Overfull Sets of One-Factors

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Abstract

We define an *overfull set* of one-factors of K_{2n} to be a set of one-factors that between them cover all the edges of K_{2n}, but contain no one-factorization of K_{2n}. We address the question: how many members can such a set contain?

1 Introduction

If G is any graph, then a *one-factor* of G is a subgraph with vertex-set $V(G)$ that is a regular graph of degree 1. In other words, a one-factor is a set of pairwise disjoint edges of G that between them contain every vertex. A *one-factorization* of G is a decomposition of the edge-set of G into edge-disjoint one-factors. Clearly any graph must have an even number of vertices to possess a one-factor or one-factorization.

In particular, we consider one-factors and one-factorizations of complete graphs. We denote the complete graph on $2n$ vertices by K_{2n}. We label the vertices of K_{2n} as $\infty, 0, 1, \cdots, 2n-2$; the labels are treated as integers modulo $2n-1$ with the proviso that $\infty + 1 = \infty$.

It is clear that K_{2n} has $(2n-1)!!$ one-factors, where $(2n-1)!!$ denotes the semifactorial

$$(2n-1)!! = (2n-1) \times (2n-3) \times \ldots \times 3 \times 1 = \frac{(2n)!}{n!2^n}.$$

There are $(2n-3)!!$ factors containing the edge ∞i, for any i.

We shall use one of the standard one-factorizations of K_{2n}, the *patterned factorization* $\mathcal{P}$. This factorization is illustrated in Figure 1. The figure shows a factor that we shall call P_0. To construct the factor P_1, rotate the

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diagram through a $(2n-1)$-th part of a full revolution. Similar rotations provide $P_2, P_3, \ldots$, and

$$\mathcal{P} = \{P_0, P_1, \ldots, P_{2n-2}\}.$$

Formally, P_i is defined to consist of the edges

$$\infty i, (i+1)(i-1), \ldots, (i+j)(i-j), \ldots, (i+n-1)(i-n+1). \quad (1)$$

Further information on one-factors and one-factorizations can be found in [3].

2 Overfull sets

Bonisoli [1] introduced the concept of an *excessive set* of one-factors of K_{2n}. This is a set $\mathcal{S}$ of $2n$ one-factors that covers all edges of K_{2n}, in which no $2n-1$ factors form a one-factorization. It is shown in [1] that such a set exists for all even orders $2n$ greater than 4 (orders 2 and 4 are easily seen to be impossible). In fact, such sets were available in the literature in 1984, but were not noticed. (See Section 5 below.)

We wish to address the more general question: for what values t does there exist a set $\mathcal{S}$ of one-factors of K_{2n} with the following properties:

(1) the members of $\mathcal{S}$ cover all edges of K_{2n};

(2) no $2n-1$ members of $\mathcal{S}$ form a one-factorization of K_{2n}?

Such a set is called an *overfull set* of one-factors of K_{2n} of order t. Obviously $t \geq 2n$ when $2n > 4$.

In particular we ask, what is the greatest order of an overfull set of one-factors of K_{2n}?

3 A construction

In this section we construct a set $\mathcal{T}_n$ of one-factors that covers K_{2n} and contains no one-factorization, for every n except 2. We denote by $\mathcal{F}_i$ the set of all one-factors of K_{2n} containing edge (∞, i). Then

$$\mathcal{T} = \{P_0\} \cup \mathcal{D} \cup \bigcup_{i=2}^{2n-2} \mathcal{F}_i$$

where P_0 is the first factor listed in $\mathcal{P}$:

$$P_0 = \{(\infty, 0), (-1, 1), \ldots, (-k, k), \ldots, (n-1, n)\},$$

and $\mathcal{D}$ is the set of all one-factors that contain both $(\infty, 1)$ and one or more edges from P_0.

