# NOTE ON FACTORIZATION OF COMPLETE GRAPHS INTO CATERPILLARS WITH SMALL DIAMETERS

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November 2004

ABSTRACT. We prove in this note that certain caterpillars with diameter 4 or 5 do not factorize complete graphs. This together with results by Kovarova [2,3] and Kubesa [5] gives the complete characterization of the caterpillars with diameter 4 that factorize the complete graph $K_{2n}$. For diameter 5, we again complement results by Kovarova [4] and Kubesa [6–9] to give the complete characterization for certain class of caterpillars.

## 1. INTRODUCTION

We say that a tree $T$ with $2n$ vertices factorizes the complete graph $K_{2n}$ if there exists a collection $T_1, T_2, \ldots, T_n$ of trees, all isomorphic to $T$, such that each edge of $K_{2n}$ belongs to exactly one $T_i$. Until recently, almost nothing was published on factorizations of complete graphs into isomorphic spanning trees. It is probably hopeless to expect a complete characterization any time soon. First attempts show that even for very simple classes of trees like caterpillars and lobsters the task is very complex. So far, T. Kovarova [2–4] and M. Kubesa [5–9] began investigating caterpillars with small diameters. They proved existence of factorizations for infinite classes of caterpillars with diameter 4, and with diameter 5 and at least one vertex of degree 2. However, there were still some classes in doubt. We show in this note that these classes do not allow factorizations and therefore complement their results to obtain the complete characterization for the above mentioned types of caterpillars.

A *caterpillar* is a tree in which each vertex of degree 1 is adjacent to a vertex of a path of length at least one. The path is then called the *spine* of the caterpillar. The spine of a caterpillar with diameter 4 will always consist of vertices $A, b, C$ and edges $Ab, bC$; the spine of a caterpillar with diameter 5 will consist of vertices $A, a, b, B$ and edges $Aa, ab, bB$. By a $(d_1, d_2, d_3)$-caterpillar or $(d_1, d_2, d_3, d_4)$-caterpillar we mean a caterpillar with $\deg(A) = d_1 \geq 2, \deg(b) = d_2, \deg(C) = d_3 \geq 2$ or with $\deg(A) = d_1 \geq 2, \deg(a) = d_2, \deg(b) = d_3, \deg(B) = d_4 \geq 2$, respectively. By a

1991 *Mathematics Subject Classification.* 05C70, 05C78.

*Key words and phrases.* Graph factorization, spanning trees.

Research for this article was supported by the University of Minnesota Duluth Grant 177–1009.

$[t_1, t_2, t_3]$-caterpillar (where $t_1 \geq t_2 \geq t_3 \geq 2$) we mean *any* $(t_i, t_j, t_k)$-caterpillar where $\{i, j, k\} = \{1, 2, 3\}$. The same convention holds for $[t_1, t_2, t_3, t_4]$-caterpillars.

## 2. Caterpillars with diameter 4

An easy computation shows that a complete graph with an odd number of vertices cannot be factorized into isomorphic trees. It is also obvious that if $T$ factorizes $K_{2n}$, then no vertex of $T$ can be of degree more than $n$. There are $n$ factors and every vertex is in each factor of degree at least one. If a vertex $v$ is of degree $n+1$ or more in $T_1$ and of degree at least one in each of $T_2, T_3, \ldots, T_n$, then its total degree in $K_{2n}$ is at least $2n$, which is absurd. In the following lemma we prove that every caterpillar with diameter 4 that factorizes the complete graph $K_{2n}$ *must* have one vertex of degree $n$. This also proves that the there are no caterpillars factorizing $K_{2n}$ besides those shown by Kovarova and Kubesa.

