

# MINIMAL EQUITABILITY OF HAIRY CYCLES

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**ABSTRACT.** Every labeling of the vertices of a graph with distinct natural numbers induces a natural labeling of its edges: the label of an edge  $(x, y)$  is the absolute value of the difference of the labels of  $x$  and  $y$ . By analogy with graceful labelings, we say that a labeling of the vertices of a graph of order  $n$  is minimally  $k$ -equitable if the vertices are labelled with  $1, 2, \dots, n$  and in the induced labeling of its edges every label either occurs exactly  $k$  times or does not occur at all. For  $m \geq 3$ , let  $C'_m$  (denoted also in the literature by  $C_m \circ K_1$  and called a corona graph) be a graph with  $2m$  vertices such that there is a partition of them into sets  $U$  and  $V$  of cardinality  $m$ , with the property that  $U$  spans a cycle,  $V$  is independent and the edges joining  $U$  to  $V$  form a matching. Let  $\mathcal{P}$  be the set of all pairs  $(m, k)$  of positive integers such that  $k$  is a proper divisor of  $2m$  (i.e., a divisor different from  $2m$  and  $1$ ) and  $k$  is odd if  $m$  is odd. We show that  $C'_m$  is minimally  $k$ -equitable if and only if  $(m, k) \in \mathcal{P}$ .

## 1. INTRODUCTION

A *labeling* of a graph  $G$  is an assignment of distinct natural numbers to the vertices of  $G$ . Every labeling induces a natural labeling of the edges: the label of an edge  $(x, y)$  is the absolute value of the difference of the labels of  $x$  and  $y$ . Bloom [4] defined a labeling of a graph to be  *$k$ -equitable* if in the induced labeling of its edges, every label occurs exactly  $k$  times, if at all. Furthermore, a  $k$ -equitable labeling of a graph of order  $n$  is said to be *minimal* if the vertices are labelled with  $1, 2, \dots, n$ . A graph is *minimally  $k$ -equitable* if it has a minimal  $k$ -equitable labeling.

The notion of minimally  $k$ -equitable labelings is a generalization of special labelings of trees called graceful. A labeling of a tree is *graceful* if and only if it is minimally 1-equitable. (Note that the induced labels of the edges must be then  $1, 2, \dots, n - 1$ .) Graceful labelings were defined by Rosa [6] in connection with a famous and very difficult conjecture of Ringel and Kotzig (see Ringel [5]) concerning decompositions of complete graphs with odd number of vertices into subgraphs isomorphic to trees. The open conjecture that every tree has a graceful labeling implies the conjecture of Ringel and Kotzig.

The notion of minimal  $k$ -equitability was first applied to cycles. Let  $C_n$  be the cycle on  $n$  vertices. Given natural numbers  $n$  and  $k$ ,  $n \geq 3$ , it is easy to see that if the cycle  $C_n$  is minimally  $k$ -equitable, then  $k$  is a proper divisor of  $n$  (that is,  $k$  divides  $n$ ,  $k \neq n$ , and  $k \neq 1$ ). Answering a question posed by Bloom [4], Wojciechowski [7] proved that this necessary condition is also sufficient.

Barrientos, Dejter and Hevia [3] proved a number of results concerning  $k$ -equitability of forests.

Another class of graphs to which the concept of minimal  $k$ -equitability was recently applied is the class of graphs whose vertices could be partitioned into two sets  $U$  and  $V$  such that  $U$  induces a cycle,  $V$  induces no edges, and the edges between  $U$  and  $V$  form a matching (in particular,  $U$  and  $V$  are of the same cardinality). We will call such graphs *hairy cycles* and denote them by  $C'_m$  ( $m$  is the cardinality of  $U$ ).

There is a general construction that, given graphs  $G$  and  $H$ , produces the *corona* graph  $G \circ H$ . The hairy cycle  $C'_m$  is obtained by applying that construction to the cycle  $C_m$  and the graph  $K_1$  consisting of a single vertex. Therefore  $C'_m$  is the corona graph  $C_m \circ K_1$ .

Again it is easy to see that given natural numbers  $m$  and  $k$ ,  $m \geq 3$ , if the hairy cycle  $C'_m$  is minimally  $k$ -equitable, then  $k$  is a proper divisor of  $2m$  which is the number of vertices of  $C'_m$ . Going a little further, we get the following result.

**Proposition 1.1.** *Let  $m \geq 3$  and  $k \geq 1$  be integers. If  $m$  is odd and the hairy cycle  $C'_m$  is minimally  $k$ -equitable, then  $k$  is also odd.*

*Proof.* Assume that  $m \geq 3$  is arbitrary and the hairy cycle  $C'_m$  is minimally  $k$ -equitable. Let  $U$  and  $V$  be the sets that form a partition of the vertex set of  $C'_m$  such that  $U$  spans a cycle,  $V$  spans no edges, and the edges between  $U$  and  $V$  form a matching. Let  $E$  be the set of edges of the cycle spanned by  $U$  and  $F$  be the set of the edges of the matching. Let

$$g : U \cup V \rightarrow \{1, 2, \dots, 2m\}$$

be any labeling of  $C'_m$ , and  $h : E \cup F \rightarrow \mathbb{N}$  be the induced labeling of the edges of  $C'_m$ . It is clear that  $\sum_{e \in E} h(e)$  is even.

We claim that

$$(1) \quad \sum_{e \in F} h(e) \equiv m \pmod{2}.$$

If  $h(e) = 1$  for every  $e \in F$ , then  $\sum_{e \in F} h(e) = m$  so (1) holds. Otherwise, there are vertices  $u \in U$  and  $v \in V$  such that  $|g(u) - g(v)| \geq 2$ . Let  $w, y$  be vertices of  $C'_m$  such that  $g(w)$  is between  $g(u)$  and  $g(v)$  in the standard ordering of integers, and  $y$  is joined to  $w$  by an edge of  $F$ . Let  $g' : U \cup V \rightarrow \mathbb{N}$  be the labeling of  $C'_m$  obtained from  $g$  by exchanging  $\min\{g(w), g(y)\}$  with

$\max \{g(u), g(v)\}$ . For example, if

$$g(u) < g(w) < g(v) < g(y),$$

then we exchange the labels of  $w$  and  $v$ .

Let  $h' : E \cup F \rightarrow \mathbb{N}$  be the labeling of the edges of  $C'_m$  induced by  $g'$ . Then

$$\sum_{e \in F} h'(e) < \sum_{e \in F} h(e)$$

and

$$\sum_{e \in F} h'(e) \equiv \sum_{e \in F} h(e) \pmod{2}.$$

Therefore, induction on  $\sum_{e \in F} h(e)$  can be used to prove that (1) holds in general.

If  $m$  is odd and  $g$  is a minimal  $k$ -equitable labeling, then  $\sum_{e \in E \cup F} h(e)$  is odd and divisible by  $k$  implying that  $k$  is odd. ■

Let  $\mathcal{P}$  be the set of all pairs  $(m, k)$  of positive integers such that  $m \geq 3$ ,  $k$  is a proper divisor of  $2m$  and  $k$  is odd if  $m$  is odd. Then for the graph  $C'_m$  to be minimally  $k$ -equitable it is necessary that  $(m, k) \in \mathcal{P}$ . Acharya and Bhat-Nayak [1] [2] proved that the condition  $(m, k) \in \mathcal{P}$  is sufficient for  $C'_m$  to be minimally  $k$ -equitable when  $k \in \{3, 4\}$ . We will prove that this condition is sufficient in general.

**Theorem 1.2.** *Let  $m \geq 3$  and  $k \geq 2$  be integers. If  $(m, k) \in \mathcal{P}$ , then the graph  $C'_m$  is minimally  $k$ -equitable.*

We shall call a graph  $G$  an *integer graph* if its vertex set is a finite subset of  $\mathbb{N}$ . If  $e = (u, v)$  is an edge of  $G$ , then we will say that  $e$  has *length*  $|u - v|$ . Observe that a finite graph  $H$  has a  $k$ -equitable labeling if and only if it is isomorphic to an integer graph  $G$  with either 0 or  $k$  edges of any length. We will call such  $G$  a  *$k$ -equitable representation* of the graph  $H$ . Note also that a finite graph  $H$  is minimally  $k$ -equitable if it has a  $k$ -equitable representation  $G$  whose vertices are consecutive integers. We will call such  $G$  a *minimal  $k$ -equitable representation* of  $H$ .

