Dense near polygons with two types of quads and three types of hexes

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Abstract

We show that the number of points at distance i from a given point x in a dense near polygon only depends on i and not on the point x. We give a number of easy corollaries of this result. Subsequently, we look to the case of dense near polygons $\mathcal{S}$ with an order in which there are two possibilities for t_Q, where Q is a quad of $\mathcal{S}$, and three possibilities for (t_H, v_H), where H is a hex of $\mathcal{S}$. Using the above-mentioned results, we will show that the number of quads of each type through a point is constant. We will also show that the number of hexes of each type through a point is constant if a certain matrix is nonsingular. If each hex is a regular near hexagon, a glued near hexagon or a product near hexagon, then that matrix turns out to be nonsingular in all but one of the eight possible cases. For the exceptional case, however, we provide an example of a near polygon that does not have a constant number of hexes of each type through each point.

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1 Introduction

1.1 Basic definitions and properties

Let $\Gamma = (V, E)$ be a graph. A *clique* of Γ is a set of mutually adjacent vertices. A clique is called *maximal* if it is not properly contained in another clique. We will denote the distance between two vertices x and y of Γ by $\mathrm{d}(x, y)$. If X_1 and X_2 are two sets of vertices, then we denote by $\mathrm{d}(X_1, X_2)$

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the minimal distance between a vertex of X_1 and a vertex of X_2. If X_1 is a singleton $\{x\}$, then we will also write $\mathrm{d}(x, X_2)$ instead of $\mathrm{d}(\{x\}, X_2)$. For every $i \in \mathbb{N}$ and every nonempty set of vertices X, we denote by $\Gamma_i(X)$ the set of all vertices y for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, then we also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A *near* $2d$*-gon* is a connected graph of diameter d with the property that for every vertex x and every maximal clique M there exists a unique point x' in M nearest to x. A near 0-gon consists of one vertex and a near 2-gon is a maximal clique with at least two vertices.

A *point-line incidence structure* is a triple $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, where $\mathcal{P}$ is a nonempty set whose elements are called *points*, $\mathcal{L}$ is a (possibly empty) set whose elements are called *lines* and where I is a subset of $\mathcal{P} \times \mathcal{L}$, called the *incidence relation*. If $(p, L) \in \mathrm{I}$, then we say that p is incident with L, that p is contained in L, that L contains p, etc.. A point-line incidence structure is called a *partial linear space* (respectively a *linear space*) if every two different points are contained in at most (respectively exactly) one line. The *point graph* or *collinearity graph* of a point-line incidence structure $\mathcal{S}$ is the graph whose vertices are the points of $\mathcal{S}$ with two different vertices adjacent whenever they are *collinear*, i.e. whenever they are incident with the same line.

There is a bijective correspondence between the class of near polygons and a class of partial linear spaces. If a graph Γ is a near polygon, then the point-line incidence structure whose points, respectively lines, are the vertices, respectively maximal cliques, of Γ (natural incidence) is a partial linear space $\mathcal{S}$. The graph Γ can easily be retrieved from $\mathcal{S}$: Γ is the point graph of $\mathcal{S}$. Because of this bijective correspondence, the partial linear spaces which correspond with near polygons are usually also called near polygons. In the sequel we will always adopt the geometrical point of view. A near 0-gon is then a point and a near 2-gon a line. In the sequel we will denote the line with $s + 1$ points by L_{s+1}.

A near polygon is said to have *order* (s, t) if every line is incident with exactly $s+1$ points and if every point is incident with exactly $t+1$ lines. A near $2d$-gon, $d \geq 2$, is called a *generalized* $2d$*-gon* if for every $i \in \{1, \dots, d-1\}$ and for every two points x and y at distance i, $|\Gamma_1(y) \cap \Gamma_{i-1}(x)| = 1$. This condition is always satisfied if $d = 2$; so, the generalized quadrangles are precisely the near quadrangles. A generalized $2d$-gon is called *degenerate* if it does not contain ordinary $2d$-gons as subgeometries. For more background information on generalized quadrangles and generalized polygons, we refer to [8] and [10].

A nonempty set X of points of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *subspace* if every line meeting X in at least two points is completely contained in X. A subspace X is called *geodetically closed* if every point on a shortest path between two points of X is as well contained in X. Having a subspace X, we can define a subgeometry $\mathcal{S}_X$ of $\mathcal{S}$ by considering only those points and lines of $\mathcal{S}$ which are completely contained in X. If $\mathcal{S}_X$ is a near polygon, then we call $\mathcal{S}_X$ a *subgon* of $\mathcal{S}$. If X is geodetically closed, then $\mathcal{S}_X$ clearly is a subgon of $\mathcal{S}$. If a geodetically closed subgon $\mathcal{S}_X$ is a nondegenerate generalized quadrangle, then X (and often also $\mathcal{S}_X$) will be called a *quad*. Sufficient conditions for the existence of quads were given in [9]. Every nonempty set X of points is contained in a unique minimal geodetically closed subgon $\mathcal{C}(X)$, namely the intersection of all geodetically closed subgons containing X. We define $\mathcal{C}(\emptyset) := \emptyset$. If $X_1, \ldots, X_k$ are sets of points, then $\mathcal{C}(X_1 \cup \cdots \cup X_k)$ is also denoted by $\mathcal{C}(X_1, \ldots, X_k)$. If one of the arguments of $\mathcal{C}$ is a singleton $\{x\}$, we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \cdots)$.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. Geodetically closed subhexagons of a dense near polygon are called *hexes*. Dense near polygons satisfy several nice properties, see [2] for an overview. We collect some of these properties in the following lemma.