**Lemma 1.** *Let $R_{2n}$ be a $[t_1, t_2, t_3]$-caterpillar with $2n$ vertices and diameter 4 that factorizes the complete graph $K_{2n}$. Then $t_1 = n$.*

*Proof.* There are $n$ copies of $R_{2n}$ that factorize $K_{2n}$, call them $F_1, F_2, \ldots, F_n$. For a vertex $v$ we denote the degree of $v$ in $F_i$ by $\deg_i(v)$. We assume that $\deg_1(x) = t_1, \deg_1(y) = t_2, \deg_1(z) = t_3$. We have shown above that $\deg_1(x) = t_1 \leq n$. Now we show that $t_1 = n$. Suppose to the contrary that $t_1 < n$. Because we have $\sum_{i=1}^{n} \deg_i(x) = 2n - 1$, there must be a factor $F_j$, say $F_2$, such that $\deg_2(x) \geq 2$, otherwise we again get a contradiction. Then $\deg_2(x) = t_2$ or $\deg_2(x) = t_3$. In both cases we have

$$2n - 1 = \sum_{i=1}^{n} \deg_i(x) \geq t_1 + t_3 + \sum_{i=3}^{n} \deg_i(x) \geq t_1 + t_3 + (n-2)$$

and $n + 1 \geq t_1 + t_3$. But the sum of all degrees of vertices of $R_{2n}$ is $4n - 2$ and hence

$$t_1 + t_2 + t_3 + (2n-3) = 4n - 2,$$

which yields

$$t_1 + t_2 + t_3 = 2n + 1.$$

Since we have shown above that $t_1 + t_3 \leq n + 1$, it follows that $t_2 \geq n$, which contradicts our assumption that $n > t_1 \geq t_2$. □

It was proved by Eldergill [1] that a $(d_1, 2, d_3)$-caterpillar of diameter 4 does not factorize $K_{2n}$ for any $n$. He also showed that the $(2,3,2)$-caterpillar does not factorize $K_6$. On the other hand, the following result was proved by Kubesa [5] for $n$ odd and by Kovarova [2,3] for $n$ even.

**Theorem 2.** (Kovarova, Kubesa) *Let $n \geq 4$. Then every $(n, d_2, d_3)$-caterpillar with $3 \leq d_2 \leq n-1$ and $d_2 + d_3 = n+1$ and every $(d_1, n, d_3)$-caterpillar with $2 \leq d_1 \leq n-1$ and $d_1 + d_3 = n+1$ factorizes $K_{2n}$.*

So the following holds.

**Theorem 3.** *A $[t_1, t_2, t_3]$-caterpillar with diameter 4 factorizes $K_{2n}$ if and only if $t_1 = n \geq 4$ and it is not the $(n, 2, n-1)$-caterpillar.*

## 3. Caterpillars with diameter 5

In the following lemmas we show two classes of caterpillars of diameter 5 that do not factorize the complete graphs $K_{2n}$. This result completes the results by Kovarova and Kubesa on the characterization of caterpillars with diameter 5 and at least one vertex of degree 2 that factorize complete graphs.

There is no tree with less than 6 vertices of diameter 5 and the only caterpillar of diameter 5 with 6 vertices is the path $P_6$ that clearly factorizes $K_6$. Therefore, we will further assume that $n \geq 4$. We also observe that if a caterpillar $R_{2n}$ with diameter 5 factorizes $K_{2n}$ and $n \geq 5$, then there are at least two vertices in $R_{2n}$ of degree more than 2. In the opposite case the only vertex of degree more than 2 has degree $2n-4 > n$, which is impossible.

**Lemma 4.** *Let $R_{2n}$ be an $(n, 2, 2, n-2)$-caterpillar with $2n$ vertices, $n \geq 5$. Then $R_{2n}$ does not factorize $K_{2n}$.*

*Proof.* We proceed by contradiction. Suppose it is not the case and there exists a factorization into factors $F_1, F_2, \ldots, F_n$.