In the following proofs, to show that  $C'_m$  is  $k$ -equitable, we will construct an integer graph  $G_m^k$  that will be a minimal  $k$ -equitable representation of  $C'_m$ . In the included figures of the graphs  $G_m^k$  the vertices will be placed on the real line, the edges that are part of the cycle (*cycle edges*) will be marked by thick lines below the real line and the edges of the matching (*matching edges*) above the real line.

The proof of Theorem 1.2 will be split into several lemmas. The proofs of these lemmas will be given in the remaining sections. Assume that  $(m, k) \in \mathcal{P}$ . We have then two possibilities: either  $k$  divides  $m$  or it does not divide  $m$ . In section 2 we will present the starting point of our construction in the case when  $k$  divides  $m$ , namely, we will show that  $C'_m$  is minimally

$k$ -equitable when  $m = k$  or  $m = 2k$ . The proof that  $C'_{\ell k}$  is minimally  $k$ -equitable for  $\ell \geq 3$  will be given in section 4 for even  $k$  and in section 5 for odd  $k$ . If  $k$  does not divide  $m$ , then the definition of  $\mathcal{P}$  implies that  $k \equiv 0 \pmod 4$  and  $m = \ell k/2$  for some odd integer  $\ell \geq 3$ . Let  $k \equiv 0 \pmod 4$ . We will prove in section 3 that the graph  $C'_{3k/2}$  is minimally  $k$ -equitable. The proof that  $C'_{\ell k/2}$  is minimally  $k$ -equitable for odd  $\ell \geq 5$  will be given in section 4.

## 2. MINIMAL $k$ -EQUITABILITY OF $C'_k$ AND $C'_{2k}$

**Lemma 2.1.** *Let  $k \geq 3$  be an integer. The graph  $C'_k$  is minimally  $k$ -equitable.*

*Proof.* Let  $G_k^k$  be the integer graph with the vertex set  $\{1, 2, \dots, 2k\}$  and the edge set consisting of all the edges listed below:

- $(1, 2), (2, 3), \dots, (k, k + 1)$

— the above edges are  $k$  edges of length 1, the first of which is a matching edge, and all the others are cycle edges;

- $(2, k + 1), (3, k + 2), \dots, (k + 1, 2k)$

— the above edges are  $k$  edges of length  $k - 1$ , the first of which is a cycle edge, and all the others are matching edges.

Figure 1 shows the graph  $G_3^3$  and Figure 2 shows the graph  $G_k^k$  for arbitrary  $k \geq 3$ .

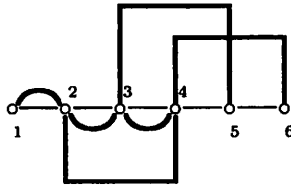


FIGURE 1. The graph  $G_3^3$

It is clear that  $G_k^k$  is a minimal  $k$ -equitable representation of  $C'_k$  implying that  $C'_k$  is minimally  $k$ -equitable. ■

**Lemma 2.2.** *Let  $k \geq 3$  be an odd integer. The graph  $C'_{2k}$  is minimally  $k$ -equitable.*

*Proof.* Let  $G_{2k}^k$  be the integer graph with the vertex set  $\{1, 2, \dots, 4k\}$  and the edge set consisting of all the edges listed below:

- $(1, \frac{k+3}{2}), (2, \frac{k+5}{2}), \dots, (\frac{k+1}{2}, k + 1)$

— the edges listed above are  $\frac{k+1}{2}$  edges of length  $\frac{k+1}{2}$ , all of which are matching edges;

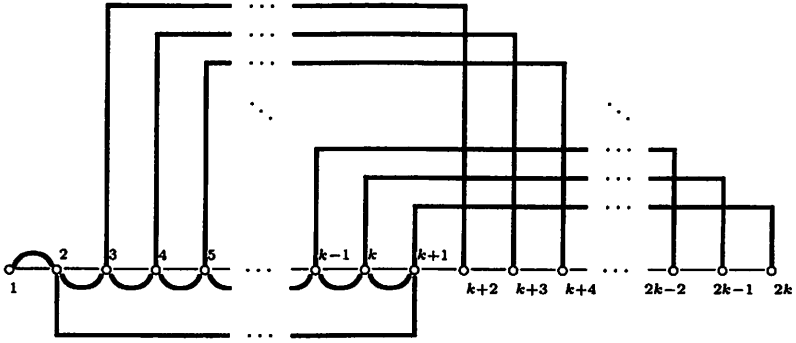


FIGURE 2. The graph  $G_k^k$

- $(k+2, \frac{3k+5}{2}), (k+3, \frac{3k+7}{2}), \dots, (\frac{3k+1}{2}, 2k+1)$
- the edges listed above are  $\frac{k-1}{2}$  edges of length  $\frac{k+1}{2}$ , the last of which is a cycle edge, and all the others are matching edges;
- $(\frac{k+3}{2}, \frac{k+5}{2}), (\frac{k+7}{2}, \frac{k+9}{2}), \dots, (\frac{3k+1}{2}, \frac{3k+3}{2})$
- these are  $\frac{k+1}{2}$  edges of length 1, the last of which is a matching edge, and all the others are cycle edges;
- $(2k+2, 2k+3), (2k+4, 2k+5), \dots, (3k-1, 3k)$
- these are  $\frac{k-1}{2}$  edges of length 1, all of which are cycle edges;
- $(\frac{k+3}{2}, 2k+1), (\frac{k+5}{2}, 2k+2), \dots, (\frac{3k+1}{2}, 3k)$
- these are  $k$  edges of length  $\frac{3k-1}{2}$ , all of which are cycle edges;
- $(2k+1, 3k+1), (2k+2, 3k+2), \dots, (3k, 4k)$
- these are  $k$  edges of length  $k$ , all of which are matching edges.

Figures 3, 4, and 5 show the graphs  $G_6^3$ ,  $G_{10}^5$ , and  $G_{14}^7$  respectively; figures 6 and 7 show the general graph  $G_{2k}^k$  for arbitrary  $k \equiv 1 \pmod{4}$ , and  $k \equiv 3 \pmod{4}$  respectively.

