A Tight Bound on the Cardinalities of Maximum Alliance-Free and Minimum Alliance-Cover Sets

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Abstract

A *defensive k-alliance* in a graph $G = (V, E)$ is a set of vertices $A \subseteq V$ such that for every vertex $v \in A$, the number of neighbors v has in A is at least k more than the number of neighbors it has in $V - A$ (k is a measure of the strength of alliance). In this paper, we deal with two types of sets associated with defensive k-alliances; maximum defensive k-alliance free and minimum defensive k-alliance cover sets. Define a set $X \subseteq V$ to be *maximum defensive k-alliance free* if X does not contain any defensive k-alliance and is a largest such set. A set $Y \subseteq V$ is called *minimum defensive k-alliance cover* if Y contains at least one vertex from each defensive k-alliance and is a set of minimum cardinality satisfying this property. We present bounds on the cardinalities of maximum defensive k-alliance free and minimum defensive k-alliance cover sets.

1 Introduction

Alliances in graphs were first introduced by Hedetniemi, et al.[3]. They proposed different types of alliances, namely (strong) defensive alliances, (strong) offensive alliances[1], global alliances[2], etc. A generalization of these alliances called k-alliances was presented in [5], where the strength of an alliance is related to the value of parameter k.

Consider a graph $G = (V, E)$ without loops or multiple edges. A vertex v in set $A \subseteq V$ is said to be k- satisfied with respect to A if

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$\deg_A(v) \geq \deg_{V-A}(v) + k$, where $\deg_A(v) = |N(v) \cap A| = |N_A(v)| = \deg(v) - \deg_{V-A}(v)$. A set A is a *defensive k- alliance* if all vertices in A are k-satisfied with respect to A, where $-\Delta < k \leq \Delta$. Note that a defensive (-1)-alliance is a "defensive alliance" (as defined in [3]), and a defensive 0-alliance is a "strong defensive alliance" or "cohesive set" [4].

A set $X \subseteq V$ is *defensive k-alliance free* (k-daf) if for all defensive k-alliances A, $A - X \neq \emptyset$, i.e., X does not contain any defensive k-alliance as a subset. A defensive k-alliance free set X is maximal if $\forall v \notin X$, $\exists S \subseteq X$ such that $S \cup \{v\}$ is a defensive k-alliance. A maximum k-daf set is a maximal k-daf set of largest cardinality. Let $\phi_k(G)$ be the cardinality of a maximum k-daf set of graph G. For simplicity of notation, we will refer to a maximum k-daf set of G as a $\phi_k(G)$-set. If a graph G does not have a defensive k-alliance (for some k), we say that $\phi_k(G)= n$, for example, $\phi_k(P_n)= n$, $\forall k > 1$. Since $\forall k_1 \geq k_2$, a defensive k_2-alliance free set is also defensive k_1-alliance free, we have $\phi_{k_1}(G) \geq \phi_{k_2}(G)$ if and only if $k_1 \geq k_2$.

We define a set $Y \subseteq V$ to be a *defensive k-alliance cover* (k-dac) if for all defensive k-alliances A, $A \cap Y \neq \emptyset$, i.e., Y contains at least one vertex from each defensive k-alliance of G. A k-dac set Y is minimal if no proper subset of Y is a defensive k-alliance cover. A minimum k-dac set is a minimal cover of smallest cardinality. Let $\zeta_k(G)$ be the cardinality of a minimum k-dac set of graph G. We refer to minimum k-dac set of G as a $\zeta_k(G)$-set. When G does not have a defensive k-alliance (for some k), we say that $\zeta_k(G)= 0$. We proved the following theorem in [5];

Theorem 1. *$X \subseteq V$ is a defensive k-alliance cover if and only if $V - X$ is defensive k-alliance free.*

Corollary 2. *$\phi_k(G)+\zeta_k(G) = n$.*

Corollary 3. *$\zeta_{k_1}(G) \leq \zeta_{k_2}(G)$ if and only if $k_1 \geq k_2$.*

Corollary 4. *If V' is a minimal k-dac then, $\forall v \in V'$, there exists a defensive k-alliance $S(v)$ for which $S(v) \cap V' = \{v\}$.*

2 A Tight Bound on $\phi_k(G)$

We first present a bound on $\phi_k(G)$ when $k = 0$. The result is then generalized to $k \geq 0$ in Theorem 14.

Theorem 5. *If G is a connected graph then $\phi_0(G) \geq \lfloor \frac{n}{2} \rfloor$.*

Proof. Let A be a $\phi_0(G)-$set of a connected graph G and assume, to the contrary, that $\phi_0(G) < \lfloor \frac{n}{2} \rfloor$. Let $B = V(G) - A$, hence $|B| = \zeta_0(G) > \lceil \frac{n}{2} \rceil$.