For $n = 4$ and 5 the result was proved by Eldergill [1]. Hence, we suppose that $n \geq 6$. Let $\deg_1(x) = n$ and $\deg_1(y) = n-2$. Then $x$ and $y$ are not adjacent in $F_1$. Because $\sum_{i=1}^{n} \deg_i(x) = 2n-1$, obviously $\deg_i(x) = 1$ for $i = 2, 3, \ldots, n$. Similarly, as $\sum_{i=1}^{n} \deg_i(y) = 2n-1$ and $n-2 > 2$, we can see that $\sum_{i=2}^{n} \deg_i(y) = n+1$, and because $y$ is never isolated in any factor, it follows that $\deg_i(y) \leq 3$ for $i \geq 2$. But in each factor every vertex of degree one is adjacent either to a vertex of degree $n$ or of degree $n-2 \geq 4$. Since $x$ is in each of the factors $F_2, F_3, \ldots, F_n$ of degree 1 while $y$ is there of degree less than $n-2$, they can never be adjacent in any of these factors. We have noticed above that they are not adjacent in $F_1$ either and hence the edge $xy$ does not appear in any factor. This is a contradiction and the proof is complete. □

**Lemma 5.** *Let $R_{2n}$ be an $(n, 2, n-m, m)$-caterpillar with the spine $A, a, b, B$ where $2 \leq m \leq n-3$ and $n \geq 4$. Then $R_{2n}$ does not factorize $K_{2n}$.*

*Proof.* For $n = 4, 5$ the result was proved by Eldergill [1]. Hence, we assume $n \geq 6$. Suppose such a factorization exists and the factors are $F_1, F_2, \ldots, F_n$.

Clearly, each vertex can be of degree $n$ only in one of the factors $F_1, F_2, \ldots, F_n$. Therefore, let $x_1, x_2, \ldots, x_n$ be vertices such that $\deg_i(x_i) = n$. For every $j \neq i$ it follows that $\deg_j(x_i) = 1$. Hence, the edge $x_i x_k$ can only appear in either $F_i$ or in $F_k$. If it is not the case and $x_i x_k$ belongs to $F_s, s \neq i, k$, then $\deg_s(x_i) = \deg_s(x_k) = 1$ and $F_s$ is disconnected, which is impossible. Let $y_1, y_2, \ldots, y_n$ be the remaining vertices of $K_{2n}$.

We define $\phi_i$ to be the mapping that takes $R_{2n}$ onto $F_i$ and assume that $\phi_i(A) = x_i$. Now we define for every vertex $y_j, j = 1, 2, \ldots, n$, an (unordered) multiset $P(y_j)$ of all pre-images of $y_j$ in the mappings $\phi_i, i = 1, 2, \ldots, n$. Clearly, each $P(y_j)$ can contain only vertices $a, b, B$ and vertices of degree 1. If some $P(y_j)$ does not contain $B$, then another one, $P_k$, contains $B$ at least twice. The same holds for $b$. We now prove several claims related to this property.

*Claim 1.* If $F_1, F_2, \ldots, F_n$ is the desired factorization, then there is no $P(y_j)$ that contains $B$ exactly twice while $b \notin P(y_j)$.

Suppose the contrary and assume without loss of generality (WLOG) that $\phi_1(B) = \phi_2(B) = y_1$ and recall that $\phi_1(A) = x_1, \phi_2(A) = x_2$. Every edge $x_i y_j$ of

$K_{2n}$ is in each factor an image of one of the edges $Aa, AA', bb', BB'$, where each of $A', b', B'$ is a vertex of degree one in $R_{2n}$. Obviously, $x_1y_1 \notin F_1$ and hence it cannot be an image of $Aa$ or $AA'$. As $y_1$ is not an image of $b$ in any factor, $x_1y_1$ cannot be the image of $bb'$ either. Hence, it must be the image $\phi_2(BB')$. So $x_1y_1 \in F_2$. Therefore, $F_2$ cannot contain the edge $x_1x_2$, as $x_1$ is of degree 1 in $F_2$. Similarly, $x_2y_1 \notin F_2$ and therefore it is not an image of $Aa$ or $AA'$. Again it is not an image of $bb'$ since $b$ is never a pre-image of $y_1$. Thus $x_2y_1 = \phi_1(BB') \in F_1$. As above, $F_1$ cannot contain the edge $x_1x_2$. Now the edge $x_1x_2$ is neither in $F_1$ nor in $F_2$. This is impossible, because in all other factors both $x_1$ and $x_2$ are of degree 1 and $x_1x_2$ would have to be isolated in whatever factor it would belong to. Therefore, no vertex $y_j$ can be an image of $B$ in exactly two factors unless it is also an image of $b$.