Lemma 1 (*i*) (Lemma 19 of [2]) *Every point of a dense near polygon $\mathcal{S}$ is incident with the same number of lines. We denote this number by $t_{\mathcal{S}} + 1$.*

(*ii*) (Theorem 4 of [2]) *If x and y are two points of a dense near polygon, then $\mathcal{C}(x, y)$ is the unique geodetically closed sub-$[2 \cdot d(x, y)]$-gon containing x and y. So, if x and y are two points at distance 2 in a dense near polygon, then these points are contained in a unique quad.*

(*iii*) ([2]) *Let $\mathcal{S}$ be a dense near 2d-gon, $d \geq 1$, let $\mathcal{F}$ be a geodetically closed sub-2i-gon, $i \in \{0, \ldots, d-1\}$, of $\mathcal{S}$ and let L be a line which intersects $\mathcal{F}$ in a point. Then $\mathcal{C}(\mathcal{F}, L)$ is a geodetically closed sub-$2(i+1)$-gon.*

1.2 Short overview

In the present paper we will show that a dense near polygon also satisfies the following nice property.

Theorem 1 *Let $\mathcal{S}$ be a dense near 2d-gon and let $i \in \{0, \ldots, d\}$. Then $|\Gamma_i(x)|$ only depends on i and not on the chosen point x of $\mathcal{S}$.*

In Section 5, we will derive some easy corollaries of Theorem 1. In Section 6, we look to the case of dense near polygons $\mathcal{S}$ with an order in which there are at most two possibilities for t_Q, where Q is a quad of $\mathcal{S}$, and at most three possibilities for (t_H, v_H), where H is a hex of $\mathcal{S}$. Using Theorem

1 and the derived corollaries, we will show that the number of hexes of each type through a point is constant if a certain matrix is nonsingular. If each hex is a regular near hexagon, a glued near hexagon or a product near hexagon, then that matrix turns out to be nonsingular in all but one of the eight possible cases. For the exceptional case, however, we provide an example of a near polygon that does not have a constant number of hexes of each type through each point.

2 Three classes of near polygons

Regular near polygons

A near $2n$-gon is called *regular* if it has an order (s,t) and if there exists constants t_i, $i \in \{0,1,\ldots,n\}$, such that for every two points x and y at distance i, there are precisely t_i+1 lines through y containing a point at distance $i-1$ from x. Obviously, $t_0=-1$, $t_1=0$ and $t_d=t$. The numbers s, t, t_i ($i \in \{2,\ldots,n-1\}$) are called the parameters of $\mathcal{S}$. The regular near polygons are precisely those near polygons whose point graph is a so-called *distance-regular graph* ([1]).

Product near polygons

Let $\mathcal{S}_1=(\mathcal{P}_1,\mathcal{L}_1,\mathrm{I}_1)$ and $\mathcal{S}_2=(\mathcal{P}_2,\mathcal{L}_2,\mathrm{I}_2)$ be two near polygons. A new near polygon $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ can be derived from $\mathcal{S}_1$ and $\mathcal{S}_2$. It is called the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$ and is denoted by $\mathcal{S}_1\times\mathcal{S}_2$. We have: $\mathcal{P}=\mathcal{P}_1\times\mathcal{P}_2$, $\mathcal{L}=(\mathcal{P}_1\times\mathcal{L}_2)\cup(\mathcal{L}_1\times\mathcal{P}_2)$, the point (x,y) of $\mathcal{S}_1\times\mathcal{S}_2$ is incident with the line $(z,L)\in\mathcal{P}_1\times\mathcal{L}_2$ if and only if $x=z$ and $y\ \mathrm{I}_2\ L$, the point (x,y) of $\mathcal{S}_1\times\mathcal{S}_2$ is incident with the line $(M,u)\in\mathcal{L}_1\times\mathcal{P}_2$ if and only if $x\ \mathrm{I}_1\ M$ and $y=u$. If $\mathrm{d}_i(\cdot,\cdot)$, $i\in\{1,2\}$, denotes the distance in $\mathcal{S}_i$ and if $\mathrm{d}(\cdot,\cdot)$ denotes the distance in $\mathcal{S}_1\times\mathcal{S}_2$, then $\mathrm{d}[(x_1,x_2),(y_1,y_2)]=\mathrm{d}_1(x_1,y_1)+\mathrm{d}_2(x_2,y_2)$ for all points (x_1,x_2) and (y_1,y_2) of $\mathcal{S}_1\times\mathcal{S}_2$. If $\mathcal{S}_i$, $i\in\{1,2\}$, is a near $2n_i$-gon then the direct product $\mathcal{S}=\mathcal{S}_1\times\mathcal{S}_2$ is a near $2(n_1+n_2)$-gon. Since $\mathcal{S}_1\times\mathcal{S}_2\cong\mathcal{S}_2\times\mathcal{S}_1$ and $(\mathcal{S}_1\times\mathcal{S}_2)\times\mathcal{S}_3\cong\mathcal{S}_1\times(\mathcal{S}_2\times\mathcal{S}_3)$, also the direct product of $k\geq 3$ near polygons $\mathcal{S}_1,\ldots,\mathcal{S}_k$ is well-defined.

Lemma 2 (Theorem 1 of [2]) *Let $\mathcal{S}$ be a near polygon with the property that every two points at distance 2 have at least two common neighbours. If $k\geq 2$ different line sizes occur in $\mathcal{S}$, then $\mathcal{S}$ is isomorphic to a direct product of k near polygons each of which has constant line size.*

Glued near polygons

A *spread* of a near polygon is a set of lines partitioning the point set. A spread S of a near polygon $\mathcal{A}$ is called a *spread of symmetry* if for every line

K of S and for all point pairs $k_1, k_2 \in K$ there exists an automorphism of $\mathcal{A}$ mapping k_1 to k_2. Two spreads of symmetry S_1 and S_2 of a near polygon $\mathcal{A}$ are called *compatible* if the group of automorphisms of $\mathcal{A}$ fixing each line of S_1 commutes with the group of automorphisms of $\mathcal{A}$ fixing each line of S_2.