Since B is a $0-$dac, $\forall v \in B$ there exists a defensive $0-$alliance $S(v)$ such that $S(v) \cap B = \{v\}$. Hence, $\forall v \in B$, $\deg_A(v) \geq \deg_B(v)$. If B does not contain a defensive $0-$alliance, then B is a $0-$daf set, which is contradiction since, $|B| > \lceil \frac{n}{2} \rceil > \phi_0(G)$. Hence, B must contain a minimal defensive $0-$alliance T. If $v \in T$ then $\deg_B(v) = \deg_A(v)$. Hence, $N_B(T) = T$.

Suppose T is the only minimal defensive $0-$alliance in B. Then, for any vertex $x \in T$, the set $B - \{x\}$ is a defensive $0-$alliance free set and $|B - \{x\}| > \phi_0(G)$, a contradiction. Thus there are at least two disjoint defensive $0-$alliances in B.

Now, assume that the number of disjoint minimal defensive $0-$alliances in B is minimum among all such sets. For each $v \in B$, let $S(v)$ be a minimal defensive $0-$alliance such that $S(v) \cap B = \{v\}$. Also, define:
$D = \{v \in B | \deg_B(v) = \deg_A(v)\}$,
$R = \{v \in A | \deg_A(v) = \deg_B(v)\}$,
$R^- = \{v \in A | \deg_A(v) < \deg_B(v)\}$, and
$R^+ = \{v \in A | \deg_A(v) > \deg_B(v)\}$.

Let $T_1, T_2, \ldots, T_r$ be the disjoint minimal defensive $0-$alliances in B. By the above arguments, $r \geq 2$ and $\forall i$, $N_B(T_i) = T_i \subseteq D$.

We now present a sequence of lemmas which culminate in the rest of the proof of Theorem 5.

Lemma 6. *For $1 \leq i \leq r$ and each $x \in T_i$, $N_A(x) \subseteq S(x) \cap R^-$.*

Proof. Suppose $x \in T_i$ and let $y \in N_A(x)$. Since $x \in T_i \subseteq D$, $\deg_B(x) = \deg_A(x)$. Hence, $N_A(x) \subseteq S(x)$ and $y \in S(x)$. Assume to the contrary that $y \notin R^-$, i.e., $\deg_A(y) \geq \deg_B(y)$. Let $A' = A \cup \{x\} - \{y\}$ and suppose $S' \subseteq A'$ is a defensive $0-$alliance. Since $\deg_{A'}(x) < \deg_{B'}(x)$, $x \notin S'$. But, then $S' \subseteq A$, which contradicts A being $0-$daf. Hence, A' is defensive $0-$alliance free and $B' = V - A'$ is a $0-$dac. Since T_i is a minimal defensive $0-$alliance in B, $T_i - \{x\}$ is not a defensive $0-$alliance in B'. Also, $\deg_{B'}(y) < \deg_{A'}(y)$ implies that $y \notin T'$, where T' is a defensive $0-$alliance in B'. But then the number of disjoint minimal defensive $0-$alliances in B' is $r-1$, which contradicts the assumption that B has a minimum number of disjoint minimal defensive $0-$alliances. □

Lemma 7. *For $i \neq j$ and every $x_1 \in T_i$ and $x_2 \in T_j$, $N(x_1) \cap N(x_2) = \emptyset$.*

Proof. Suppose $i \neq j$ and there exist $x_1 \in T_i$ and $x_2 \in T_j$ such that $y \in N(x_1) \cap N(x_2)$. Since $T_i \cap T_j = \emptyset$ and $N_B(T_i) = T_i$, we have that $y \in A$. From Lemma 6, we know that $y \in R^- \cap S(x_1) \cap S(x_2)$. Consider the sets $A' = A \cup \{x_1, x_2\} - \{y\}$ and $B' = V - A'$. Since $|A'| = |A| + 1$ and A is a $\phi_0(G)-$set, A' must contain a defensive $0-$alliance S'. However,

$\deg_A(x_l) = \deg_B(x_l)$, $l \in \{1,2\}$ and $x_1x_2 \notin E(G)$. Therefore, $\deg_{A'}(x_l) = \deg_{B'}(x_l) - 1$ and, hence, $\{x_1, x_2\} \cap S' = \emptyset$. This implies that $S' \subseteq A$, and contradicts A being a defensive 0−alliance free set. □

Lemma 8. *For every $x \in T_i$*

(i) $S(x) \subseteq N_A(x) \cup R \cup \{x\}$,

(ii) $S(x)$ *is the unique minimal defensive* 0−*alliance in* $A \cup \{x\}$, *and*

(iii) $N_{A\cup\{x\}}(S(x)) = S(x)$.

