On the Order of a Graph with a given Degree Set

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Abstract

The *degree set* of a finite simple graph G is the set of distinct degrees of vertices of G. For any given finite set $\mathcal{D}$ of positive integers, we determine all positive integers n such that $\mathcal{D}$ is the degree set of some simple graph with n vertices. This extends a theorem of Kapoor, Polimeni & Wall (1977) which shows that the least such n is $1 + \max(\mathcal{D})$.

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A finite sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers is said to be *graphic* if there exists a finite simple graph G with vertices $v_1, v_2, \ldots, v_n$ such that each v_i has degree d_i. Two obvious necessary conditions for such a sequence to be graphic are: (1) $d_i < n$ for each i, and (2) $\sum_{i=1}^{n} d_i$ is *even*. However, these two conditions together do not ensure that a sequence will be graphic. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphic are well known. Two such characterizations of graphic sequences are a rather involved explicit characterization by Erdös & Gallai [4], and an elegant recursive characterization by Havel [5] and

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later, but independently, by Hakimi [6].

The *degree set* of a simple graph G is the set $\mathcal{D}(G)$ comprising the distinct degrees of vertices in G, that is, $\mathcal{D}(G) = \{d : deg\, v = d$ for some vertex v in $G\}$. In fact it has long been known that every finite simple graph G of order $n \geq 2$ must have two vertices of the same degree [1], so $|\mathcal{D}(G)| < n$ when $n \geq 2$. Moreover, there is an explicit characterization of graphic degree sequences which utilizes a proper subset of the Erdös-Gallai conditions, one condition for each member of the degree set ([3],[8]). It is natural to ask when a set of positive integers forms the degree set of a graph, and then to investigate the order and size of such graphs. A significant step in that direction is the result

Theorem KPW (Kapoor, Polimeni & Wall [7]) *For each nonempty finite set $\mathcal{D}$ of positive integers, there exists a finite simple graph G for which $\mathcal{D}(G) = \mathcal{D}$. Moreover, there is always such a graph of order $\Delta + 1$, where $\Delta = \max(\mathcal{D})$, and there is no such graph of smaller order.*

We say that a graph G is an $(n, \mathcal{D})$-graph if has order n and degree set $\mathcal{D}$. Our goal here is to determine all n for which there exists an $(n, \mathcal{D})$-graph corresponding to a given finite set $\mathcal{D}$ of positive integers. The main result is the following.

Theorem. *Let $\mathcal{D}$ be any nonempty finite set of positive integers, and let $\Delta = \max(\mathcal{D})$. Then $\mathcal{D}$ is the degree set of infinitely many distinct finite simple graphs. If all members of $\mathcal{D}$ are odd, there exists an $(n, \mathcal{D})$-graph if and only if $n > \Delta$ and n is even; otherwise, there exists an $(n, \mathcal{D})$-graph if and only if $n > \Delta$, provided also that $n \neq \Delta + 2$ in the special case where $\mathcal{D} = \{1, \Delta\}$ for any even integer $\Delta \geq 4$.*

We need two lemmas to prove our theorem. First recall that a *matching* in a graph is any collection of edges no two of which have a vertex in common. A matching is *maximal* if it is not properly contained in any larger matching in the graph.

Lemma 1. *Every maximal matching in a simple graph with minimum degree δ has size at least $\lceil \delta/2 \rceil$.*

Proof. Any matching $\mathcal{M}$ of size less than $\lceil \delta/2 \rceil$ is incident with fewer than δ vertices. The graph has at least $\delta + 1$ vertices; choose any vertex v not incident with any edge in $\mathcal{M}$. Since v has degree at least δ, it has at least one neighbour w which is not incident with any edge in $\mathcal{M}$. Then $\mathcal{M} \cup \{vw\}$ is a matching that properly contains $\mathcal{M}$, so $\mathcal{M}$ is not maximal. □

Lemma 2. *Let $\mathcal{D}$ be any nonempty finite set of positive integers, with $\delta = \min(\mathcal{D})$, and suppose there exists an $(n, \mathcal{D})$-graph. If δ is even there exists an $(n+1, \mathcal{D})$-graph, while if δ is odd there exists an $(n+2, \mathcal{D})$-graph. Moreover, if δ is odd and $\delta + 1 \in \mathcal{D}$, there exists an $(n+1, \mathcal{D})$-graph.*