In [3], it was explained how a certain partial linear space $\mathcal{S}$ can be constructed from any 7-tuple $(\mathcal{A}_1, \mathcal{A}_2, S_1, S_2, K_1, K_2, \theta)$, with $\mathcal{A}_1$ and $\mathcal{A}_2$ near polygons of diameter at least 2, S_i $(i \in \{1,2\})$ a spread of symmetry of $\mathcal{A}_i$, K_i $(i \in \{1,2\})$ a line of S_i and θ a bijection from K_1 to K_2. The resulting partial linear space $\mathcal{S}$ is not necessarily a near polygon, but when it is, it is called a *glued near polygon of type* $\mathcal{A}_1 \otimes \mathcal{A}_2$. We refer to [3] for necessary and sufficient conditions. If $\mathcal{S}$ is a glued near polygon, then there exists a partition T_1 of $\mathcal{S}$ into geodetically closed subgons isomorphic to $\mathcal{A}_1$ and a partition T_2 of $\mathcal{S}$ into geodetically closed subgons isomorphic to $\mathcal{A}_2$ such that

- every element of T_1 intersects every element of T_2 in a line,
- every line is contained in an element of T_1 or an element of T_2.

3 Some properties of near polygons

Let $\mathcal{S}$ be a dense near $2d$-gon, let $\mathcal{F}$ be a geodetically closed subgon of $\mathcal{S}$ and let $i \in \{0, \ldots, d\}$. Then we define the following sets:

- $\mathcal{W}(\mathcal{S})$: the set of all geodetically closed subgons of $\mathcal{S}$,
- $\mathcal{W}_i(\mathcal{S})$: the set of all geodetically closed sub-$2i$-gons of $\mathcal{S}$,
- $\mathcal{W}_i(\mathcal{S}, \mathcal{F})$: the set of all geodetically closed sub-$2i$-gons of $\mathcal{S}$ containing $\mathcal{F}$.

For every point x of $\mathcal{S}$ and every $i \in \{0, \ldots, d\}$, we define $\mathcal{W}_i(\mathcal{S}, x) := \mathcal{W}_i(\mathcal{S}, \{x\})$. If U is a geodetically closed subgon containing x, then $n_i(U, x)$ denotes the number of points of U at distance i from x. Obviously, $n_i(\mathcal{S}, x) = |\Gamma_i(x)|$. By property (ii) of Lemma 1,

$$n_i(\mathcal{S}, x) = \sum_{U \in \mathcal{W}_i(\mathcal{S}, x)} n_i(U, x). \tag{1}$$

Lemma 3 *Let $\mathcal{S}$ be a dense near $2d$-gon, $d \geq 2$, of order (s,t) and let x be a point of $\mathcal{S}$. Then*

$$n_d(\mathcal{S}, x) = v - \sum_{i=0}^{d-1} n_i(\mathcal{S}, x), \tag{2}$$

$$n_{d-1}(\mathcal{S},x) = \frac{1}{s+1}\left(v+\sum_{i=0}^{d-2}[(-s)^{d-i}-1]n_i(\mathcal{S},x)\right), \quad (3)$$

where v denotes the total number of point of $\mathcal{S}$. As a consequence, if the numbers $n_i(\mathcal{S},x)$, $i \in \{0,\ldots,d-2\}$, do not depend on x, then also the numbers $n_{d-1}(\mathcal{S},x)$ and $n_d(\mathcal{S},x)$ do not depend on x.

Proof. Obviously, $v = \sum_{i=0}^{d} n_i(\mathcal{S},x)$ and so equation (2) holds. Every line L of $\mathcal{S}$ contains a unique point nearest to x and hence $\sum_{y\in L}(-s)^{d-\mathrm{d}(x,y)} = 0$. So, $0 = \sum_{L\in\mathcal{L}}\sum_{y\in L}(-s)^{d-\mathrm{d}(x,y)} = \sum_{y\in\mathcal{P}}\sum_{LIy}(-s)^{d-\mathrm{d}(x,y)} = (t+1)\sum_{y\in\mathcal{P}}(-s)^{d-\mathrm{d}(x,y)}$. Hence

$$\sum_{i=0}^{d}(-s)^{d-i}\cdot n_i(\mathcal{S},x) = 0. \quad (4)$$

Equation (3) now easily follows from equations (2) and (4). □

If $\mathcal{S}$ is a dense near polygon of order (s,t), then we can apply Lemma 3 not only to $\mathcal{S}$, but also to every geodetically closed subgon of $\mathcal{S}$ (see property (i) of Lemma 1). Together with the properties of geodetically closed subgons and equation (1) this will allow us to prove Theorem 1 for dense near polygons with an order. For dense near polygons without an order, we will need Lemma 2.

4 Proof of Theorem 1

Theorem 1 says the following.

> Let $\mathcal{S}$ be a dense near $2d$-gon, $d \geq 0$ and let $i \in \{0,\ldots,d\}$. Then the numbers $n_i(\mathcal{S},x)$ do not depend on the point x of $\mathcal{S}$.

We will prove this result by induction on the pair (d,i). (We say that $(d',i') < (d,i)$ if either $d' < d$ or ($d' = d$ and $i' < i$).) Obviously, Theorem 1 is true if $(d,i) \leq (2,0)$. Suppose therefore that $(d,i) > (2,0)$ and that Theorem 1 holds for any dense near $2d'$-gon $\mathcal{S}$ and any $i' \in \{0,\ldots,d'\}$ satisfying $(d',i') < (d,i)$ (= Induction Hypothesis).