It is clear that $\mathcal{T}$ contains no one-factorization, because such a factorization would include disjoint factors containing $(\infty, 0)$ and $(\infty, 1)$. The first must be P_0, and no member of $\mathcal{T}$ contains $(\infty, 1)$ but is disjoint from P_0.

To prove that $\mathcal{T}$ covers K_{2n}, we show that $\mathcal{T}$ contains an excessive set of one-factors. In the case $2n = 6$, such a set is

$$\begin{array}{llllll}
(\infty,0) & (1,4) & (2,3) & (\infty,3) & (0,1) & (2,4) \\
(\infty,1) & (0,4) & (2,3) & (\infty,4) & (0,2) & (1,3) \\
(\infty,2) & (0,1) & (3,4) & (\infty,4) & (0,3) & (1,2).
\end{array}$$

If $2n > 6$ we define two special one-factors F_1^* and F_2^* by

$$\begin{aligned}
F_1^* &= (\infty,1)\,(0,2)\,(-1,3)\,(-2,4)\,\ldots\,(1-k,1+k)\,\ldots \\
&\qquad (n-2,n+3)\,(n-1,n)\,(n+1,n+2), \\
F_2^* &= (\infty,2)\,(1,3)\,(0,4)\,(-1,5)\,\ldots\,(2-k,2+k)\,\ldots \\
&\qquad (n-1,n+2)\,(n,n+1)\,(n+3,n+4).
\end{aligned}$$

Factor F_1^* is the same as P_1, except that edges $(n-1, n+2)$ and $(n, n+1)$ of P_1 are replaced by $(n-1, n)$ and $(n+1, n+2)$ in F_1^*. The two missing edges are however contained in F_2^*. So $\{P_0, F_1^*, F_2^*, P_2, \ldots, P_{2n-2}\}$ covers all edges of K_{2n}, and it contains no one-factorization.

How large is $\mathcal{T}$? Each set $\mathcal{F}_i$ has $(2n-3)!!$ members. To enumerate $\mathcal{D}$ we use the principle of inclusion and exclusion. Write $\mathcal{D}_j$ for the set of all one-factors containing both $(\infty, 1)$ and $(j, -j)$. Then

$$|\mathcal{D}| = |\bigcup_{j=2}^{n-1} \mathcal{D}_j|$$

$$
\begin{aligned}
&= \sum_{j=2}^{n-1} |\mathcal{D}_j| - \sum_{2\le j_1<j_2}^{n-1} |\mathcal{D}_{j_1} \cap \mathcal{D}_{j_2}| + \dots \\
&\quad +(-1)^{k+1} \sum_{2\le j_1<j_2<\dots<j_k}^{n-1} |\mathcal{D}_{j_1} \cap \mathcal{D}_{j_2} \cap \dots \cap \mathcal{D}_{j_k}| \dots \\
&\quad +(-1)^{n-1} |\mathcal{D}_2 \cap \mathcal{D}_3 \cap \dots \cap \mathcal{D}_{n-1}|
\end{aligned}
$$

Now the intersection of any k of the $\mathcal{D}_j$ is the set of all factors containing $k+1$ specific edges, so it has $(2n-2k-3)!!$ elements, and

$$
\begin{aligned}
|\mathcal{D}| &= (n-2)(2n-5)!! - \binom{n-2}{2}(2n-7)!! + \dots \\
&\quad +(-1)^{k+1}\binom{n-2}{k}(2n-2k-3)!! \dots + (-1)^n \\
&= \sum_{k=1}^{n-2} (-1)^{k+1}\binom{n-2}{k}(2n-2k-3)!!.
\end{aligned}
$$

So $\mathcal{T}$ has order

$$
T(2n) = 1 + (2n-3)(2n-3)!! + \sum_{k=1}^{n-2} (-1)^{k+1}\binom{n-2}{k}(2n-2k-3)!!.
$$

Theorem 1 *There is an overfull set of order t in K_{2n} whenever $2n \le t \le T(2n)$.*

Proof. For $t = 2n$, use the excessive set in $\mathcal{T}$. If $t = 2n + s$, add to that set any s further members of $\mathcal{T}$.