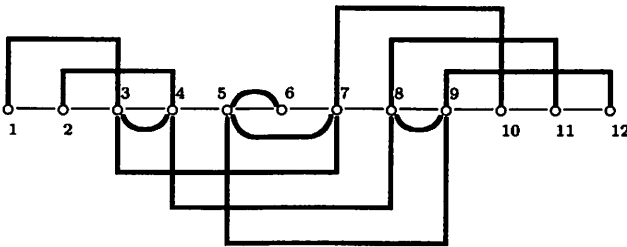


FIGURE 3. The graph  $G_6^3$

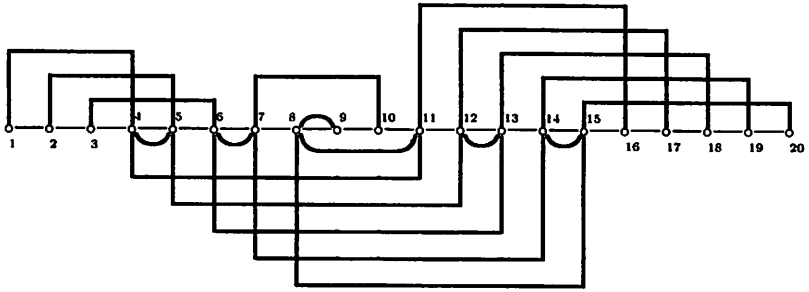


FIGURE 4. The graph  $G_{10}^5$

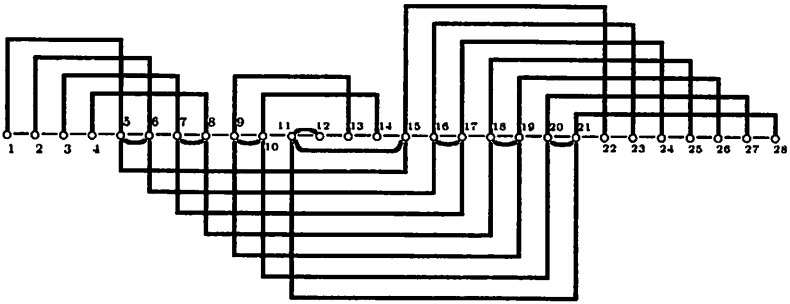


FIGURE 5. The graph  $G_{14}^7$

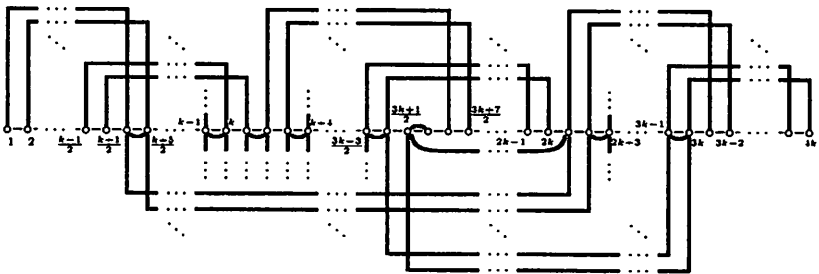


FIGURE 6. The graph  $G_{2k}^k$  when  $k \equiv 1 \pmod{4}$

It is clear that  $G_{2k}^k$  is a minimal  $k$ -equitable representation of  $C'_{2k}$  implying that  $C'_{2k}$  is minimally  $k$ -equitable. ■

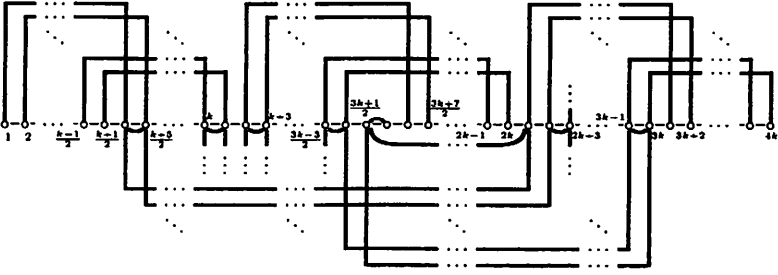


FIGURE 7. The graph  $G_{2k}^k$  when  $k \equiv 3 \pmod{4}$

**Lemma 2.3.** *Let  $k \geq 2$  be an even integer. The graph  $C'_{2k}$  is minimally  $k$ -equitable.*

*Proof.* Let  $G_4^2$  be the integer graph with the vertex set  $\{1, 2, \dots, 8\}$  and the edge set consisting of the following edges (see Figure 8):

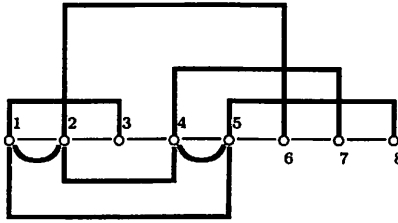


FIGURE 8. The graph  $G_4^2$

- $(1, 2), (4, 5)$  — 2 edges of length 1 — both of them cycle edges;
- $(1, 3), (2, 4)$  — 2 edges of length 2 — the first a matching edge, the second a cycle edge;
- $(4, 7), (5, 8)$  — 2 edges of length 3 — both of which being matching edges;
- $(1, 5), (2, 6)$  — 2 edges of length 4 — the first a cycle edge, the second a matching edge.

For  $k \geq 4$  let  $G_{2k}^k$  be the integer graph with the vertex set  $\{1, 2, \dots, 4k\}$  and the edge set consisting of all the edges listed below:

- $(1, \frac{k+2}{2}), (2, \frac{k+4}{2}), \dots, (\frac{k}{2}, k)$
- these are  $\frac{k}{2}$  edges of length  $\frac{k}{2}$ , all of which are matching edges;
- $(k+1, \frac{3k+2}{2}), (k+2, \frac{3k+4}{2}), \dots, (\frac{3k}{2}, 2k)$
- these are  $\frac{k}{2}$  edges of length  $\frac{k}{2}$ , the last of which is a cycle edge, and all the others are matching edges;

- $(\frac{k+2}{2}, \frac{k+4}{2}), (\frac{k+6}{2}, \frac{k+8}{2}), \dots, (\frac{3k-2}{2}, \frac{3k}{2})$
- these are  $\frac{k}{2}$  edges of length 1, all of which are cycle edges;
- $(2k, 2k+1), (2k+2, 2k+3), \dots, (3k-2, 3k-1)$
- these are  $\frac{k}{2}$  edges of length 1, all of which are cycle edges;
- $(\frac{k+2}{2}, 2k+1), (\frac{k+4}{2}, 2k+2), \dots, (\frac{3k}{2}, 3k)$
- these are  $k$  edges of length  $\frac{3k}{2}$ , the last of which is a matching edge, and all the others are cycle edges;
- $(2k, 3k+1), (2k+1, 3k+2), \dots, (3k-1, 4k)$
- these are  $k$  edges of length  $k+1$ , all of which are matching edges.

Figures 9 and 10 show the graphs  $G_8^4$  and  $G_{12}^6$  respectively; Figures 11 and 12 show the general graph  $G_{2k}^k$  for arbitrary  $k \equiv 0 \pmod{4}$ , and  $k \equiv 2 \pmod{4}$  respectively.

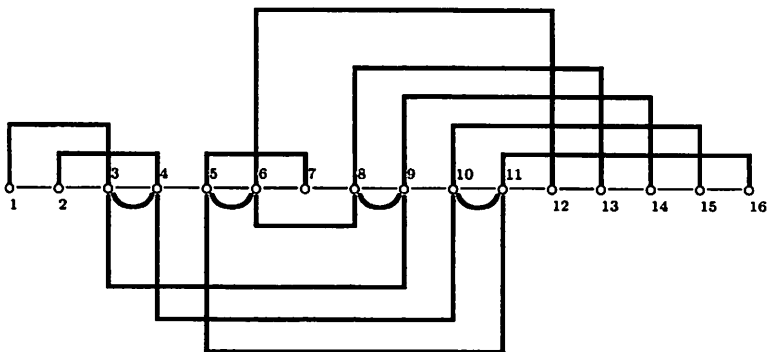


FIGURE 9. The graph  $G_8^4$

It is clear that, for every even  $k \geq 2$ , the graph  $G_{2k}^k$  is a minimal  $k$ -equitable representation of  $C'_{2k}$ , implying that  $C'_{2k}$  is minimally  $k$ -equitable. ■

### 3. MINIMAL $k$ -EQUITABILITY OF $C'_{3k/2}$ WHEN $k \equiv 0 \pmod{4}$

**Lemma 3.1.** *Let  $k \geq 4$  be an integer such that  $k \equiv 0 \pmod{4}$ . The graph  $C'_{3k/2}$  is minimally  $k$ -equitable.*

*Proof.* Let  $G_{3k/2}^k$  be the integer graph with the vertex set  $\{1, 2, \dots, 3k\}$  and the edge set consisting of all the edges listed below:

- $(1, \frac{k+2}{2}), (2, \frac{k+4}{2}), \dots, (\frac{k-2}{2}, k-1)$
- these are  $\frac{k-2}{2}$  edges of length  $\frac{k}{2}$ , all of which are matching edges;
- $(\frac{3k+4}{2}, 2k+2), (\frac{3k+6}{2}, 2k+3), \dots, (2k-1, \frac{5k-2}{2})$