(I) Suppose that not every line of $\mathcal{S}$ is incident with the same number of points. Then, by Lemma 2, there exist dense near polygons $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfying $\mathcal{S} \cong \mathcal{S}_1 \times \mathcal{S}_2$, $\mathrm{diam}(\mathcal{S}_1) < \mathrm{diam}(\mathcal{S})$ and $\mathrm{diam}(\mathcal{S}_2) < \mathrm{diam}(\mathcal{S})$. For every point (x,y) of $\mathcal{S}_1 \times \mathcal{S}_2$, we have $n_i(\mathcal{S}_1\times\mathcal{S}_2,(x,y)) = \sum_{j=0}^{i} n_j(\mathcal{S}_1,x)\cdot n_{i-j}(\mathcal{S}_2,y)$ and, by the Induction Hypothesis, this number does not depend on the chosen point (x,y).

(II) Suppose that $\mathcal{S}$ has order (s,t), where $t = t_{\mathcal{S}}$ (see Lemma 1) and $s+1$ is the constant number of points on a line. By connectedness of $\mathcal{S}$, it suffices to show that $n_i(\mathcal{S},x) = n_i(\mathcal{S},y)$ for arbitrary collinear points x and y. By Lemma 3 and the Induction Hypothesis, this property holds if $i \in \{d-1,d\}$. So, suppose that $i \leq d-2$. By equation (1), we have

$$n_i(\mathcal{S},x) = \sum_{U\in\mathcal{W}_i(\mathcal{S},xy)} n_i(U,x) + \sum_{U\in\mathcal{W}_i(\mathcal{S},x)\setminus\mathcal{W}_i(\mathcal{S},xy)} n_i(U,x). \qquad (5)$$

By Property (iii) of Lemma 1, we have

$$\sum_{U\in\mathcal{W}_i(\mathcal{S},x)\setminus\mathcal{W}_i(\mathcal{S},xy)} n_i(U,x) = \sum_{V\in\mathcal{W}_{i+1}(\mathcal{S},xy)} \; \sum_{U\in\mathcal{W}_i(V,x)\setminus\mathcal{W}_i(V,xy)} n_i(U,x).$$

By equation (5),

$$\sum_{U\in\mathcal{W}_i(V,x)\setminus\mathcal{W}_i(V,xy)} n_i(U,x) = n_i(V,x) - \sum_{U\in\mathcal{W}_i(V,xy)} n_i(U,x).$$

Summarizing, we have

$$n_i(\mathcal{S},x) = \sum_{U\in\mathcal{W}_i(\mathcal{S},xy)} n_i(U,x) + \sum_{V\in\mathcal{W}_{i+1}(\mathcal{S},xy)} \left(n_i(V,x) - \sum_{U\in\mathcal{W}_i(V,xy)} n_i(U,x) \right). \qquad (6)$$

Now, all geodetically closed subgons occuring in the right hand side of equation (6) contain the point y and have diameter at most $i+1$. Now, $i+1 \leq d-1$. So, from the Induction Hypothesis, it follows that

$$n_i(\mathcal{S},x) = \sum_{U\in\mathcal{W}_i(\mathcal{S},xy)} n_i(U,y) + \sum_{V\in\mathcal{W}_{i+1}(\mathcal{S},xy)} \left(n_i(V,y) - \sum_{U\in\mathcal{W}_i(V,xy)} n_i(U,y) \right).$$

Now, applying equation (6) with y instead of x, we obtain that

$$n_i(\mathcal{S},x) = n_i(\mathcal{S},y).$$

As mentioned earlier, Theorem 1 now follows from the connectedness of $\mathcal{S}$.

5 Some easy corollaries

Lemma 4 (Proposition 2.6 of [9]) *If (x,Q) is a point-quad pair of a dense near polygon, then precisely one of the following holds.*

(a) *There is a unique point x' in Q nearest to x and $d(x,y) = d(x,x') + d(x',y)$ for every point y of Q. In this case we will say that x is* classical *with respect to Q.*

(b) *The points in Q nearest to x form an ovoid O_x of Q. In this case we will say that x is* ovoidal *with respect to Q.*

Definition. For every quad Q of a dense near polygon $\mathcal{S}$ and for every $i \in \mathbb{N}$, let $M_{i,C}(Q)$, respectively $M_{i,O}(Q)$, be the number of points of $\Gamma_i(Q)$ which are classical, respectively ovoidal, with respect to Q. By Lemma 4, $|\Gamma_i(Q)| = M_{i,C}(Q) + M_{i,O}(Q)$.

Corollary 1 *Let $\mathcal{S}$ be a dense near 2d-gon of order (s,t). Then the following holds.*

(a) *The number of points at distance $i \in \{0,\ldots,d-1\}$ from a line L only depends on i and not on the chosen line L.*

(b) *The number of lines at distance i from a point x only depends on i and not on x.*

(c) *For every $i \in \{1,\ldots,d-1\}$ and for every quad Q, $st_Q \cdot M_{i-1,C} + M_{i,C}(Q) + (1+st_Q)\cdot M_{i,O}(Q)$ only depends on i and t_Q. So, if Q_1 and Q_2 are two quads of order (s,α) for a certain $\alpha \in \mathbb{N}\setminus\{0\}$, then $|\Gamma_i(Q_1)| \equiv |\Gamma_i(Q_2)| \pmod{s\alpha}$ for every $i \in \{0,\ldots,d\}$.*

Proof.