Proof. Let G be an $(n, \mathcal{D})$-graph. By Lemma 1, G has a matching $\mathcal{M}^*$ of size $\lceil \delta/2 \rceil$. Construct the graph G^* from G by adjoining a new vertex u to $G \setminus \mathcal{M}^*$ and adjoining the new edges $\{uv : v$ is incident with some edge in $\mathcal{M}^*\}$. This preserves the degrees of all vertices that were in G, and u has even degree, either δ or $\delta + 1$. Thus G^* is an $(n+1, \mathcal{D})$-graph if δ is even or if δ is odd and $\delta + 1 \in \mathcal{D}$. Now let us further consider the case when δ is odd, regardless of whether or not $\delta + 1 \in \mathcal{D}$. Since $(\delta - 1)/2 < \lceil \delta/2 \rceil$, Lemma 1 ensures that G has a matching $\mathcal{M}$ of size $(\delta - 1)/2$. Then $G \setminus \mathcal{M}$ has minimum degree δ or $\delta - 1$, so has a matching $\mathcal{M}'$ of size $(\delta - 1)/2$, by Lemma 1. Construct G'' from $G \setminus (\mathcal{M} \cup \mathcal{M}')$ by adjoining new vertices u and u', and adding the edges $\{uv : v$ is incident with some edge in $\mathcal{M}\}$, $\{u'v : v$ is incident with some edge in $\mathcal{M}'\}$, and uu'. (If $\delta = 1$ then $\mathcal{M}$ and $\mathcal{M}'$ are empty, so $G'' = G \cup K_2$.) This preserves the degrees all vertices from G, while u and u' have degree δ, so G'' is an $(n+2, \mathcal{D})$-graph. □

Lemma 3. *Let $\mathcal{D} = \{1, 2, 2m - 1\}$, where $m \geq 2$. There exists an $(n, \mathcal{D})$-graph with unique vertex of degree 1 if and only if $n \geq 2m$.*

Proof. The necessity for $n \geq 2m$ follows from Theorem KPW. For any such n, let S be the sheaf of cycles formed from $m - 2$ cycles of order 3 and one cycle of order $n - 2m + 3$, by identifying a single vertex x in each of these cycles. Then S has order $n - 1$, the degree of x is $2m - 2$, and all other vertices have degree 2. Finally form the graph G from S by attaching a pendant vertex y by the single edge xy. Then G is an $(n, \mathcal{D})$-graph, and y is its unique vertex of degree 1. □

Proof of Theorem. Let $\mathcal{D}$ be any nonempty finite set of positive integers, with $\delta = min(\mathcal{D})$ and $\Delta = max(\mathcal{D})$.

(1) Theorem KPW ensures there is a $(\Delta + 1, \mathcal{D})$-graph but no $(n, \mathcal{D})$-graph with $n \leq \Delta$. If δ is even, or δ is odd and $\delta + 1 \in \mathcal{D}$, it follows from Lemma 2 that there is an $(n, \mathcal{D})$-graph if and only if $n \geq \Delta + 1$.

(2) Next consider the case when all members of $\mathcal{D}$ are odd. If G is any $(n, \mathcal{D})$-graph then n is even since the sum of degrees of G is necessarily even, whence Theorem KPW and Lemma 2 imply that there is an $(n, \mathcal{D})$-graph if and only if $n \geq \Delta + 1$ and n is even.

(3) For the rest of the proof we may assume that δ is odd, $\delta+1 \notin \mathcal{D}$ and $\mathcal{D}$ has at least one even member.

(4) Let us begin with $|\mathcal{D}| = 2$, so $\mathcal{D} = \{\delta, \Delta\}$, with δ odd, Δ even, and $\Delta \geq \delta + 3$.

(4a) Suppose $\delta = 1$, so $\mathcal{D} = \{1, \Delta\}$ with even $\Delta \geq 4$. Any $(\Delta+2, \mathcal{D})$-graph must have an even number of vertices of degree 1 since the sum of degrees is necessarily even. But $\Delta+2$ is even, so the number of vertices of degree Δ is even. Any two vertices u and v of degree Δ must be adjacent, for otherwise the remaining Δ vertices would all be in the common neighbourhood of u and v, so there could not be any vertices of degree 1. Hence exactly two vertices have degree 1; one is adjacent to u, the other is adjacent to v. It follows that are the only two vertices of degree Δ. But this forces $\Delta = 2$, contrary to the condition $\Delta \geq 4$, so there is no $(\Delta+2, \mathcal{D})$-graph. On the other hand, by (1) there is a $(\Delta+2, \{\Delta\})$-graph G, and the disjoint union $G \cup K_2$ is a $(\Delta+4, \mathcal{D})$-graph. From Theorem KPW and Lemma 2, it now follows that there is an $(n, \mathcal{D})$-graph if and only if $n > \Delta$ and $n \neq \Delta + 2$.

(4b) For all instances of $\mathcal{D}$ remaining in this proof it suffices to show that there is a $(\Delta+2, \mathcal{D})$-graph, for then Theorem KPW and Lemma 2 combine to ensure hat there is an $(n, \mathcal{D})$-graph if and only if $n > \Delta$.

(4c) Now suppose $\mathcal{D} = \{\delta, \Delta\}$ with odd $\delta \geq 3$, Δ even, and $\Delta \geq \delta + 3$. By (2) there is a $(\Delta, \{\delta-2\})$-graph G. Let $G \vee 2K_1$ be the graph resulting from G by adjoining two new vertices u and u', and adding the edges $\{uv, u'v : v$ is a vertex of $G\}$. Then $G \vee 2K_1$ is $(\Delta+2, \mathcal{D})$-graph and (4b) applies.