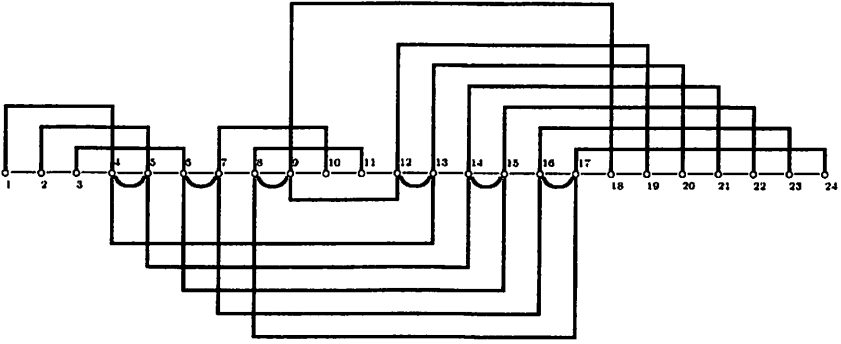


FIGURE 10. The graph  $G_{12}^6$

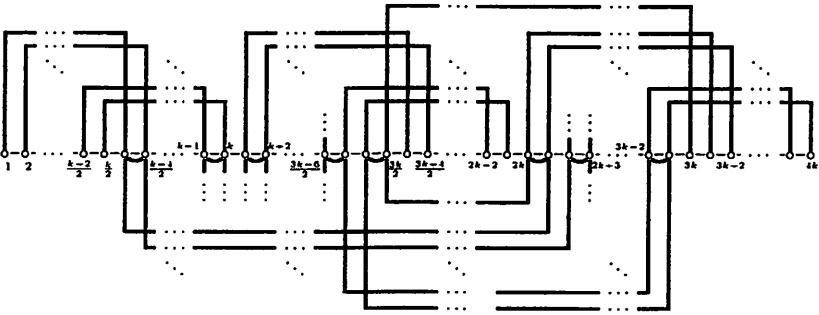


FIGURE 11. The graph  $G_{2k}^k$  when  $k \equiv 0 \pmod{4}$

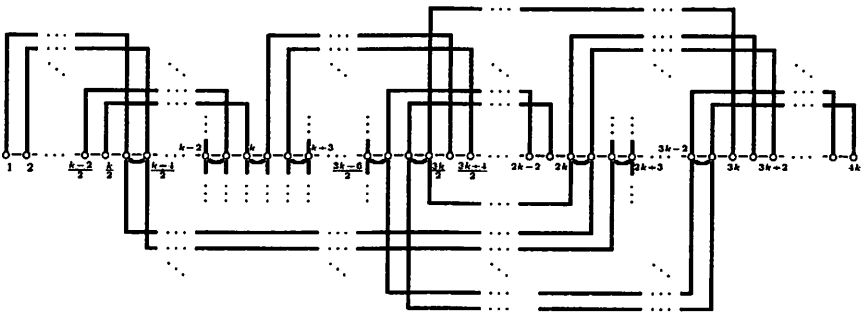


FIGURE 12. The graph  $G_{2k}^k$  when  $k \equiv 2 \pmod{4}$

— these are  $\frac{k-4}{2}$  edges of length  $\frac{k}{2}$ , all of which are matching edges, (they appear only when  $k \geq 8$ );

- $(k + 1, \frac{3k+2}{2}), (\frac{3k}{2}, 2k), (2k + 1, \frac{5k+2}{2})$
- these are 3 edges of length  $\frac{k}{2}$ , the first and the third of which are matching edges, and the second is a cycle edge;
- $(\frac{k+2}{2}, \frac{k+4}{2}), (\frac{k+6}{2}, \frac{k+8}{2}), \dots, (k - 1, k)$
- these are  $\frac{k}{4}$  edges of length 1, all of which are cycle edges;
- $(k, k + 1), (k + 1, k + 2), \dots, (\frac{3k-2}{2}, \frac{3k}{2})$
- these are  $\frac{k}{2}$  edges of length 1, all of which are cycle edges;
- $(2k, 2k + 1), (2k + 2, 2k + 3), \dots, (\frac{5k-4}{2}, \frac{5k-2}{2})$
- these are  $\frac{k}{4}$  edges of length 1, all of which are cycle edges;
- $(\frac{k}{2}, 2k), (\frac{k+2}{2}, 2k + 1), \dots, (k, \frac{5k}{2})$
- these are  $\frac{k+2}{2}$  edges of length  $\frac{3k}{2}$ , the first and the last of which are matching edges, and all the others are cycle edges;
- $(k + 2, \frac{5k+4}{2}), (k + 3, \frac{5k+6}{2}), \dots, (\frac{3k}{2}, 3k)$
- these are  $\frac{k-2}{2}$  edges of length  $\frac{3k}{2}$ , all of which are matching edges.

Figures 13, 14 and 15 show the graphs  $G_6^4$ ,  $G_{12}^8$ , and  $G_{18}^{12}$  respectively; figure 16 shows the general graph  $G_{3k/2}^k$  for arbitrary  $k \equiv 0 \pmod 4$ .

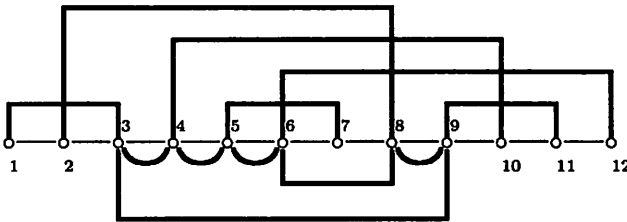


FIGURE 13. The graph  $G_6^4$

It is clear that, for every  $k \equiv 0 \pmod 4$ , the graph  $G_{3k/2}^k$  is a minimal  $k$ -equitable representation of  $C'_{3k/2}$ , implying that  $C'_{3k/2}$  is minimally  $k$ -equitable. ■

#### 4. MINIMAL $k$ -EQUITABILITY OF $C'_m$ WHEN $k$ IS EVEN

Assume that  $k \geq 2$  is even. We are going to show that  $C'_m$  is minimally  $k$ -equitable for every  $m$  such that  $(m, k) \in \mathcal{P}$ .

Given a minimal  $k$ -equitable representation  $G$  of  $C'_m$ , we say that a set  $S$  of edges of  $G$  is a  $k$ -socket if the following conditions are satisfied:

- (1)  $S$  consists of  $\frac{k}{2}$  cycle edges of length 1 whose endpoints form a set of consecutive  $k$  integers  $s, s + 1, \dots, s + k - 1$ ;

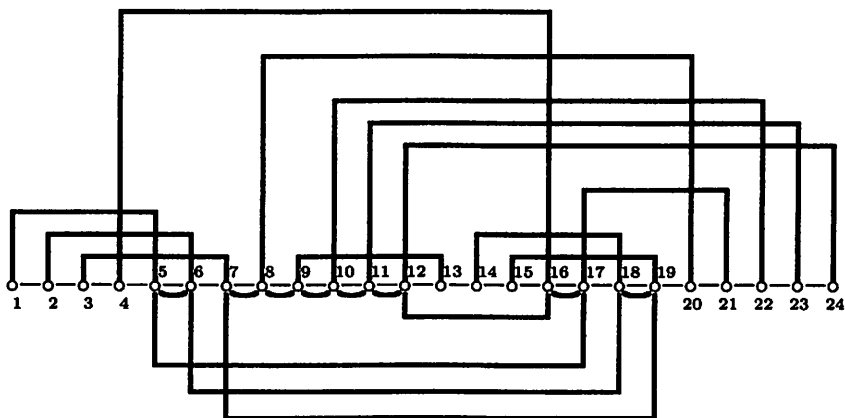


FIGURE 14. The graph  $G_{12}^8$

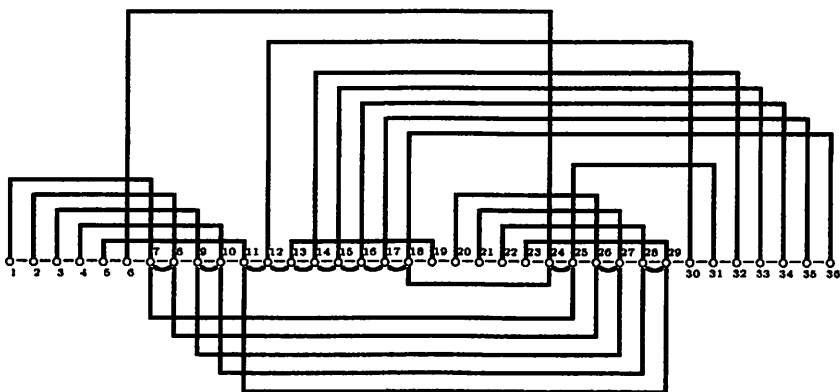


FIGURE 15. The graph  $G_{18}^{12}$

- (2) if  $a$  is the smallest integer in the vertex set of  $G$  and  $b$  is the largest integer in the vertex set of  $G$ , then either  $b - s + 1$  or  $s + k - a$  is not the length of any edge of  $G$ .