(a) Let $L = \{x_1,\ldots,x_{s+1}\}$, let $i \in \{1,\ldots,d\}$ and let n_i denote the constant number of points at distance i from a given point. If $y \in \Gamma_i(x_j)$ for a certain $j \in \{1,\ldots,s+1\}$, then either $y \in \Gamma_i(L)$ or $y \in \Gamma_{i-1}(L)$. Hence, $(s+1)n_i = |\Gamma_i(x_1)| + |\Gamma_i(x_2)| + \cdots + |\Gamma_i(x_{s+1})| = |\Gamma_i(L)| + s\cdot|\Gamma_{i-1}(L)|$ for every $i \in \{1,\ldots,d\}$. The property now easily follows if one takes into account that $|\Gamma_0(L)| = s+1$.

(b) Let m_i, $i \in \{0,\ldots,n-1\}$, denote the number of lines at distance i from x. Counting flags (y,L) with $y \in \Gamma_i(x)$ gives $n_i(t+1) = m_{i-1}\cdot s + m_i$ for every $i \in \{1,\ldots,d\}$. The property now follows from the fact that $m_0 = t+1$.

(c) Let L^* be a given line of Q. Counting pairs (x,L) with $x \in \Gamma_i(L)$ and L a line of Q gives $(1+t_Q)(1+st_Q)|\Gamma_i(L^*)| = \sum_{L\subset Q}|\Gamma_i(L)| = (1+t_Q)st_Q\cdot M_{i-1,C}(Q) + (1+t_Q)\cdot M_{i,C}(Q) + (1+t_Q)(1+st_Q)\cdot M_{i,O}$. Hence, $st_Q\cdot M_{i-1,C}(Q) + M_{i,C}(Q) + (1+st_Q)\cdot M_{i,O}(Q) = (1+st_Q)\cdot|\Gamma_i(L^*)|$ only depends on i and t_Q. □

Corollary 2 *Let $\mathcal{S}$ be a dense near 2d-gon with $d \geq 3$. Suppose that every quad has order (s, α_1) or (s, α_2) with $\alpha_1 \neq \alpha_2$. Then the number of quads of order (s, α_i) through a given point is a constant.*

Proof. Let n_2 denote the constant number of points at distance 2 from a given point. Take a point x of $\mathcal{S}$ and let λ_i, $i \in \{1, 2\}$, denote the total number of quads of order (s, α_i) containing x. Since every two lines through x are contained in a unique quad, we have

$$(\alpha_1 + 1)\alpha_1 \cdot \lambda_1 + (\alpha_2 + 1)\alpha_2 \cdot \lambda_2 = (t + 1)t. \qquad (7)$$

Since there exists a unique quad through every two points at distance 2, we have

$$s^2\alpha_1 \cdot \lambda_1 + s^2\alpha_2 \cdot \lambda_2 = n_2. \qquad (8)$$

Now, the determinant of the system (7)-(8) is nonzero. So, λ_1 and λ_2 can be written as a function of s, t, α_1, α_2, n_2 and are independent from the chosen point x. □

Corollary 3 *Let $\mathcal{S}$ be a dense near 2d-gon with $d \geq 4$. Suppose that every quad has order (s, α) and that every hex has order (s, β_1) or (s, β_2) with $\beta_1 \neq \beta_2$. Then the number of hexes of order (s, β_i) through a given point is a constant.*

Proof. Let n_3 denote the constant number of points at distance 3 from a given point. Take a point x of $\mathcal{S}$ and let μ_i, $i \in \{1, 2\}$, denote the total number of hexes of order (s, β_i) containing x. One easily calculates that the number $n_3(H_i, x)$ is equal to $\frac{s^3\beta_i(\beta_i - \alpha)}{\alpha+1}$ for every hex H_i of order (s, β_i) and every point x of H_i. Since there exists a unique hex through every two points at distance 3,

$$\frac{s^3\beta_1(\beta_1 - \alpha)}{\alpha + 1} \cdot \mu_1 + \frac{s^3\beta_2(\beta_2 - \alpha)}{\alpha + 1} \cdot \mu_2 = n_3. \qquad (9)$$

Now, if we count the number of pairs (Q, L), where Q is a quad through x and L is a line through x not contained in Q, we find by (iii) of Lemma 1 that

$$\frac{(\beta_1 + 1)\beta_1}{\alpha(\alpha + 1)}(\beta_1 - \alpha) \cdot \mu_1 + \frac{(\beta_2 + 1)\beta_2}{\alpha(\alpha + 1)}(\beta_2 - \alpha) \cdot \mu_2 = \frac{(t + 1)t}{(\alpha + 1)\alpha}(t - \alpha). \qquad (10)$$

Now, the determinant of the system (9)-(10) is nonzero. So, μ_1 and μ_2 can be written as a function of s, t, α, β_1, β_2, n_3 and are independent from the chosen point x. □

Definition. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near $2n$-gon, $n \geq 2$. A distance j-ovoid $(2 \leq j \leq n)$ of $\mathcal{S}$ is a set X of points satisfying:

(1) $\mathrm{d}(x,y) \geq j$ for all points $x, y \in X$;

(2) for every point a of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(a,x) \leq \frac{j}{2}$;

(3) for every line L of $\mathcal{S}$, there exists a point $x \in X$ such that $\mathrm{d}(L,x) \leq \frac{j-1}{2}$.

A distance 2-ovoid is just an ovoid. From (1), (2) and (3), we immediately have:

- If j is odd, then for every point a of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(a,x) \leq \frac{j-1}{2}$.
- If j is even, then for every line L of $\mathcal{S}$, there exists a unique point $x \in X$ such that $\mathrm{d}(L,x) \leq \frac{j-2}{2}$.