*Claim 2.* If $F_1, F_2, \dots, F_n$ is the desired factorization, then there is no $P(y_j)$ that contains $b$ exactly twice while $B \notin P(y_j)$.

Repeating the same argument as above, we can show that the same holds for an image of $b$.

*Claim 3.* Let $m \geq \lceil \frac{n}{2} \rceil + 1$. Then every $P(y_j)$ contains each of the vertices $a, b, B$ exactly once.

Obviously, no vertex $y_j$ can be of degree $m$ (that means, an image of $B$) in two or more different factors, for if it were, then there would be at most $n-3$ edges incident with $y_j$ in $K_{2n}$ left for the remaining $n-2$ factors, which is impossible. Hence, each $y_j$ is of degree $m$ in exactly one factor. If then $y_j$ is not of degree $n-m$ in another factor, we must have a vertex $y_t$ such that it is of degree $m$ in one factor and of degree $n-m$ (and therefore, an image of $b$) in at least two factors. But then again there are at most $(2n-1)-m-2(n-m) = m-1 \leq n-4$ edges left for the remaining $n-3$ factors, which is a contradiction. It follows that if $m \geq \lceil \frac{n}{2} \rceil + 1$, then for each $y_j$ the set $P(y_j)$ contains each of the vertices $b, B$ exactly once. Because then we need to distribute the remaining $(2n-1)-m-(n-m) = n-1$ edges among the remaining $n-2$ factors, clearly $y_j$ must be of degree 2 in exactly one of them, which proves the claim.

*Claim 4.* Let $m \leq \lfloor \frac{n}{2} \rfloor - 1$. Then every $P(y_j)$ contains each of the vertices $a, b, B$ exactly once.

The argument is similar as above, since then $n - m \geq \lceil \frac{n}{2} \rceil + 1$.

Now we need to treat the cases when $\lfloor \frac{n}{2} \rfloor \leq m \leq \lceil \frac{n}{2} \rceil$.

*Claim 5.* If $F_1, F_2, \dots, F_n$ is the desired factorization, $n = 2m$ and $m = \frac{n}{2}$, then every $P(y_j)$ contains each of the vertices $a, b, B$ exactly once.

Because $n = 2m$, it follows that $n - m = m$ and each factor contains two vertices of degree $m$. Hence, there must be a vertex $y_j$ that is of degree $m$ in at least two factors. On the other hand, no vertex can be of degree $m$ in three or more factors, otherwise there is not enough edges in $K_{2n}$ for the remaining factors at that vertex. Therefore, each vertex $y_j$ is of degree $m$ in exactly two factors. According to Claims 1 and 2, this can happen only when $y_j$ is once the image of $b$ and once of $B$. There are still $n-1$ unused edges to be distributed among the $n-2$ remaining factors. It follows that $y_j$ must be exactly once the image of $a$ since $a$ has degree 2.

*Claim 6.* If $F_1, F_2, \dots, F_n$ is the desired factorization, $n$ is odd and $\lfloor \frac{n}{2} \rfloor \leq m \leq \lceil \frac{n}{2} \rceil$, then every $P(y_j)$ contains each of the vertices $a, b, B$ exactly once.

First assume that $m = \lfloor \frac{n}{2} \rfloor$. Then $n = 2m + 1$ and $(n, 2, n - m, m) = (n, 2, m+1, m)$ which yields $\deg(b) = m+1 = \lceil \frac{n}{2} \rceil$. If we assume that $m = \lceil \frac{n}{2} \rceil$, then $n = 2m - 1$ and $(n, 2, n - m, m) = (n, 2, m - 1, m)$ which yields $\deg(B) = m = \lceil \frac{n}{2} \rceil$. In either case we have one of $b, B$ of degree $\lceil \frac{n}{2} \rceil$.