Note that if the set of endpoints of the edges of  $\mathcal{S}$  consists of either the largest  $k$  vertices of  $G$  or the smallest  $k$  vertices of  $G$ , then the second condition above is satisfied.

A minimal  $k$ -equitable representation of  $C'_m$  with a  $k$ -socket will be called a  $k$ -proper representation of  $C'_m$ .

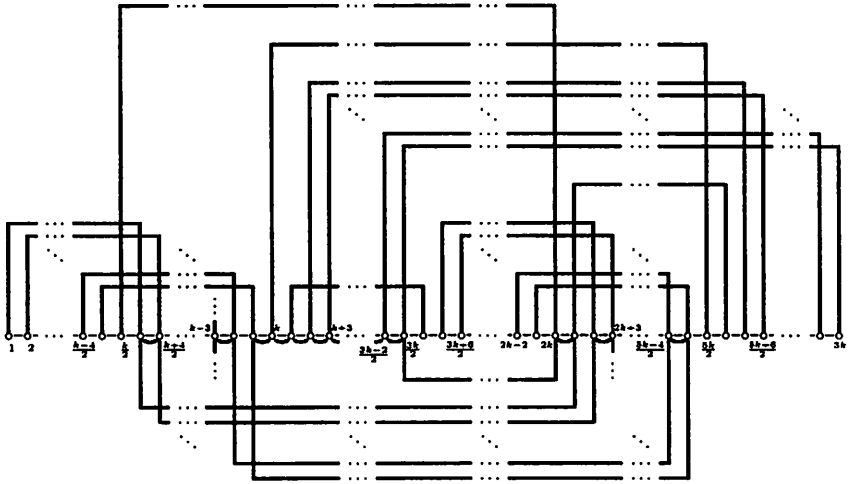


FIGURE 16. The graph  $G_{3k/2}^k$  when  $k \equiv 0 \pmod{4}$

**Lemma 4.1.** *The graph  $G_{2k}^k$  is a  $k$ -proper representation of  $C'_{2k}$  for every even  $k \geq 2$ .*

*Proof.* Let  $k \geq 2$  be an even integer. It follows from the proof of Lemma 2.3 that  $G_{2k}^k$  is a minimally  $k$ -equitable representation of  $C'_{2k}$ . If  $k = 2$ , then let  $S = \{(4, 5)\}$ . (See Figure 8.) With  $b = 8$  and  $s = 4$  the integer  $b - s + 1 = 5$  is not a length of any edge of  $G_4^2$ . Thus  $S$  is a 2-socket in  $G_4^2$  implying that  $G_4^2$  is a 2-proper representation of  $C'_4$ .

If  $k \geq 4$ , then let (see Figures 11 and 12)

$$S = \left\{ \left( \frac{k+2}{2}, \frac{k+4}{2} \right), \left( \frac{k+6}{2}, \frac{k+8}{2} \right), \dots, \left( \frac{3k-2}{2}, \frac{3k}{2} \right) \right\}.$$

With  $b = 4k$  and  $s = \frac{k+2}{2}$ , the integer  $b - s + 1 = \frac{7k}{2}$  is not a length of any edge of  $G_{2k}^k$ , implying that  $G_{2k}^k$  is a  $k$ -proper representation of  $C'_{2k}$ . ■

**Lemma 4.2.** *The graph  $G_{3k/2}^k$  is a  $k$ -proper representation of  $C'_{3k/2}$  for every integer  $k \geq 4$  such that  $k \equiv 0 \pmod{4}$ .*

*Proof.* Let  $k \geq 4$  be an integer with  $k \equiv 0 \pmod{4}$ . It follows from the proof of Lemma 3.1 that  $G_{3k/2}^k$  is a minimally  $k$ -equitable representation of  $C'_{3k/2}$ . Let (see Figure 16)

$$S = \left\{ \left( \frac{k+2}{2}, \frac{k+4}{2} \right), \left( \frac{k+6}{2}, \frac{k+8}{2} \right), \dots, \left( \frac{3k-6}{2}, \frac{3k-4}{2} \right), \left( \frac{3k-2}{2}, \frac{3k}{2} \right) \right\}$$

With  $b = 3k$  and  $s = \frac{k+2}{2}$ , the integer  $b - s + 1 = \frac{5k}{2}$  is not a length of any edge of  $G_{3k/2}^k$ , implying that  $G_{3k/2}^k$  is a  $k$ -proper representation of  $C'_{3k/2}$ . ■

**Lemma 4.3.** *If  $k$  is even and there is a  $k$ -proper representation of  $C'_m$ , then there is a  $k$ -proper representation of  $C'_{m+k}$ .*

*Proof.* Let  $G$  be a  $k$ -proper representation of  $C'_m$  with a  $k$ -socket  $\mathcal{S}$  and let  $s$  be the smallest integer in the set of endpoints of the edges in  $\mathcal{S}$ . Let  $a$  be the smallest integer which is a vertex of  $G$  and let  $b$  be the largest integer which is a vertex of  $G$ . Let  $H$  be the graph obtained from  $G$  by performing the following operations:

- remove the edges of  $\mathcal{S}$ ;
- add 2 sets of  $k$  vertices each at both ends of the graph  $G$ , namely add the vertices  $a - k, a - k + 1, \dots, a - 1$  and  $b + 1, b + 2, \dots, b + k$ ;
- add  $k$  edges of length  $b - a + k + 1$  matching the new vertices, namely add the edges  $(a - k, b + 1), (a - k + 1, b + 2), \dots, (a - 1, b + k)$ ;

**Case 1:** if  $G$  has no edges of length  $b - s + 1$ , then

- add  $k$  edges of length  $b - s + 1$  joining the endpoints of the edges in  $\mathcal{S}$  to the new vertices whose value is larger than  $b$ , namely add the edges  $(s, b + 1), (s + 1, b + 2), \dots, (s + k - 1, b + k)$ ;
- add  $\frac{k}{2}$  edges of length 1 that form a matching of the set of all the new vertices whose value is larger than  $b$ , namely add the edges  $(b + 1, b + 2), (b + 3, b + 4), \dots, (b + k - 1, b + k)$ .

Figure 17 shows the new edges of the graph  $H$  in case 1.

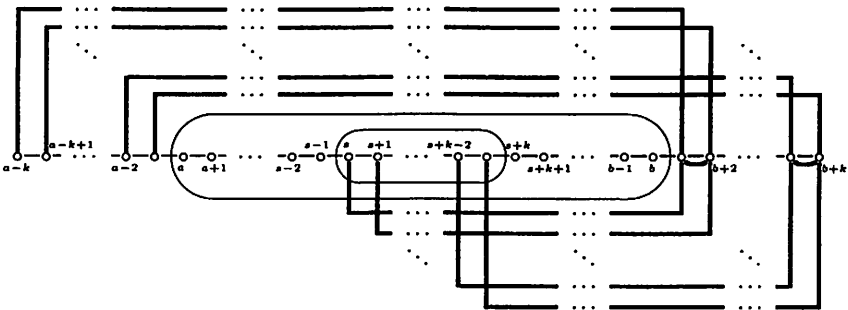


FIGURE 17. The new edges of the graph  $H$  in case 1

**Case 2:** if  $G$  has some edge of length  $b - s + 1$ , (so that it has no edges of length  $s + k - a$ ), then

- add  $k$  edges of length  $s + k - a$  joining the endpoints of the edges in  $\mathcal{S}$  to the new vertices whose value is smaller than  $a$ , namely add the edges  $(a - k, s)$ ,  $(a - k + 1, s + 1)$ ,  $\dots$ ,  $(a - 1, s + k - 1)$ ;
- add  $\frac{k}{2}$  edges of length 1 that form a matching of the set of the new vertices whose value is smaller than  $a$ , namely add the edges  $(a - k, a - k + 1)$ ,  $(a - k + 2, a - k + 3)$ ,  $\dots$ ,  $(a - 2, a - 1)$ .