Proof. Let $x \in T_i$ and perform the following procedure:

$S' \leftarrow N_A(x) \cup \{x\}$
While $N_A(S') \subseteq N_A(x) \cup R$ *and* $N_A(S') - S' \neq \emptyset$
Begin
$S' \leftarrow S' \cup N_A(S')$
End

Since G is finite, the procedure will terminate with either $N_A(S') - S' = \emptyset$, or with a vertex $z \in N_A(S') - S'$ such that $z \notin R$. Assume $N_A(S') - S' \neq \emptyset$. By construction, $S' \cup N_A(S') \cup \{z\} \subseteq S(x)$ for every $S(x)$ that is a defensive 0−alliance and for which $S(x) \cap B = \{x\}$. There are two cases.

Case 1. $z \in R^-$: This implies that $\deg_{A\cup\{x\}}(z) < \deg_{B-\{x\}}(z)$ and contradicts the assumption that $S(x)$ is a defensive 0−alliance containing z.

Case 2. $z \in R^+$: The set $A' = (A \cup \{x\}) - \{z\}$ is a $\phi_0(G)$ −set, otherwise there is a defensive 0−alliance in $A \cup \{x\}$ not containing z. Thus, $B' = V - A'$ is a 0−dac. Since T_i is a minimal defensive 0−alliance in B, $T_i - \{x\}$ is not a defensive 0−alliance in B'. Also, $\deg_{B'}(z) < \deg_{A'}(z)$ implies that $z \notin T'$, where T' is a defensive 0−alliance in B'. But, then the number of disjoint minimal defensive 0−alliances in B' is $r-1$, contradicting the assumption that B has minimum number of disjoint minimal defensive 0−alliances.

Since both cases lead to a contradiction, we conclude that $N_A(S') - S' = \emptyset$. Hence, $S' = S(x) \subseteq N_A(x) \cup R \cup \{x\}$ and, by the construction, $S(x) = S'$ is the unique minimal defensive 0−alliance in $A \cup \{x\}$. Also, since $v \in S(x)$ implies $\deg_{A\cup\{x\}}(v) = \deg_{B-\{x\}}(v)$, we must conclude that $N_{A\cup\{x\}}(S(x)) = S(x)$. □

Lemma 9. *For $i \neq j$ and every $x_1 \in T_i$ and $x_2 \in T_j$, $S(x_1) \cap S(x_2) = \emptyset$.*

Proof. Suppose $i \neq j$, $x_1 \in T_i$, and $x_2 \in T_j$. Assume, to the contrary, that $z \in S(x_1) \cap S(x_2)$. By Lemmas 6, 7 and 8, we know that $N_A(x_1) \subseteq S(x_1) \cap R^-$, $N_A(x_1) \cap N_A(x_2) = \emptyset$, and $S(x_2) \subseteq N_A(x_2) \cup R \cup \{x_2\}$. Hence, $N_A(x_1) \cap S(x_2) = \emptyset$. Since $S(x_1)$ is a minimal defensive 0–alliance, $G[S(x_1)]$, the subgraph of G induced by $S(x_1)$, is connected. Hence, there is a path P in $G[S(x_1)]$ between z and a vertex $y \in N_A(x_1)$ that does not contain x_1. From Lemma 8, $N_{A\cup\{x_2\}}(S(x_2)) = S(x_2)$ and, hence, $y \in N_A(x_1) \cap S(x_2)$, a contradiction. □

Corollary 10. *For $i \neq j$ and any $x_1 \in T_i$ and $x_2 \in T_j$, every path between $S(x_1)$ and $S(x_2)$ contains a vertex not in A.*

Lemma 11. *If $i \neq j$ then there is no path between T_i and T_j.*

Proof. Assume to the contrary that such a path exists. Recall that $T_i \cap T_j = \emptyset$ and $N_B(T_i) = T_i$. Hence, any path P from T_i to T_j must have an even number of edges in common with the edge cutset $\langle A, B\rangle$. Let the number of common edges between the edge cutset $F = \langle A, B\rangle$ and the path P be $|F \cap P| \geq 2$ and assume that $|F \cap P|$ is minimum for all such bipartitions. Now we have two cases:

Case 1: $|F \cap P| = 2$. Let $F \cap P = \{x_1a_1, a_2x_2\}$, where $x_1 \in T_i$, $a_1 \in N_A(x_1) \subseteq S(x_1)$, $x_2 \in T_j$, and $a_2 \in N_A(x_2) \subseteq S(x_2)$. By Lemma 9, $S(x_1) \cap S(x_2) = \emptyset$ and, by Corollary 10, there is no path from $S(x_1)$ to $S(x_2)$ consisting of only vertices in A, a contradiction.