Suppose then that a vertex $y_j$ is of degree $\lceil \frac{n}{2} \rceil$ more than once. Then of course it can happen at most twice as otherwise there is not enough edges for the remaining factors incident with $y_j$. If $y_j$ is of degree $\lceil \frac{n}{2} \rceil$ in exactly two factors, then in each of the remaining $n - 2$ factors $y_j$ is of degree 1, since we have used $2\lceil \frac{n}{2} \rceil = n+1$ edges already. But this is impossible by Claims 1 and 2. Hence, every $y_j$ is of degree $\lceil \frac{n}{2} \rceil$ in one factor and of degree $\lfloor \frac{n}{2} \rfloor$ in another. The remaining $n - 1$ edges incident with $y_j$ can be distributed into the $n - 2$ remaining factors only such that $y_j$ is of degree 2 in exactly one factor and the claim is proved.

By now we have shown that if the desired factorization exists, then in any case every $P(y_j)$ contains each of the vertices $a, b, B$ exactly once. In the following claim we show that even then the existence of such a factorization is impossible.

*Claim 7.* Let $F_1, F_2, \ldots, F_n$ be factors of $K_{2n}$, all isomorphic to the caterpillar $R_{2n}$ as described in the assumption of this lemma. Let every $P(y_j)$ contain each of the vertices $a, b, B$ exactly once. Then the factors $F_1, F_2, \ldots, F_n$ do *not* form a factorization of $K_{2n}$.

Let again $R_{2n}$ be the $(n, 2, n - m, m)$-caterpillar with the spine $A, a, b, B$, edges $Aa, ab, bB$, and $\deg(A) = n$. Denote by $b'_k, k = 1, 2, \ldots, n - m - 2$ the neighbors of $b$ of degree 1. Since $m \leq n - 3$, at least one such $b'_k$ exists. Similarly denote by $B'_t, t = 1, 2, \ldots, m-1$ the neighbors of $B$ of degree 1. Since $m \geq 2$, at least one such $B'_t$ exists. We suppose that $\phi_i(A) = x_i$ and $\phi_i(b) = y_i$ for $i = 1, 2, \ldots, n$. Then of course $x_i y_i \notin F_i$ for any $i$. We can also WLOG assume that the factors are ordered such that $\phi_1(B) = y_2$ and in general $\phi_i(B) = y_{i+1}$ and $\phi_s(B) = y_1$, for some $s \leq n$. We observe that $s \geq 3$, since if $s = 2$ then the edge $y_1 y_2$ appears in two factors $F_1$ and $F_2$.

Because every vertex $x_i$ is of degree 1 in each factor except for $F_i$, it is clear that the edge $x_1 x_2$ must belong to either $F_1$ or to $F_2$. For the same reason, every edge $x_i y_i$ must be an image of the edge $BB'_t$ for some $t$. But the vertex $y_2$ is an image of $B$ exactly once, namely in $F_1$, and hence $F_1$ contains the edge $x_2 y_2$. Therefore, $F_1$ does not contain $x_1 x_2$ and it must belong to $F_2$. Now we will see that there is no factor that contains the edge $x_1 y_2$. This edge cannot belong to $F_1$, because $x_1 = \phi_1(A)$ and $y_2 = \phi_1(B)$ and $A$ and $B$ are not adjacent in $R_{2n}$. In all other factors $x_1$ is of degree 1 and therefore $x_1 y_2$ can only be the image of one of the edges $bb'_k, BB'_t$. Because $y_2$ is the image of $B$ only in $F_1$ and we have shown that $x_1 y_2$ cannot belong to $F_1$, it must be the image of $bb'_k$. But $y_2$ is the image of $b$ only in $F_2$. At the same time in $F_2$ there is the edge $x_1 x_2$ and therefore $x_1$ cannot be incident with another edge. Therefore, $x_1 y_2$ is not contained in any factor and the claim is proved.