(5) Next consider $|\mathcal{D}| = 3$. We may assume $\mathcal{D} = \{\delta, d, \Delta\}$ with odd δ, $d \geq \delta + 2$, and at least one of d and Δ even.

(5a) Suppose $\delta = 1$. If d is odd then Δ must be even, so in any case (1) and (2) ensure the existence of a $(\Delta, \{d-2\})$-graph G. Let $G \vee^* P_4$ be the graph formed by adjoining an order 4 path P_4 with internal vertices u and u', and adding the edges $\{uv, u'v : v$ is a vertex of $G\}$. Then $G \vee^* P_4$ is a $(\Delta+2, \mathcal{D})$-graph and (4b) applies.

(5b) Now suppose $\delta \geq 3$, and assume inductively that the theorem holds for any degree set with exactly 3 elements and odd minimum element less than δ. Let $\mathcal{D}^* = \{\delta-2, d-2, \Delta-2\}$. If d is odd then Δ must be even, so in any case by hypothesis there is a $(\Delta, \mathcal{D}^*)$-graph G. Then $G \vee 2K_1$ is a $(\Delta+2, \mathcal{D})$-graph, so (4b) applies. We have now shown that the theorem holds for every degree set with at most 3 elements.

(6) Now suppose $|\mathcal{D}| = k \geq 4$, and assume inductively that the theorem holds for all degree sets with fewer than k elements. Let $\mathcal{D}$ be a set of k positive integers with δ odd, $\delta + 1 \notin \mathcal{D}$ and at least one even member.

(6a) Suppose $k = 4$, $\delta = 1$ and $\mathcal{D} = \{1, 3, 2m, \Delta\}$, where $4 \leq 2m < \Delta$. Let $\mathcal{D}' = \{1, 2, 2m-1\}$. By Lemma 3 there is a $(\Delta+1, \mathcal{D}')$-graph G with exactly one vertex y of degree 1. Then there is a $(\Delta+2, \mathcal{D})$-graph $G \vee^* K_1$ formed from G by adjoining a single vertex u and adding the edges $\{uv : v$ is a vertex of G, $v \neq y\}$. Now (4b) applies.

(6b) Suppose $k \geq 4$, $\delta = 1$ and $\mathcal{D}$ is not of the form $\{1, 3, 2m, \Delta\}$, where $4 \leq 2m < \Delta$. Let $\mathcal{D}' = \{d-2 : d \in \mathcal{D}, 1 < d < \Delta\}$. Since Δ must be even if all members of $\mathcal{D}^*$ are odd, in all cases it follows from the induction hypothesis that there is a $(\Delta-2, \mathcal{D}')$-graph G. Then $G \vee^* P_4$ is a $(\Delta+2, \mathcal{D})$-graph, and (4b) applies.

(6c) Finally suppose $k \geq 4$ and $\delta \geq 3$. Assume the theorem holds for all degree sets with exactly k elements and odd smallest member less than δ, with base case guaranteed by (6a) and (6b). Now let $\mathcal{D}^* = \{d-2 : d \in \mathcal{D}\}$. By hypothesis, there is a $(\Delta, \mathcal{D}^*)$-graph G. Then $G \vee 2K_1$ is a $(\Delta+2, \mathcal{D})$-graph, and (4b) applies. □

An $(n, \mathcal{D})$-graph G is a *progenitor* for the degree set $\mathcal{D}$ if there is no $(m, \mathcal{D})$-graph with $m < n$ and $m \equiv n \pmod 2$. Note that there is an $(m, \mathcal{D})$-graph for every $m > n$ with $m \equiv n \pmod 2$, by Lemma 2.

Corollary 1. *Every nonempty finite set of positive integers* $\mathcal{D}$ *with* $\max(\mathcal{D}) = \Delta$ *has a progenitor of order* $\Delta + 1$. *Moreover* $\mathcal{D}$ *has a progenitor of order* $\Delta + 2$, *with the following exceptions:* (a) *if all members of* $\mathcal{D}$ *are odd, it has no progenitor of odd order;* (b) *if* $\mathcal{D} = \{1, \Delta\}$, *with even* $\Delta \geq 4$, *then* $\mathcal{D}$ *has a progenitor of order* $\Delta + 4$.

If G and H are two graphs with the same degree set, $\mathcal{D}(G) = \mathcal{D}(H)$, their degree sequences can only differ with regard to the multiplicity with which any degree $d \in \mathcal{D}$ occurs. In particular, G and H are *degree sequence equivalent* (mod d) for some $d \in \mathcal{D}$ if G and H have the same multiplicities for each degree in $\mathcal{D} \setminus \{d\}$. Then Lemma 2 and the main theorem imply:

Corollary 2. *Let* $\mathcal{D}$ *be any finite set of positive integers with* $\min(\mathcal{D}) = \delta$, *and let* G *be a progenitor for* $\mathcal{D}$. *If* order $(G)=n$, *then for every* $m > n$ *with* $m \equiv n \pmod 2$ *there is an* $(m, \mathcal{D})$*-graph* H *which is degree sequence equivalent to* G *(mod* δ*).*

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