4 Conjecture

We have tried several other techniques to find an overfull set with more than $T(2n)$ members, without success. One can check by hand that there is no overfull set for $2n = 6$ with more than 11 members. For $2n = 8$, $T(2n) = 81$; we have found an alternative construction for 81 factors, but cannot improve it; and the generalization of that construction is not as good as the one given in the previous section. Accordingly we diffidently conjecture that $T(2n)$ is best-possible. We would be happy to be proven wrong.

5 Historical note

In [2], the authors sought designs to use in the construction of Room squares with subsquares. They defined a *house* to be a $2n \times 2n$ array whose cells either were empty or contained unordered pairs from a set S of order $2n$, with the following properties. The pairs in any row or column constituted a one-factor; the factors in the rows constitute a one-factorization with one factor repeated, while each factor in the columns meets the repeated factor in exactly one edge. It is easy to see that the columns must form an excessive factorization. However, this was not noticed in [2].

References

[1] A. Bonisoli, *Excessive factorizations of complete graphs,* preprint.

[2] D. R. Stinson and W. D. Wallis, *An even-side analogue of Room squares,* Aeq. Math. **27** (1984), 201–213.

[3] W. D. Wallis, *One-factorizations.* (Kluwer, Dortrecht, Netherlands, 1997).

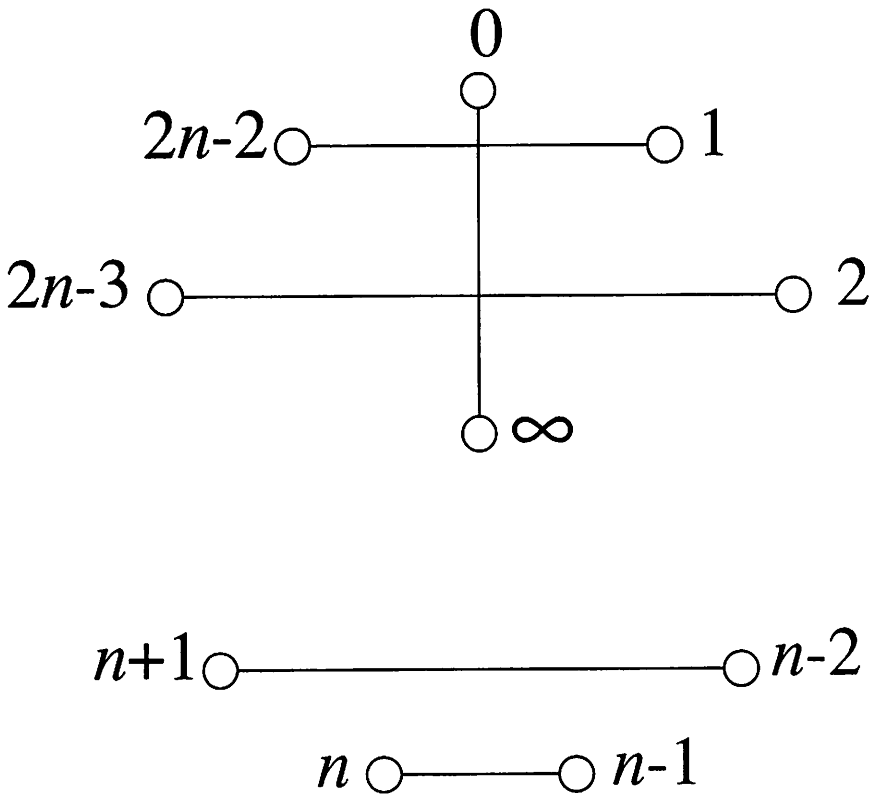

Figure 1: The factor P_0 in the patterned starter on K_{2n}.