Figure 18 shows the new edges of the graph  $H$  in case 2. Figures 19 and 20 show the results of applying the above construction to the graphs  $G_4^2$  and  $G_6^4$  respectively.

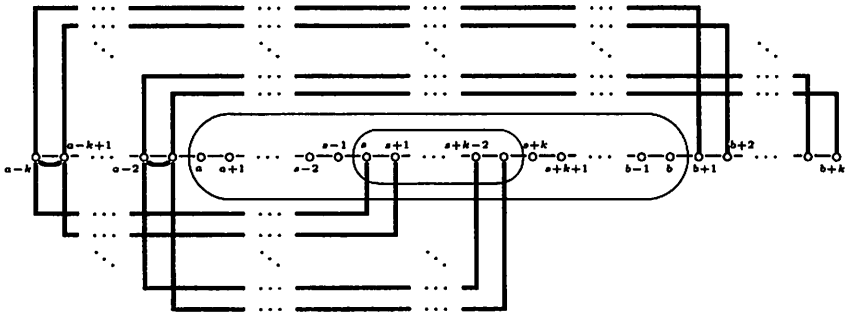


FIGURE 18. The new edges of the graph  $H$  in case 2

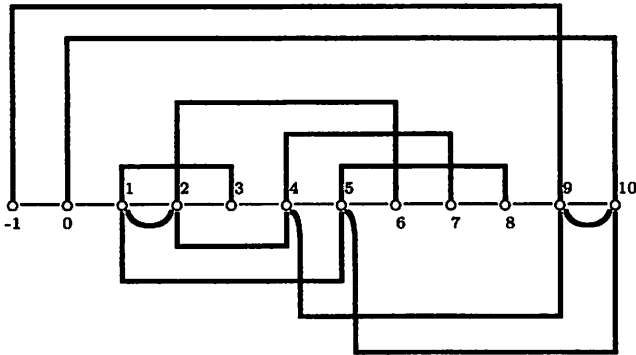


FIGURE 19. A 2-proper representation of  $C'_6$  obtained from  $G_4^2$

Since each edge of  $\mathcal{S}$  in  $G$  is replaced by a path of length 3 in  $H$ , the cycle of  $G$  gives rise to a cycle of length  $m + k$  in  $H$ . Moreover, the new  $k$  edges of length  $b - a + k + 1$  in  $H$  are matching edges implying that  $H$  is isomorphic

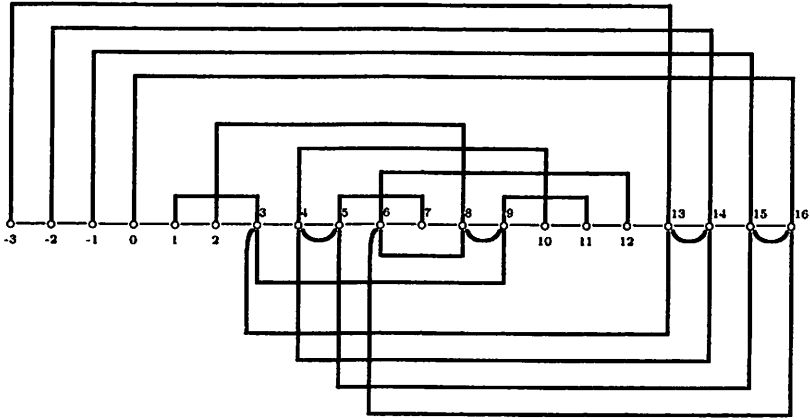


FIGURE 20. A 2-proper representation of  $C'_{10}$  obtained from  $G^4_6$

to  $C'_{m+k}$ . It is clear that  $H$  is a minimally  $k$ -equitable representation of  $C'_{m+k}$ . The set of the new edges of length 1 in  $H$  is a  $k$ -socket since the set of endpoints of these edges consists of either the  $k$  vertices of  $H$  having the smallest label or the  $k$  vertices of  $H$  having the largest label. Thus  $H$  is a  $k$ -proper representation of  $C'_{m+k}$ . ■

**Proposition 4.4.** *Let  $(m, k) \in \mathcal{P}$  with  $k$  being even. Then  $C'_m$  is minimally  $k$ -equitable.*

*Proof.* Assume that  $m = \ell k$  for some integer  $\ell$ . The minimal  $k$ -equitability of  $C'_m$  follows from Lemma 2.1 when  $\ell = 1$ , and from Lemma 2.3 when  $\ell = 2$ . If  $\ell \geq 3$ , then the minimal  $k$ -equitability of  $C'_m$  follows by induction on  $\ell$  using Lemmas 4.1 and 4.3.

If  $k$  is not a divisor of  $m$ , then  $k \equiv 0 \pmod{4}$  and  $m = \ell k/2$  for some odd integer  $\ell \geq 3$ . The minimal  $k$ -equitability of  $C'_m$  follows from Lemma 3.1 when  $\ell = 3$ , and follows by induction using Lemmas 4.2 and 4.3 when  $\ell \geq 5$ . ■

## 5. MINIMAL $k$ -EQUITABILITY OF $C'_m$ WHEN $k$ IS ODD

Assume that  $k \geq 3$  is odd. We are going to show that  $C'_m$  is minimally  $k$ -equitable for every  $m$  such that  $(m, k) \in \mathcal{P}$ .

If  $e = (u, v)$  is an edge of an integer graph and  $u < v$ , then we say that  $u$  is the *left endpoint* of  $e$  and  $v$  is the *right endpoint* of  $e$ . Given a minimal  $k$ -equitable representation  $G$  of  $C'_m$ , we say that a pair  $\mathbb{T} = (T, C)$  is a  $k$ -*thread* in  $G$  if the following conditions are satisfied:

- (1)  $\mathcal{C}$  consists of  $k$  edges that are cycle edges and have the same length;
- (2)  $T$  is either the set of all left endpoints or the set of all right endpoints of the edges in  $\mathcal{C}$  and it consists of  $k$  consecutive integers  $t, t + 1, \dots, t + k - 1$ ;
- (3) if  $a$  is the vertex of  $G$  with the smallest label,  $b$  is the vertex of  $G$  with the largest label,  $s$  is the other endpoint of the edge in  $\mathcal{C}$  that has  $t$  as one of its endpoints,  $A_G$  is the set of lengths of the edges of  $G$ , and  $R_{G,T}, L_{G,T}, W_G$  are infinite sets of integers defined as follows:

$$R_{G,T} = \{b - s + 1\} \cup \{b + 2ik - t + 1 : i \geq 0\} \\ \cup \{t - a + 2ik : i \geq 1\},$$

$$L_{G,T} = \{s - a + k\} \cup \{b + (2i + 1)k - t + 1 : i \geq 0\} \\ \cup \{t - a + (2i + 1)k : i \geq 0\},$$

$$W_G = \{b - a + ik + 1 : i \geq 1\},$$

then either  $L_{G,T}$  or  $R_{G,T}$  is disjoint with  $A_G \cup W_G$ .

The following three figures illustrate the definition of a  $k$ -thread in a graph  $G$  when  $t < s$ ; the pictures require obvious modifications when  $s < t$ . Figure 21 shows the edges of the graph  $G$  that belong to the set  $\mathcal{C}$ . Figure 22 shows the relationship between the sets  $R_{G,T}, W_G$  and the vertices of the graph  $G$ , where the integers in  $R_{G,T}$  and  $W_G$  are represented by edges of the corresponding lengths. The set  $R_{G,T}$  is represented above the line containing the vertices of  $G$  and the set  $W_G$  below it. Figure 23 shows analogous relationship between the sets  $L_{G,T}, W_G$  and the graph  $G$ .

