;3) \le 6$	$5^{\bullet} \le L(28,10,6;3) \le 7$
$5^{\bullet} \le L(28,10,9;4) \le 6$	$6^{\dagger} \le L(29,7,9;3) \le 7$	$6^{\dagger} \le L(29,7,13;4) \le 7$
$5^{\bullet\ddagger} \le L(29,9,7;3) \le 7$	$5^{\bullet} \le L(29,10,9;4) \le 6$	$6^{\dagger} \le L(30,7,9;3) \le 7$
$6^{\dagger\ddagger} \le L(30,8,4;2) \le 7$	$5^{\bullet} \le L(30,10,9;4) \le 6$	$6^{\dagger\ddagger} \le L(31,8,4;2) \le 7$
$6^{\dagger\ddagger} \le L(32,8,4;2) \le 7$	$5^{\bullet} \le L(33,10,7;3) \le 6^{*}$	$6^{\dagger} \le L(34,8,9;3) \le 7$
$6^{\dagger\ddagger} \le L(34,9,4;2) \le 7^{*}$	$5^{\bullet\ddagger} \le L(34,10,7;3) \le 7^{*}$	$6^{\dagger\ddagger} \le L(35,9,4;2) \le 7^{*}$
$6^{\dagger} \le L(37,7,17;4) \le 7^{*}$	$6^{\dagger\ddagger} \le L(37,10,4;2) \le 7$	$6^{\dagger\ddagger} \le L(38,7,17;4) \le 7^{*}$
$5^{\bullet} \le L(38,9,9;3) \le 7^{*}$	$6^{\dagger\ddagger} \le L(38,10,4;2) \le 7$	$6^{\dagger\ddagger} \le L(39,10,4;2) \le 7$
$6^{\dagger\ddagger} \le L(41,9,5;2) \le 7^{*}$	$6^{\dagger\ddagger} \le L(42,9,5;2) \le 7^{*}$	$6^{\dagger} \le L(46,10,10;3) \le 7^{*}$
$6^{\dagger} \le L(68,10,15;3) \le 7^{*}$	$6^{\dagger\ddagger} \le L(69,10,15;3) \le 7^{*}$	$6^{\dagger\ddagger} \le L(70,10,15;3) \le 7^{*}$

Table 6.5: Lotteries $\langle m, n, t; k\rangle$, $(m > 20,\ t \neq k)$ for which the optimality of the design upper bound $L(m,n,t;k) \le \ell$ for $\ell = 6, 7$ could not be established. Largest lower bounds are given for which the characterisation search technique in §2 could be executed within a reasonable amount of time. $^{\bullet}$The characterisation procedure could not be implemented up to level 5. †The characterisation procedure could not be implemented up to level 6. ‡By Theorem 1(a). *See Table 4.3.

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