If X is a distance j-ovoid, then the map $\mathcal{P} \to \mathbb{N}; x \mapsto \mathrm{d}(x,X)$ is a so-called *valuation* of $\mathcal{S}$, see [5]. Valuations are very important objects for classifying near polygons. These objects will be used in [7] to classify all dense near octagons with three points per line.

Corollary 4 *If $\mathcal{S}$ is a dense near $2n$-gon and if X_1 and X_2 are two distance j-ovoids of $\mathcal{S}$, then X_1 and X_2 have the same number of points.*

Proof. Let v, respectively b, denote the the total number of points, respectively lines, of $\mathcal{S}$. Let n_i, $i \in \{0,\ldots,n\}$, denote the constant number of points at distance i from a given point. Let m_j, $j \in \{0,\ldots,n-1\}$, denote the total number of lines at distance j from a given point (see Corollary 1). By the above remark, we have that every distance j-ovoid contains $\frac{v}{n_0+n_1+\cdots+n_{\frac{j-1}{2}}}$ or $\frac{b}{m_0+m_1+\cdots+m_{\frac{j-2}{2}}}$ points depending on j odd or even. □

6 Dense near polygons with two types of quads and three types of hexes

In this section, let $\mathcal{S}$ be a dense near polygon for which the following holds:

- every quad of $\mathcal{S}$ has order (s,α_1) or (s,α_2),
- (t_H, v_H) is equal to (β_1, v_1), (β_2, v_2) or (β_3, v_3) for every hex H of $\mathcal{S}$.

Here v_H denotes the total number of points of H and s, α_1, α_2, β_1, β_2, β_3, v_1, v_2 and v_3 are nonnegative integers such that $\alpha_1 \neq \alpha_2$ and $(\beta_1, v_1) \neq (\beta_2, v_2) \neq (\beta_3, v_3) \neq (\beta_1, v_1)$. We say that a hex is of type (i), $i \in \{1,2,3\}$, if $(t_H, v_H) = (\beta_i, v_i)$. By Corollary 2, there exist constants λ_1 and λ_2 such that each point of S is contained in λ_i quads of order (s, α_i) $(i \in \{1, 2\})$. By equations (2) and (3), it follows that $n_2(H, x)$ and $n_3(H, x)$ only depend on the type of the hex H and not on the particular choice of the hex H or the point x in H. By equations (7) and (8), it follows that there exist numbers $\lambda_j^{(i)}$ $(1 \leq i \leq 3$ and $1 \leq j \leq 2)$ such that for every hex H of type (i) and every point $x \in H$, there are $\lambda_j^{(i)}$ quads of order (s, α_j) through x contained in H. By equation (4) it follows that for every hex H of type (i) and every point x of H, $n_3(H, x) = s^3(\lambda_1^{(i)}\alpha_1 + \lambda_2^{(i)}\alpha_2 - \beta_i)$. Let x be a point and let μ_i, $i \in \{1, 2, 3\}$, denote the total number of hexes of type (i) containing x. Counting points at distance 3 from x, we obtain that $\frac{n_3}{s^3}$ is equal to

$$(\lambda_1^{(1)}\alpha_1 + \lambda_2^{(1)}\alpha_2 - \beta_1) \cdot \mu_1 + (\lambda_1^{(2)}\alpha_1 + \lambda_2^{(2)}\alpha_2 - \beta_2) \cdot \mu_2 + (\lambda_1^{(3)}\alpha_1 + \lambda_2^{(3)}\alpha_2 - \beta_3) \cdot \mu_3.$$

Counting pairs (Q, L) where Q is a quad of order (s, α_j) through x and L is a line through x not contained in Q, gives

$$\lambda_1^{(1)}(\beta_1 - \alpha_1) \cdot \mu_1 + \lambda_1^{(2)}(\beta_2 - \alpha_1) \cdot \mu_2 + \lambda_1^{(3)}(\beta_3 - \alpha_1) \cdot \mu_3 = \lambda_1(t - \alpha_1),$$
$$\lambda_2^{(1)}(\beta_1 - \alpha_2) \cdot \mu_1 + \lambda_2^{(2)}(\beta_2 - \alpha_2) \cdot \mu_2 + \lambda_2^{(3)}(\beta_3 - \alpha_2) \cdot \mu_3 = \lambda_2(t - \alpha_2).$$

Let $M(s, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, v_1, v_2, v_3)$ be the following matrix

$$\begin{bmatrix} \lambda_1^{(1)}(\beta_1 - \alpha_1) & \lambda_1^{(2)}(\beta_2 - \alpha_1) & \lambda_1^{(3)}(\beta_3 - \alpha_1) \\ \lambda_2^{(1)}(\beta_1 - \alpha_2) & \lambda_2^{(2)}(\beta_2 - \alpha_2) & \lambda_2^{(3)}(\beta_3 - \alpha_2) \\ \lambda_1^{(1)}\alpha_1 + \lambda_2^{(1)}\alpha_2 - \beta_1 & \lambda_1^{(2)}\alpha_1 + \lambda_2^{(2)}\alpha_2 - \beta_2 & \lambda_1^{(3)}\alpha_1 + \lambda_2^{(3)}\alpha_2 - \beta_3 \end{bmatrix}.$$

The above equations provide a proof of the following result.

Theorem 2 *If* $\det[M(s, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, v_1, v_2, v_3)] \neq 0$, *then the number of hexes of each type through a point is a constant.*

For every hex H of type (i), we define

$$N(H) := \begin{bmatrix} \lambda_1^{(i)}(\beta_i - \alpha_1) \\ \lambda_2^{(i)}(\beta_i - \alpha_2) \\ \lambda_1^{(i)}\alpha_1 + \lambda_2^{(i)}\alpha_2 - \beta_i \end{bmatrix}.$$

Except for three examples when $s = 2$, every known dense near hexagon is either regular, glued or a direct product of a line and a generalized quadrangle. Consider now the following classes of near hexagons.