So we have shown in Claim 6 that if there is a factorization of $K_{2n}$ into $n$ copies of $R_{2n}$, then each vertex $y_j$ is an image of each of $a, b, B$ in exactly one factor. On the other hand, in Claim 7 we have proved that if there are factors $F_1, F_2, \ldots, F_n$ such that every $y_j$ is an image of each of $a, b, B$ in exactly one of them, then the factors $F_1, F_2, \ldots, F_n$ do not form a factorization of $K_{2n}$. This contradiction concludes the proof of the Lemma. □

We can summarize the above two lemmas into one result.

**Theorem 6.** *An $(n, 2, n-m, m)$-caterpillar with $2n$ vertices and $2 \le m \le n-2$ does not factorize $K_{2n}$ for any $n$.*

For $n$ odd it was also shown by Kubesa [6,7,8] that all caterpillars of diameter 5 with one vertex of degree $n$ and one or two vertices of degree 2 different from the ones in Theorem 6 factorize $K_{2n}$. Analogical result for $n$ even was proved by Kovarova [4]. We include their results in one theorem.

**Theorem 7.** (Kovarova, Kubesa) *Let $n \ge 5, n \ne 2^k$ and $R_{2n}$ be an $[n, t_2, t_3, 2]$-caterpillar with $2n$ vertices and $t_2 \ge 3$. Then $R_{2n}$ factorizes $K_{2n}$ if it is not an $(n, 2, n-m, m)$-caterpillar for $2 \le m \le n-2$.*

Therefore, the following corollary holds.

**Corollary 8.** *An $[n, t_2, t_3, 2]$-caterpillar $R_{2n}$ with diameter 5 and $n \ge 5, n \ne 2^k$, factorizes $K_{2n}$ into $n$ isomorphic copies if and only if $R_{2n}$ is not an $(n, 2, n-m, m)$-caterpillar for $2 \le m \le n-2$.*

It was also shown by Kovarova [4] and Kubesa [9] that all $[n-1, t_2, t_3, 2]$-caterpillars factorize $K_{2n}$ for any admissible pair $t_2 \ge t_3 \ge 2$ .

**Theorem 9.** (Kovarova, Kubesa) *All $[n-1, t_2, t_3, 2]$-caterpillars with $2n$ vertices factorize $K_{2n}$ for every $n \ge 3, n \ne 2^k$.*

By proving the following lemma we show that Theorems 8 and 9 actually give the complete characterization of $[t_1, t_2, t_3, 2]$-caterpillars with with $2n$ vertices that factorize $K_{2n}$.

**Lemma 10.** *If a $[t_1, t_2, t_3, 2]$-caterpillar $R_{2n}$ factorizes $K_{2n}$, then $t_1 \ge n-1$.*

*Proof.* For $n = 3$ the assertion is obvious. Therefore, we assume now that $n \ge 4$. Let $x_1, x_2, \ldots, x_{2n}$ be the vertices of $R_{2n}$. Then indeed $\sum_{i=1}^{2n} \deg(x_i) = t_1 + t_2 + t_3 + 2 + (2n-4) = 4n-2$, which yields $t_1 + t_2 + t_3 = 2n$. Therefore, $t_1 > 2n/3$ and no vertex $x_i$ can be of degree $t_1$ in more than one factor. We can suppose WLOG that $\deg_i(x_i) = t_1$ for $i = 1, 2, \ldots, n$.

Suppose now to the contrary that $t_1 \le n-2$. Then we can assume $t_1 = n-q$, where $q \ge 2$. Because $t_2 \le t_1$, then $t_1 + t_2 \le 2n - 2q$ and hence $t_3 \ge 2q$. It follows that no vertex $x_j$ for $j \le n$ can be of degree $t_2$ or $t_3$ in any factor, otherwise $\sum_{i=1}^{n} \deg_i(x_j) \ge (n-q) + 2q + (n-2) = 2n + q - 2 \ge 2n$, which is impossible. Because $\deg_1(x_1) = n - q \le n-2$, there must be at least two factors in which $x_1$ is of degree 2, otherwise $\sum_{i=1}^{n} \deg_i(x_1) \le n - q + 2 + (n-2) = 2n - q \le 2n-2$, a contradiction. But if $x_1$ is of degree 2 in two or more factors, another vertex $x_j \in \{x_2, x_3, \ldots, x_n\}$ is in $F_j$ of degree $t_1 = n-q \le n-2$ and in all $n-1$ remaining factors of degree 1. Thus $\sum_{i=1}^{n} \deg_i(x_j) = (n-q) + (n-1) \le 2n-3$, which is the final contradiction showing that the maximum degree $t_1$ must be at least $n-1$. □

The characterization mentioned above immediately follows.