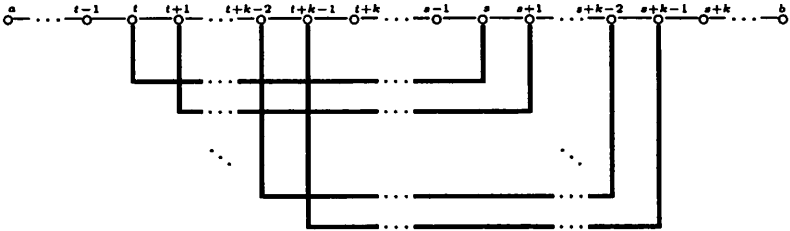


FIGURE 21. Edges of the graph  $G$  that belong to  $\mathcal{C}$

A minimal  $k$ -equitable representation of  $C'_m$  with a  $k$ -thread will be called a  $k$ -proper representation of  $C'_m$ .

**Lemma 5.1.** *The graph  $G_{2k}^k$  is a  $k$ -proper representation of  $C'_{2k}$  for every odd  $k \geq 3$ .*

*Proof.* Let  $k \geq 3$  be an odd integer. It follows from the proof of Lemma 2.2 that  $G_{2k}^k$  is a minimally  $k$ -equitable representation of  $C'_{2k}$ .



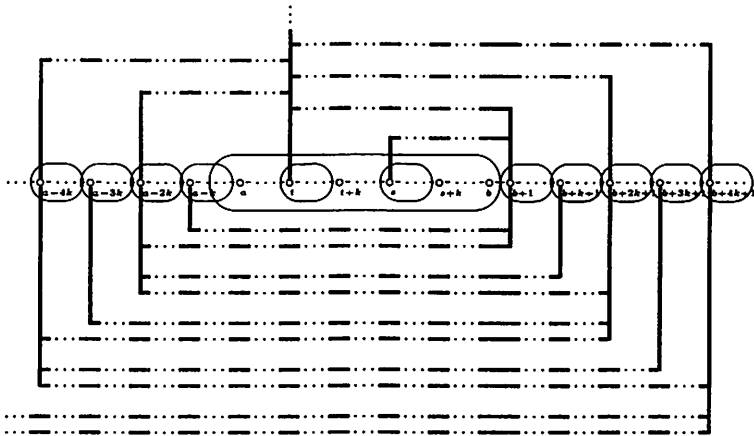


FIGURE 22. Edges whose length is in  $R_{G,T}$  or  $W_G$

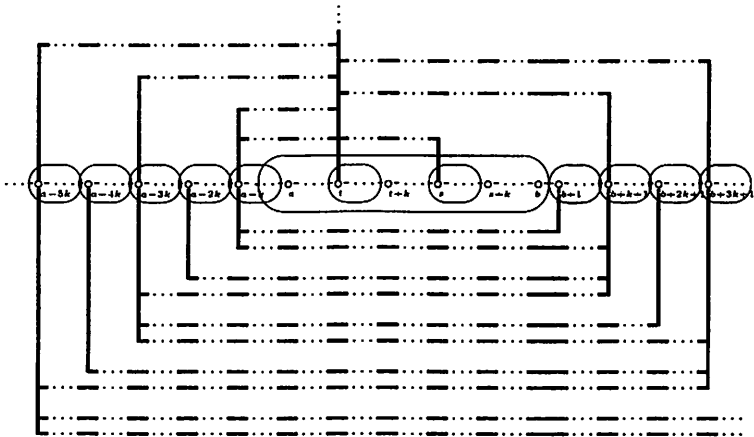


FIGURE 23. Edges whose length is in  $L_{G,T}$  or  $W_G$

Let (see Figures 6 and 7)

$$C = \left\{ \left( \frac{k+3}{2}, 2k+1 \right), \left( \frac{k+5}{2}, 2k+2 \right), \dots, \left( \frac{3k+1}{2}, 3k \right) \right\},$$

and

$$T = \left\{ \frac{k+3}{2}, \frac{k+5}{2}, \dots, \frac{3k+1}{2} \right\}.$$

With  $\mathbb{T} = (T, C)$ ,  $a = 1$ ,  $b = 4k$ ,  $t = \frac{k+3}{2}$  and  $s = 2k + 1$ , we have

$$R_{G, \mathbb{T}} = \{2k\} \cup \left\{ \frac{(4i-1)k-1}{2} : i \geq 2 \right\} \cup \left\{ \frac{(4i+1)k+1}{2} : i \geq 1 \right\}.$$

Since

$$W_G = \{ik : i \geq 5\},$$

and

$$A_G = \left\{ 1, \frac{k+1}{2}, k, \frac{3k-1}{2} \right\},$$

it is clear that

$$R_{G, \mathbb{T}} \cap (A_G \cup W_G) = \emptyset.$$

It follows that  $\mathbb{T}$  is a  $k$ -thread in  $G_{2k}^k$ , implying that  $G_{2k}^k$  is a  $k$ -proper representation of  $C'_{2k}$ . ■

**Lemma 5.2.** *If  $k$  is odd and there is a  $k$ -proper representation of  $C'_m$ , then there is a  $k$ -proper representation of  $C'_{m+k}$ .*

*Proof.* Let  $G$  be a  $k$ -proper representation of  $C'_m$  with a  $k$ -thread  $\mathbb{T} = (T, C)$ , let  $t$  be the smallest label of  $T$ , let  $s$  be the other endpoint of the edge in  $C$  that has  $t$  as one of its endpoints, and let

$$S = \{s, s+1, \dots, s+k-1\}.$$

Let  $a$  be the smallest label of the vertices of  $G$  and let  $b$  be the largest label of the vertices of  $G$ . Let  $H$  be the graph obtained from  $G$  by performing the following operations:

- remove the edges of  $C$ ;
- add the vertices  $a-k, a-k+1, \dots, a-1$  and  $b+1, b+2, \dots, b+k$ ;
- add  $k$  edges of length  $b-a+k+1$  matching the new vertices, namely add the edges  $(a-k, b+1), (a-k+1, b+2), \dots, (a-1, b+k)$ ;

**Case 1:** if  $R_{G, \mathbb{T}} \cap (A_G \cup W_G) = \emptyset$ , then

- add  $k$  edges of length  $b-s+1$  joining the vertices in  $S$  to the new vertices whose label is larger than  $b$ , namely add the edges  $(s, b+1), (s+1, b+2), \dots, (s+k-1, b+k)$ ;
- add  $k$  edges of length  $b-t+1$  joining the vertices in  $T$  to the new vertices whose label is larger than  $b$ , namely add the edges  $(t, b+1), (t+1, b+2), \dots, (t+k-1, b+k)$ .

Figure 24 shows the new edges of the graph  $H$  in case 1.

**Case 2:** if  $R_{G, \mathbb{T}} \cap (A_G \cup W_G) \neq \emptyset$ , (so that  $L_{G, \mathbb{T}} \cap (A_G \cup W_G) = \emptyset$ ) then

- add  $k$  edges of length  $s+k-a$  joining the vertices in  $S$  to the new vertices whose label is smaller than  $a$ , namely add the edges  $(a-k, s), (a-k+1, s+1), \dots, (a-1, s+k-1)$ ;

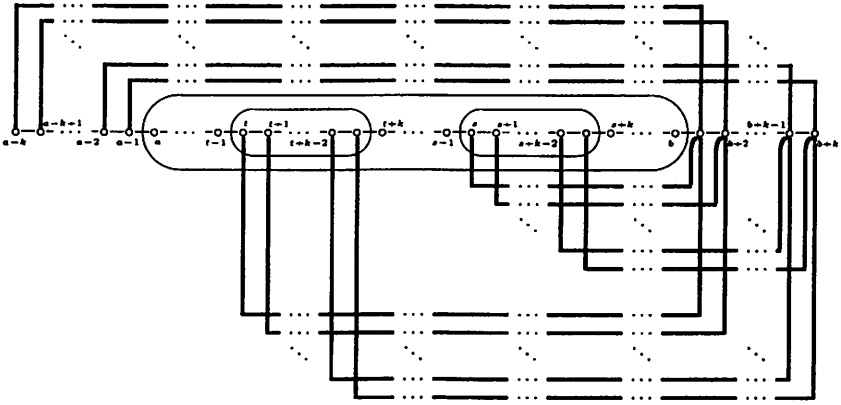


FIGURE 24. The new edges of the graph  $H$  in case 1

- add  $k$  edges of length  $t + k - a$  joining the vertices in  $T$  to the new vertices whose label is smaller than  $a$ , namely add the edges  $(a - k, t)$ ,  $(a - k + 1, t + 1)$ ,  $\dots$ ,  $(a - 1, t + k - 1)$ .