- For all $s, t_2 \in \mathbb{N} \setminus \{0, 1\}$, let $C_1(s, t_2)$ denote the class of all near hexagons which are isomorphic to $L_{s+1} \times Q$ for some generalized quadrangle Q of order (s, t_2).
- For all $s, t_2 \in \mathbb{N} \setminus \{0, 1\}$, let $C_2(s, t)$ denote the class of all near hexagons which are isomorphic to a glued near hexagon of type $Q \otimes Q'$ for some generalized quadrangles Q and Q' of order (s, t_2).
- For all s, t_2, t with $s \geq 2$ and $t, t_2 \in \mathbb{N} \setminus \{0\}$, let $C_3(s, t, t_2)$ denote the class of all regular near hexagons with parameters (s, t, t_2).

In the following table, we list the parameters for each of these classes.

class	α_1	α_2	β	λ_1	λ_2
$C_1(s, t_2)$	1	t_2	$t_2 + 1$	$t_2 + 1$	1
$C_2(s, t_2)$	1	t_2	$2t_2$	t_2^2	2
$C_3(s, t_2, t)$	t_2	—	t	$\frac{t(t+1)}{t_2(t_2+1)}$	0

If H belongs to one of these classes, then $N(H)$ is equal to one of the following matrices :

$$N_1(s, t_2) := \begin{bmatrix} t_2(t_2 + 1) \\ 1 \\ t_2 \end{bmatrix},$$

$$N_2(s, t_2) := \begin{bmatrix} t_2^2(2t_2 - 1) \\ 2t_2 \\ t_2^2 \end{bmatrix},$$

$$N_3(s, t, t_2) := \begin{bmatrix} \frac{t(t+1)(t-t_2)}{t_2(t_2+1)} \\ 0 \\ \frac{t(t-t_2)}{t_2+1} \end{bmatrix}.$$

If S is a dense near hexagon with two types of quads and three types of hexes such that every hex belongs to one of the above classes, then we have to consider the following possibilities.

(1) Every hex belongs to $C_1(s, t_2) \cup C_2(s, t_2) \cup C_3(s, t_2, t)$, $t_2 \neq 1$.

(2) Every hex belongs to $C_1(s, t_2) \cup C_2(s, t_2) \cup C_3(s, 1, t)$, $t_2 \neq 1$.

(3) Every hex belongs to $C_1(s, t_2) \cup C_3(s, t_2, t) \cup C_3(s, 1, t')$, $t_2 \neq 1$.

(4) Every hex belongs to $C_1(s, t_2) \cup C_3(s, t_2, t) \cup C_3(s, t_2, t')$, $t_2 \neq 1$ and $t \neq t'$.

(5) Every hex belongs to $C_1(s,t_2) \cup C_3(s,1,t) \cup C_3(s,1,t')$, $t_2 \neq 1$ and $t \neq t'$.

(6) Every hex belongs to $C_2(s,t_2) \cup C_3(s,t_2,t) \cup C_3(s,1,t')$, $t_2 \neq 1$.

(7) Every hex belongs to $C_2(s,t_2) \cup C_3(s,t_2,t) \cup C_3(s,t_2,t')$, $t_2 \neq 1$ and $t \neq t'$.

(8) Every hex belongs to $C_2(s,t_2) \cup C_3(s,1,t) \cup C_3(s,1,t')$, $t_2 \neq 1$ and $t \neq t'$.

We will now treat each of these cases separately.

Case (1):

In this case, we have

$$M = \begin{bmatrix} t_2(t_2+1) & t_2^2(2t_2-1) & 0 \\ 1 & 2t_2 & \frac{t(t+1)(t-t_2)}{t_2(t_2+1)} \\ t_2 & t_2^2 & \frac{t(t-t_2)}{t_2+1} \end{bmatrix}.$$

One easily calculates that $\det(M) = \frac{t_2^2 t(t-t_2)}{t_2+1}[t(t_2-2)+t_2+1]$. Since $t_2 \geq 2$, $\det(M)$ is always nonzero. So, the number of hexes of each type through a point is a constant. This case includes the near polygons in which each hex is isomorphic to either $Q(5,q) \times L_{q+1}$, a glued near polygon of type $Q(5,q) \otimes Q(5,q)$ or $H^D(5,q^2)$. All the near polygons which have only these hexes will be classified in [6]. [Recall: $Q(5,q)$ is the generalized quadrangle of the points and lines of a nonsingular elliptic quadric in $\mathrm{PG}(5,q)$. $H^D(5,q^2)$ is the near hexagon whose points, respectively lines, are the two-, respectively one-dimensional, subspaces of a nonsingular hermitean variety in $\mathrm{PG}(5,q^2)$, with reverse containment as incidence relation.]