**Theorem 11.** *A $[t_1, t_2, t_3, 2]$-caterpillar $R_{2n}$ with $n \ge 3$, $n \ne 2^k, t_1 \ge t_2 \ge t_3$ and diameter 5 factorizes $K_{2n}$ into $n$ isomorphic copies if and only if $t_1 + t_2 + t_3 = 2n$, $n \ge t_1 \ge n-1$, and $R_{2n}$ is not an $(n, 2, n-m, m)$-caterpillar with $2 \le m \le n-2$.*

## 4. Concluding remarks

The resaerch dealing with the remaining class of caterpillars with diameter 5, namely the caterpillars with no vertices of degree 2, is still in progress. However, we restrict here the class of possible candidates by proving the following lemma.

**Lemma 12.** *If a $[t_1, t_2, t_3, t_4]$-caterpillar $R_{2n}$ with $t_4 \geq 3$ factorizes $K_{2n}$ into $n$ isomorphic copies, then either $t_1 = n$ and $t_2 + t_3 + t_4 = n + 2$ or $t_1 + t_4 = t_2 + t_3 = n + 1$.*

*Proof.* Obviously, $t \geq 3$ yields $n \geq 5$. We can suppose that there is a vertex $x_1$ such that $\deg_1 x_1 = t_1$ and $\deg_i x_1 \geq \deg_{i+1} x_1$ for $i = 1, 2, \ldots, n-1$. By the necessary condition, $t_1 \leq n$. If $t_1 = n$, then obviously $t_2 + t_3 + t_4 = n + 2$, since $t_1 + t_2 + t_3 + t_4 + (2n - 4) = 4n - 2$. Let $t_1 \leq n - 1$. Hence, $\deg_2 x_1 \geq 2$, otherwise $\sum_{i=1}^{n} \deg_i(x_1) = t_1 + (n-1) \leq 2n - 2$, which is impossible. Since there is no vertex of degree 2 in our $R_{2n}$, we have $\deg_2 x_1 \geq t_4$. If now $t_1 + t_4 \geq n + 2$, then $\sum_{i=1}^{n} \deg_i(x_1) \geq t_1 + t_4 + (n-2) \geq 2n$, which is again a contradiction.

Therefore, $t_1 + t_4 \leq n + 1$. If $t_1 + t_4 \leq n$, then of course $t_2 + t_3 \geq n + 2$, and consequently $t_1 + t_2 \geq t_1 + t_3 \geq n + 2$. In that case, if $\deg_i x_1 = t_2$ or $t_3$ for some $i$, we again get $\sum_{i=1}^{n} \deg_i(x_1) \geq t_1 + t_3 + (n-2) \geq 2n$, the same contradiction as above. More generally, no vertex can be of degree $t_1$ in one factor and of degree more than $t_4$ in another. So we must have $\deg_2 x_1 = t_4$ and also $\deg_3 x_1 = t_4$. But then there is another vertex, say $x_2$, such that it is of degree $t_1 \leq n - 1$ in one factor and of degree 1 in all remaining factors, having the total degree $\sum_{i=1}^{n} \deg_i(x_2) = t_1 + (n-1) \leq 2n - 2$, which is the final contradiction. The only remaining possibility is $t_1 + t_4 = n + 1$, which instantly yields $t_2 + t_3 = n + 1$. □

We remark that for diameter 5 there are still two gaps to be filled. The caterpillars with $2^k$ vertices in general, and the caterpillars with no vertex of degree 2. These classes are currently being studied but the results are too incomplete to be stated here.

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