Figure 25 shows the new edges of the graph  $H$  in case 2. Figure 26 shows the results of applying the above construction to the graph  $G_8^3$ .

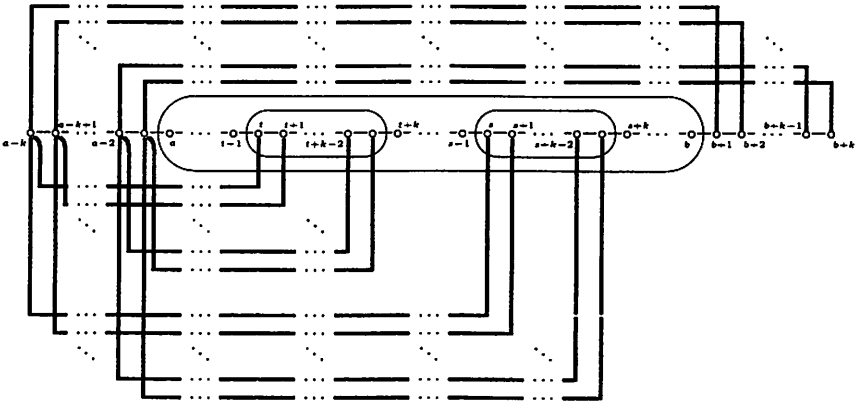


FIGURE 25. The new edges of the graph  $H$  in case 2

Since each edge of  $C$  in  $G$  is replaced by a path of length 2 in  $H$ , the cycle of  $G$  gives rise to a cycle of length  $m + k$  in  $H$ . Moreover, the new  $k$  edges of length  $b - a + k + 1$  in  $H$  are matching edges, implying that  $H$  is isomorphic to  $C'_{m+k}$ . Since

$$b - a + k + 1 > b - a,$$

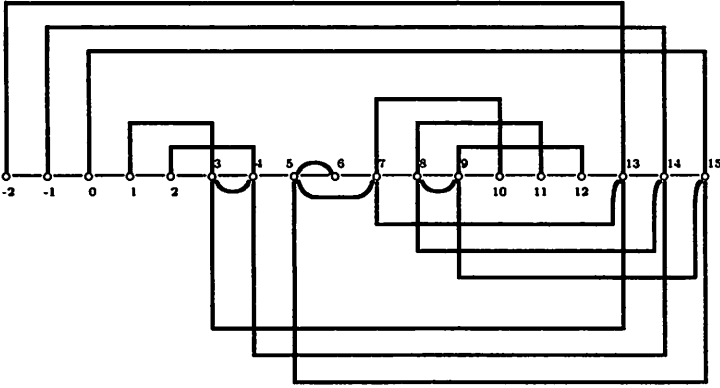


FIGURE 26. A 3-proper representation of  $C'_9$  obtained from  $G^3_6$

there are no edges in  $G$  of length  $b - a + k + 1$ . Moreover the lengths of the new cycle edges of  $H$  are in  $L_{G,T}$  in case 1 and in  $R_{G,T}$  in case 2, implying that they are not in  $A_G$ . Therefore, the graph  $H$  is a minimally  $k$ -equitable representation of  $C'_{m+k}$ . To prove that  $H$  is a  $k$ -proper representation of  $C'_{m+k}$  it remains to show that there is a  $k$ -thread in  $H$ .

In case 1, let  $C'$  be the set of the new edges of  $H$  of length  $b - t + 1$  that join the vertices in  $T$  to the new vertices whose label is larger than  $b$ . We claim that the pair  $T' = (T, C')$  is a  $k$ -thread in the graph  $H$ . Let  $a' = a - k$ ,  $b' = b + k$ , and let  $s' = b + 1$  be the other endpoint of the edge of  $C'$  that has  $t$  as one of its endpoints. We have then

$$\begin{aligned}
 L_{H,T'} &= \{s' - a' + k\} \cup \{b' + (2i + 1)k - t + 1 : i \geq 0\} \\
 &\quad \cup \{t - a' + (2i + 1)k : i \geq 0\} \\
 &= \{b - a + 2k + 1\} \cup \{b + 2ik - t + 1 : i \geq 1\} \\
 &\quad \cup \{t - a + 2ik : i \geq 1\} \\
 &= R_{G,T} \cup \{b - a + 2k + 1\} \setminus \{b - s + 1, b - t + 1\}.
 \end{aligned}$$

Moreover

$$A_H = A_G \cup \{b - a + k + 1, b - s + 1, b - t + 1\} \setminus \{|t - s|\},$$

and

$$\begin{aligned}
 W_H &= \{b' - a' + ik + 1 : i \geq 1\} \\
 &= \{b - a + ik + 1 : i \geq 3\} \\
 &= W_G \setminus \{b - a + k + 1, b - a + 2k + 1\},
 \end{aligned}$$

so

$$A_H \cup W_H = (A_G \cup W_G) \cup \{b - s + 1, b - t + 1\} \setminus \{b - a + 2k + 1, |t - s|\}.$$

Since  $R_{G,T} \cap (A_G \cup W_G) = \emptyset$ , it follows that

$$L_{H,T'} \cap (A_H \cup W_H) = \emptyset,$$

so the pair  $T' = (T, C')$  is a  $k$ -thread in the graph  $H$ .

In case 2, let  $C'$  be the set of the new edges of  $H$  of length  $t + k - a$  that join the vertices in  $T$  to the new vertices whose label is smaller than  $a$ . We claim that the pair  $T' = (T, C')$  is a  $k$ -thread in the graph  $H$ . Let  $a' = a - k$ ,  $b' = b + k$ , and let  $s' = a - k$  be the other endpoint of the edge of  $C'$  that has  $t$  as one of its endpoints. Then

$$\begin{aligned} R_{H,T'} &= \{b' - s' + 1\} \cup \{b' + 2ik - t + 1 : i \geq 0\} \cup \{t - a' + 2ik : i \geq 1\} \\ &= \{b - a + 2k + 1\} \cup \{b + (2i + 1)k - t + 1 : i \geq 0\} \\ &\quad \cup \{t - a + (2i + 1)k : i \geq 1\} \\ &= L_{G,T} \cup \{b - a + 2k + 1\} \setminus \{s + k - a, t + k - a\}. \end{aligned}$$

Moreover

$$A_H = A_G \cup \{b - a + k + 1, s + k - a, t + k - a\} \setminus \{|t - s|\},$$

and

$$W_H = W_G \setminus \{b - a + k + 1, b - a + 2k + 1\}$$

so

$$A_H \cup W_H = (A_G \cup W_G) \cup \{s + k - a, t + k - a\} \setminus \{b - a + 2k + 1, |t - s|\}.$$

Since  $L_{G,T} \cap (A_G \cup W_G) = \emptyset$ , it follows that

$$R_{H,T'} \cap (A_H \cup W_H) = \emptyset,$$

so the pair  $T' = (T, C')$  is a  $k$ -thread in the graph  $H$ .

Thus  $H$  is a  $k$ -proper representation of  $C'_{m+k}$ . ■

**Proof of Theorem 1.2.** Assume the  $(m, k) \in \mathcal{P}$ . If  $k$  is even then it follows from Proposition 4.4 that  $C'_m$  is  $k$ -equitable. If  $k$  is odd, then it is a divisor of  $m$ . The  $k$ -equitability of  $C'_m$  follows from Lemma 2.1 if  $k = m$ , and it follows by induction using Lemmas 5.1 and 5.2 if  $k$  is a proper divisor of  $m$ .

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