Case (2):

In this case, we have

$$M = \begin{bmatrix} t_2(t_2+1) & t_2^2(2t_2-1) & \frac{t(t+1)(t-1)}{2} \\ 1 & 2t_2 & 0 \\ t_2 & t_2^2 & \frac{t(t-1)}{2} \end{bmatrix}.$$

One calculates that $\det(M) = -\frac{t(t-1)(t-2)}{2}t_2^2$. If $t \neq 2$, then the number of hexes of each type through a point is a constant. This property does not necessarily hold if $t = 2$. In [7] it is shown that there exist two glued near octagons of type $(Q(5,2) \otimes Q(5,2)) \otimes Q(5,2)$ and one of these near octagons does not have the required property: there are points which are contained in three hexes of type $Q(5,2) \otimes Q(5,2)$ and there are points which are contained in only two such hexes. The construction of the near octagon given in [7] can be generalized to generalized quadrangles with a pair of compatible spreads of symmetry. If Q is a generalized quadrangle of order (s,t) having a pair (S_1, S_2) of compatible spreads of symmetry such that $|S_1 \cap S_2| \in \{1, s+1\}$, then there exists a glued near octagon of type $(Q \otimes Q) \otimes Q$ which contains points x_2 and x_3 such that x_i, $i \in \{2,3\}$, is contained in precisely i hexes of type $Q \otimes Q$. Among all known examples of generalized quadrangles with a spread of symmetry only the generalized quadrangles $Q(5,q)$ have a pair (S_1, S_2) of compatible spreads of symmetry such that $|S_1 \cap S_2| \in \{1, s+1\}$. We refer to [4] for more details.

Case (3):

In this case, we have

$$M = \begin{bmatrix} t_2(t_2+1) & 0 & \frac{t'(t'+1)(t'-1)}{2} \\ 1 & \frac{t(t+1)(t-t_2)}{t_2(t_2+1)} & 0 \\ t_2 & \frac{t(t-t_2)}{t_2+1} & \frac{t'(t'-1)}{2} \end{bmatrix}.$$

One calculates that $\det(M) = \frac{tt'(t'-1)(t-t_2)}{2(t_2+1)}[(t_2+1)(t+1) - (t'+1)t]$. If $\det(M) = 0$, then t would divide $t_2 + 1$ and hence $t \leq t_2 + 1$. Now, by the corollary on p158 of [2] we have $t \geq t_2(t_2+1)$ for any regular near hexagon with parameters (s, t_2, t). Since $t_2 \geq 2$, we have $t > t_2 + 1$. So, $\det(M) \neq 0$ and the number of hexes of each type through a point is a constant.

Case (4):

In this case, we have

$$M = \begin{bmatrix} t_2(t_2+1) & 0 & 0 \\ 1 & \frac{t(t+1)(t-t_2)}{t_2(t_2+1)} & \frac{t'(t'+1)(t'-t_2)}{t_2(t_2+1)} \\ t_2 & \frac{t(t-t_2)}{t_2+1} & \frac{t'(t'-t_2)}{t_2+1} \end{bmatrix}.$$

One calculates that $\det(M) = \frac{tt'(t-t_2)(t'-t_2)(t-t')}{t_2+1}$. So, the number of hexes of each type through a point is a constant.

Case (5):

In this case, we have

$$M = \begin{bmatrix} t_2(t_2+1) & \frac{t(t+1)(t-1)}{2} & \frac{t'(t'+1)(t'-1)}{2} \\ 1 & 0 & 0 \\ t_2 & \frac{t(t-1)}{2} & \frac{t'(t'-1)}{2} \end{bmatrix}.$$

One calculates that $\det(M) = \frac{tt'(t-1)(t'-1)(t-t')}{4} \neq 0$. So, the number of hexes of each type through a point is a constant.

Case (6):

In this case, we have

$$M = \begin{bmatrix} t_2^2(2t_2-1) & 0 & \frac{t'(t'+1)(t'-1)}{2} \\ 2t_2 & \frac{t(t+1)(t-t_2)}{t_2(t_2+1)} & 0 \\ t_2^2 & \frac{t(t-t_2)}{t_2+1} & \frac{t'(t'-1)}{2} \end{bmatrix}.$$

One calculates that $\det(M) = \frac{t_2tt'(t-t_2)(t'-1)}{2(t_2+1)}[(t+1)(2t_2-1)-(t'+1)(t-1)]$. If $\det(M) = 0$, then $t'+1 = 2t_2-1+\frac{4t_2-2}{t-1}$. Hence, $\frac{4t_2-2}{t-1} \in \mathbb{N}$ and $t+1 \leq 4t_2$. Since $st_2 \geq 4$, we have $t+1 < (st_2+1)(t_2+1)$. From Theorem 5 of [2], it then follows that $t = t_2^2 + t_2$. From $t+1 \leq 4t_2$, it then follows that $t_2 = 2$ and $t = 6$, but this is impossible since we should have that $\frac{4t_2-2}{t-1} \in \mathbb{N}$. So, $\det(M)$ is always nonzero and the number of hexes of each type through a point is a constant.

Case (7):

In this case, we have

$$M = \begin{bmatrix} t_2^2(2t_2-1) & 0 & 0 \\ 2t_2 & \frac{t(t+1)(t-t_2)}{t_2(t_2+1)} & \frac{t'(t'+1)(t'-t_2)}{t_2(t_2+1)} \\ t_2^2 & \frac{t(t-t_2)}{t_2+1} & \frac{t'(t'-t_2)}{t_2+1} \end{bmatrix}.$$

One calculates that $\det(M) = \frac{t_2(2t_2-1)tt'(t-t_2)(t'-t_2)(t-t')}{(t_2+1)^2} \neq 0$. So, the number of hexes of each type through a point is a constant.

Case (8):

In this case, we have

$$M = \begin{bmatrix} t_2^2(2t_2-1) & \frac{t(t+1)(t-1)}{2} & \frac{t'(t'+1)(t'-1)}{2} \\ 2t_2 & 0 & 0 \\ t_2^2 & \frac{t(t-1)}{2} & \frac{t'(t'-1)}{2} \end{bmatrix}.$$

One calculates that $\det(M) = \frac{t_2 tt'(t-1)(t'-1)(t-t')}{2} \neq 0$. So, the number of hexes of each type through a point is a constant.

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