Case 2: $|F \cap P| > 2$. Let $F \cap P = \{x_1a_1, a_2x_2, x_3a_3, \ldots, a_{2s+2}x_{2s+2}\}$, $s \geq 1$, where $x_1 \in T_i$, $a_1 \in N_A(x_1)$, $a_2 \in S(x_1)$, $x_2 \in N_B(a_2)$, ..., $a_{2s+2} \in N_A(x_{2s+2})$ and $x_{2s+2} \in T_j$. Further, for $1 \leq l \leq 2s+2$, $a_l \in A$ and $x_l \in B$. We claim for $2 \leq l \leq 2s+1$, that $x_l \notin T_u$, $1 \leq u \leq r$. Otherwise, suppose that $x_l \in T_u$. Without loss of generality, assume $u \neq i$, then there is a path from T_i to T_u such that $|F \cap P| \leq 2s$, which is contrary to P minimizing $|F \cap P|$.

Since $a_2 \in S(x_1)$, by Lemma 8, the set $A' = A \cup \{x_1\} - \{a_2\}$ is a $\phi_0(G)$–set and the set $B' = V - A'$ is a 0–dac. Let $F' = \langle A', B'\rangle$. Suppose there is no defensive 0–alliance T' in B' such that $a_2 \in T'$. Then there are $r-1$ disjoint minimal defensive 0–alliances in B', which is a contradiction since B has the minimum number of disjoint minimal defensive 0–alliances. Thus, there is a defensive 0–alliance $T' \subseteq B'$ which contains a_2 and is disjoint from sets $T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_k$. But, then there is a path P' between T' and T_j such that $|F' \cap P'| = 2s$, which is again a contradiction.

Since both cases lead to contradictions, there is no path P between T_i and T_j whenever $i \neq j$. □

From Lemma 11, we conclude that G is disconnected, a contradiction. Therefore, the set B must be defensive 0−alliance free and, hence, $\phi_0(G) \geq |B| > |A| = \phi_0(G)$, again a contradiction. Thus, $\phi_0(G) \geq \lfloor \frac{n}{2} \rfloor$, which completes the proof of Theorem 5. □

Corollary 12. *If G is a connected Eulerian graph then $\phi_{-1}(G) \geq \lfloor \frac{n}{2} \rfloor$.*

We have also shown the following statement to be true.

Theorem 13. *For connected graphs G, $\phi_0(G) < \zeta_0(G)$ if and only if every block of G is either an odd clique or an odd cycle.*

Theorem 14. *For every connected graph G and $0 \leq k \leq \Delta$, $\phi_k(G) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor$.*

Proof. By Theorem 5, the statement is true for $k = 0$. Since every k−daf set is also $(k+1)$−daf, $\phi_1(G) \geq \phi_0(G) \geq \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{1}{2} \rfloor$, i.e., the statement is also true for $k = 1$. Hence, we may proceed by induction on k.

Assume that the statement is true for $k \leq M$ for arbitrary $M > 1$. Let A be a $\phi_M(G)$−set of a graph G. Again, A is also $(M+2)$−daf of graph G. By the induction hypothesis, $\phi_{M+2}(G) \geq |A| = \phi_M(G) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{M}{2} \rfloor$. If there exists a vertex $v \in V - A$ such that $A \cup \{v\}$ is $(M+2)$−daf, then $\phi_{M+2}(G) \geq |A \cup \{v\}| \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{M}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{M+2}{2} \rfloor$. Suppose no such vertex exists. Then, $\forall v \in V - A$ there exists a defensive $(M+2)$−alliance $S(v)$ such that $S(v) \cap (V - A) = \{v\}$. But, then $\forall w \in S(v)$, $\deg_{S(v)-\{v\}}(w) \geq \deg_{V-S(v)-\{v\}}(w) + M$ which is contrary to the assumption that A is M-daf. □

The bound of Theorem 14 is also sharp and is achieved by the complete graphs of even order. We believe (but have been unable to prove) the following extension of the above theorem:

Conjecture 1. *If G is a connected graph and $-\delta(G) < k \leq \Delta(G)$ then*

$$\phi_k(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor.$$

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